ON THE DEGENERATE CAUCHY PROBLEM

R. W. CARROLL AND C. L. WANG

1. The problem treated here is an abstract version of the Cauchy problem for an equation of mixed type in the hyperbolic region with initial data on the parabolic line (cf. **2**, **3**, **5**, **11**, **13**, **14**, **15**, **16**, **21**, **27**). A more complete bibliography may be found in (**3**, **5**, **18**). We begin with the equation (**6**)

$$u'' + \Lambda^{\alpha} S(t) u' + \Lambda^{\beta} R(t) u + \Lambda q(\Sigma) u = f, \tag{1.1}$$

where Λ is a (closed) densely defined self-adjoint operator in a separable Hilbert space H with $(\Lambda u, u) \geqslant c\|u\|^2$, $c > 0$, $\Sigma = \Lambda^{-1} \in \mathfrak{L}(H)$ ($\mathfrak{L}(H)$ is the space of continuous linear maps $H \to H$), $q(\Sigma) = a(t) + B(t)\Sigma$ ($a(t)$, which vanishes as $t \to 0$, being a function of t whereas $B(t) \in \mathfrak{L}(H)$ for now), and $S(t) \in \mathfrak{L}(H)$, $R(t) \in \mathfrak{L}(H)$. It is assumed that all operators commute, and we seek $u \in \mathfrak{C}^2(H)$ ($\mathfrak{C}^m(H)$ is the space of m-times continuously differentiable functions of t with values in H) satisfying (1.1) with

$$u(0) = 0, \qquad u'(0) = 0. \tag{1.2}$$

Precise hypotheses will be given later. We note in passing the possibility of exploiting techniques of the type developed in (**20**) to our problems; this will be considered in subsequent work.

Existence and uniqueness theorems will be obtained for (1.1)–(1.2), under suitable hypotheses, by applying spectral techniques developed in (**6**, **7**). We obtain results similar to those of (**15**) in the special case when $a = t^m$, $R(t) = Rr(t)$, $r = t^n$ (other assumptions on $S(t)$, etc. also holding); we require slightly more in this case but our solution is stronger. This situation corresponds to the case

$$\int_{\tau}^{l} \frac{|r|^2}{a} < \infty \qquad \text{as } \tau \to 0.$$

When

$$\int_{\tau}^{l} \frac{|r|^2}{a} \to \infty$$

some interesting new phenomena occur; it is possible to allow a to be non-monotone (**5**) if much more is required of f and, of course, less of r since

$$\int_{\tau}^{l} \frac{|r|^2}{a} \to \infty.$$

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We have not tried to compare the results to those of **(5)** since the solutions are of a different nature, those of the present paper being stronger (i.e. more regular); on the other hand the conditions of **(5)** are weaker in general.

2. In order to apply spectral methods we assume first that $S(t) = Ss(t)$, $R(t) = Rr(t)$, $B(t) = Bb(t)$, where B, R, S, Σ commute and are bounded normal with $b, r, s \in C^0[0, l]$, $a \in C'[0, l]$ (also assume that Λ commutes with B, R, S). The case of $\Lambda^{\alpha-\frac{1}{2}}S$ and $\Lambda^{\beta-1}R$ bounded normal, for example, can also be treated (see **6**). Let $\mathfrak{A}$ be the uniformly closed $*$ algebra generated by Σ, B, R, S, B^*, R^*, S^*, and I; we associate with these operators the complex spectral variables $z_0, z_1, \ldots, z_6$ (I omitted; cf. **6, 8**). Then the map $\alpha\colon \Phi_{\mathfrak{A}} \to \mathbf{C}^7$ given by $\alpha(\phi) = (\hat{\Sigma}(\phi), \hat{B}(\phi), \ldots, \hat{S}^*(\phi))$ is a homeomorphism of the carrier space $\Phi_{\mathfrak{A}}$ with the joint spectrum σ of the elements $\Sigma, B, R, \ldots, S^*$; cf. **(1, 6, 22)**. We consider now in connection with (1.1) the equation ($\lambda = 1/z_0$; z_0 is real)

$$u'' + \lambda^\alpha z_3 s(t)u' + \lambda^\beta z_2 r(t)u + \lambda[a(t) + z_0 z_1 b(t)]u = 0. \tag{2.1}$$

Solutions $Z(t, \tau, z_i, \lambda)$ and $Y(t, \tau, z_i, \lambda)$ of (2.1) with $Z(\tau, \tau) = 1$, $Z_t(\tau, \tau) = 0$, $Y(\tau, \tau) = 0$, $Y_t(\tau, \tau) = 1$ (cf. **7**) will give rise to operators in the von Neumann algebra $\mathfrak{A}''$ if for example Y and Z are continuous in (z_i, λ) for $|z_i| \leqslant c_1$ ($i = 1, \ldots, 6$), $|\lambda| \leqslant R_0$ (R_0 arbitrary), and bounded for $|z_i| \leqslant c_1$, $|z_0| \leqslant 1/c$ (this is proved in **(6)**). The constant c_1 is chosen so that

$$c_1 \geqslant \max(||B||, ||R||, ||S||)$$

and then the joint spectrum σ lies within the region $|z_i| \leqslant c_1$ ($i = 1, \ldots, 6$), $|z_0| \leqslant 1/c$ (note that $\lambda \to \infty$ corresponds to $z_0 \to 0$).

We know by classical results (cf. **10, 12**) that for $0 \leqslant \tau \leqslant t \leqslant l < \infty$ there exist unique Z and Y as required, continuous in (t, τ, z_i, λ) in the region $0 \leqslant \tau \leqslant t \leqslant l < \infty$, $|z_i| \leqslant c_1$ ($i = 1, \ldots, 6$), $0 < z_0 \leqslant 1/c$ (note Z, Y are not analytic single-valued in z_i, λ because α, β may be fractional). Thus the Green's operator associated with (2.1) will be

$$\mathfrak{g} = \begin{pmatrix} Z & \sqrt{\lambda}\, Y \\ \dfrac{1}{\sqrt{\lambda}} Z_t & Y_t \end{pmatrix}, \qquad \mathfrak{g}(\tau, \tau) = I, \tag{2.2}$$

and will satisfy the first-order equation

$$\partial \mathfrak{g}/\partial t + \lambda^{\frac{1}{2}} \mathfrak{h}(t)\mathfrak{g} = 0, \tag{2.3}$$

where **(6, 9)**

$$\mathfrak{h} = \begin{pmatrix} 0 & -1 \\ a(t) + z_0 z_1 b(t) + \lambda^{\beta-1} z_2 r(t) & \lambda^{\alpha-\frac{1}{2}} z_3 s(t) \end{pmatrix}. \tag{2.4}$$

The problem now is to find suitable bounds for $\mathfrak{g}$. Such estimates will be based on a method developed in **(7, 8)**. First note **(24)** that

$$\partial\mathfrak{g}(t,\tau)/\partial\tau - \lambda^{\frac{1}{2}}\mathfrak{g}(t,\tau)\mathfrak{h}(\tau) = 0. \tag{2.5}$$

Hence if $\mathbf{u}_t + \lambda^{\frac{1}{2}}\mathfrak{h}(t)\mathbf{u} = \mathbf{f}$, then $(\mathbf{u}(\tau) = 0)$

$$\mathbf{u}(t) = \int_\tau^t \mathfrak{g}(t,\xi)\mathbf{f}(\xi)d\xi. \tag{2.6}$$

Therefore recalling the nature of $\mathfrak{g}$ in (2.2) and associating operators $\mathbf{Z}$, $\mathbf{Y}$, $\mathbf{G}$, $\mathbf{H}$ with Z, Y, $\mathfrak{g}$, $\mathfrak{h}$, we obtain formally for the solution of (1.1)

$$u(t) = \int_\tau^t \mathbf{Y}(t,\xi)f(\xi)d\xi \tag{2.7}$$

(here $u_1 = u$, $u_2 = u'/\sqrt{\lambda}$ **(6)**; thus $\mathbf{f}$ above corresponds to $\binom{0}{f/\sqrt{\lambda}}$). Relations for $\mathbf{Y}$ of the form derived in **(7, 8)** will also be valid.

Therefore let $Y(t, \tau, z_i, \lambda)$ be the unique solution of (2.1) with $Y(\tau,\tau) = 0$, $Y_t(\tau,\tau) = 1$. Replace t by ξ and multiply (2.1) by $\bar{Y}_\xi$. This gives, taking real parts and assuming $a(t)$ real,

$$\begin{aligned} d|Y_\xi|^2/d\xi + 2\operatorname{Re}(\lambda^\alpha z_3 s(\xi))|Y_\xi|^2 + 2\operatorname{Re}(\lambda^\beta z_2 r(\xi)Y\bar{Y}_\xi) \\ + \lambda a(\xi)d|Y|^2/d\xi + 2\operatorname{Re}(z_1 b(\xi)\,Y\bar{Y}_\xi) = 0. \end{aligned} \tag{2.8}$$

Now note that $|r\lambda^\beta Y Y_\xi| \leqslant \frac{1}{2}(|r|^2\lambda^{2\beta}|Y|^2 + |Y_\xi|^2)$ and thus, on integration,

$$\begin{aligned} |Y_t|^2 - 1 + \int_\tau^t 2\operatorname{Re}(\lambda^\alpha z_3\, s(\xi))|Y_\xi|^2\, d\xi + \lambda a(t)|Y|^2 \\ - \lambda\int_\tau^t a'|Y|^2 \leqslant \int_\tau^t |z_1|(|b|^2|Y|^2 + |Y_\xi|^2)d\xi + \int_\tau^t |z_2|(|r|^2\lambda^{2\beta}|Y|^2 + |Y_\xi|^2)d\xi. \end{aligned} \tag{2.9}$$

If now $\operatorname{Re}(z_3\, s(t)) \geqslant 0$, then the term in z_3 may be neglected; we assume this holds for the moment, and assume further that $2\beta \leqslant 1$. Recalling that $|z_i| \leqslant c_1$, $|z_0| \leqslant 1/c$, there results for $\lambda \geqslant 1$ (recall that $0 < \lambda_0 \leqslant \lambda$, $\lambda_0 = c$)

$$|Y_t|^2 + \lambda a(t)|Y|^2 \leqslant 1 + 2c_1\int_\tau^t |Y_\xi|^2\, d\xi + \lambda\int_\tau^t P|Y|^2\, d\xi, \tag{2.10}$$

where

$$P = a' + c_1\left(|r|^2 + \frac{1}{\lambda}|b|^2\right). \tag{2.11}$$

Adding now

$$2c_1\int_\tau^t \lambda a|Y|^2\, d\xi$$

to the right-hand side of (2.10), we have

$$|Y_t|^2 + \lambda a(t)|Y|^2 \leqslant 1 + \lambda\int_\tau^t P|Y|^2\, d\xi + 2c_1\int_\tau^t (|Y_\xi|^2 + \lambda a|Y|^2)d\xi \tag{2.12}$$

and to this the Gronwall lemma **(23)** may be applied to give

$$(2.13)\qquad |Y_t|^2 + \lambda a(t)|Y|^2 \leqslant \exp[2c_1(t-\tau)] + \int_\tau^t \lambda P|Y|^2 \exp[2c_1(t-\xi)]d\xi.$$

In particular we have, setting $E(t, \tau) = \exp[2c_1(t-\tau)]$,

$$(2.14)\qquad \lambda a(t)|Y|^2 \leqslant E(t,\tau) + \int_\tau^t \lambda P|Y|^2 E(t,\xi)d\xi.$$

We shall now prove a lemma which will be used to treat (2.14); for our purposes it will give a much better result than merely rough estimates for E, etc. and another application of the Gronwall lemma would produce. We remark, however, that a simultaneous bound for $|Y_t|^2 + \lambda a(t)|Y|^2$ can be obtained directly from (2.10) **(26)**.

LEMMA 1. *Given* (2.14) *with* $P \geqslant 0$, *it follows that for* $0 < \tau \leqslant t \leqslant l < \infty$ *and* $\lambda \geqslant 1$

$$(2.15)\qquad \lambda a(t)|Y|^2 \leqslant E(t,\tau)\exp\left(\int_\tau^t \frac{P}{a}d\xi\right).$$

Proof. Let

$$\chi(t,\tau) = \int_\tau^t \lambda P|Y|^2 E(t,\xi)d\xi;$$

then

$$(2.16)\qquad \chi' = \lambda P(t)E(t,t)|Y|^2 + \int_\tau^t \lambda P|Y|^2 E'(t,\xi)d\xi$$

$$= \lambda P(t)|Y|^2 + 2c_1\chi.$$

Multiplying (2.14) by λP and using (2.16), we obtain

$$(2.17)\qquad a(\chi' - 2c_1\chi) \leqslant PE + P\chi.$$

Thus defining

$$F(t,\tau) = \exp\left(-\int_\tau^t \left(\frac{P}{a} + 2c_1\right)d\xi\right),$$

we obtain from (2.17)

$$(2.18)\qquad (F\chi)' \leqslant (P/a)E(t,\tau)F(t,\tau).$$

However, clearly

$$E(t,\tau)F(t,\tau) = \exp\left(-\int_\tau^t \frac{P}{a}d\xi\right),$$

and hence (2.18) gives

$$(2.19)\qquad (F\chi)' \leqslant \left[-\exp\left(-\int_\tau^t \frac{P}{a}d\xi\right)\right].$$

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Since $F(\tau, \tau)\chi(\tau, \tau) = 0$ (recall $\tau > 0$ here), we have from (2.19)

$$F(t,\tau)\chi \leqslant 1 - \exp\left(-\int_\tau^t \frac{P}{a}\,d\xi\right), \tag{2.20}$$

which may be written

$$\chi + E(t,\tau) \leqslant E(t,\tau)\exp\left(\int_\tau^t \frac{P}{a}\,d\xi\right). \tag{2.21}$$

This yields the lemma.

Now note that

$$\frac{P}{a} = \frac{a'}{a} + \frac{c_1}{a}\left(|r|^2 + \frac{1}{\lambda}|b|^2\right)$$

and hence

$$\exp\left(\int_\tau^t \frac{P}{a}\,d\xi\right) = \frac{a(t)}{a(\tau)}\exp\left(\int_\tau^t c_1\left(\frac{|r|^2}{a} + \frac{|b|^2}{\lambda a}\right)d\xi\right). \tag{2.22}$$

If $\lambda_0 < 1$ we can carry through the estimates with $|r|^2$ replaced by $|r|^2\lambda^{2\beta-1}$ and hence there results

Proposition 1. *The function Y, solution of* (2.1) *with*

$$Y(\tau,\tau) = 0,\ Y_t(\tau,\tau) = 1 \qquad (0 \leqslant \tau \leqslant t \leqslant l < \infty),$$

satisfies the estimate for $\tau > 0$ $(a > 0)$:

$$\begin{aligned} a(\tau)|Y|^2 &\leqslant \frac{1}{\lambda}E(t,\tau)\exp\left(c_1\int_\tau^t\left(\frac{|r|^2}{a} + \frac{|b|^2}{\lambda a}\right)d\xi\right) \\ &\leqslant \frac{c_2}{\lambda}\exp\left(\tilde{c}_1\int_\tau^t\left(\frac{|r|^2}{a} + \frac{|b|^2}{\lambda a}\right)d\xi\right), \end{aligned} \tag{2.23}$$

where $\tilde{c}_1 = c_1 \max(1, \lambda_0^{2\beta-1})$.

Now besides $a(\tau) \to 0$ as $\tau \to 0$, the functions

$$\phi(t,\tau) = \exp\tilde{c}_1\int_\tau^t \frac{|r|^2}{a}\,d\xi, \qquad \psi(t,\tau) = \exp\frac{\tilde{c}_1}{\lambda}\int_\tau^t \frac{|b|^2}{a}\,d\xi \tag{2.24}$$

may become infinite as $\tau \to 0$. Thus noting that $\phi(t,\tau) \leqslant \phi(l,\tau)$, $\psi(t,\tau) \leqslant \psi(l,\tau)$, we may state, recalling that $\lambda \geqslant \lambda_0 > 0$ and observing that that $\psi(l,\tau;\lambda) \leqslant \psi(l,\tau;\lambda_0)$,

Corollary. *The function Y satisfies the estimate*

$$\phi(\tau)\psi(\tau)a(\tau)|Y|^2 \leqslant c_2/\lambda, \tag{2.25}$$

where $\phi(\tau) = \phi^{-1}(l,\tau)$, $\psi(\tau) = \psi^{-1}(l,\tau;\lambda_0)$; thus

$$\phi(\tau) = \exp\left(-\tilde{c}_1\int_\tau^l \frac{|r|^2}{a}\,d\xi\right), \qquad \psi(\tau) = \exp\left(-\frac{\tilde{c}_1}{\lambda_0}\int_\tau^l \frac{|b|^2}{a}\,d\xi\right). \tag{2.26}$$

3. It has been shown that for $0 \leqslant \tau \leqslant t \leqslant l < \infty$, $|z_i| \leqslant c_1$ $(i = 1, \ldots, 6)$ and $0 < z_0 \leqslant 1/c$, $Y(t, \tau, z_i, \lambda)$ is continuous in (t, τ, z_i, λ) (and is the unique solution of (2.1) with $Y(\tau, \tau) = 0$, $Y_t(\tau, \tau) = 1$). Moreover, for $\tau > 0$

$$|W|^2 = \phi(\tau)\psi(\tau)a(\tau)|Y|^2 \leqslant c_2/\lambda.$$

It is easily seen that this estimate holds for $\tau = 0$ as well. Hence

$$W(t, \tau, z_i, \lambda) = (\phi(\tau)\psi(\tau)a(\tau))^{\frac{1}{2}} Y$$

defines an operator $\mathbf{W} \in \mathfrak{A}''$ for example; we write $\sqrt{(\phi\psi a)} = Q$ and thus $W = QY$; cf. **(6)**. In order to exploit these facts we make use of an intermediate stage of a continuous direct sum of Hilbert spaces related to $\mathfrak{A}$ **(6)**. Thus it is known (cf. **19, 12a**) that there is a basic measure ν on σ and an isometric isomorphism θ: $H \to \mathbf{h} = \int^{\oplus} \mathbf{h}(\xi)\, d\nu(\xi)$ diagonalizing the algebra $\mathfrak{A}$. Now, for example, if $h \in H$, then $W\theta h \in \theta D(\Lambda^{\frac{1}{2}})$; this means that

$$\lambda^{\frac{1}{2}} W|\theta h|_{\mathbf{h}(\xi)} \in L^2(\nu)$$

$(D(\Lambda^{\frac{1}{2}})$ has graph topology). As in **(7, 8)** to W corresponds the operator $\mathbf{W} = \theta^{-1}W\theta$ and proceeding exactly as in **(6, 7, 8)** we have (the subscript s denotes the strong operator topology)

PROPOSITION 2. *Under the assumptions of Proposition* 1

$$(t, \xi) \to \mathbf{W}(t, \xi) \in \mathfrak{C}^0(\mathfrak{L}_s(H, D(\Lambda^{\frac{1}{2}}))),\ t \to \mathbf{W}(t, \xi) \in \mathfrak{C}^1(\mathfrak{L}_s(H)),$$

and

$$t \to \mathbf{W}(t, \xi) \in \mathfrak{C}^2(\mathfrak{L}_s(D(\Lambda^{\gamma}), H)),$$

where $\gamma = \max(\alpha, \frac{1}{2})$.

Proof. We need only check the bounds with regard to λ since the rest of the proof follows **(6, 7, 8)** exactly. The first statement has been shown; for the second we note from (2.13) that

$$|Y_t|^2 \leqslant E(t, \tau)\, \frac{a(t)}{a(\tau)}\, \phi(t, \tau)\psi(t, \tau). \tag{3.1}$$

Hence the second statement follows from

$$Q^2(\tau)|Y_t|^2 \leqslant c_3. \tag{3.2}$$

Finally for the last statement we go back to (2.1) to obtain

$$|Y_{tt}| \leqslant c_4 \lambda |Y| + c_5 \lambda^{\alpha} |Y_t|. \tag{3.3}$$

Thus (recall that $2\beta \leqslant 1$)

$$Q|Y_{tt}| \leqslant c_6 \lambda^{\frac{1}{2}} + c_7 \lambda^{\alpha}. \tag{3.4}$$

The proposition follows.

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Now we consider (2.7) and will give it meaning for certain f and show that it is the required solution of (1.1). Clearly if $h(\xi) = f(\xi)/Q(\xi)$ is continuous with values in H, then (2.7) is

$$(3.5) \qquad u(t) = \int_\tau^t \mathbf{W}(t, \xi)h(\xi)d\xi,$$

which is well defined (for integration of vector-valued functions see **4**). We need only show that it actually gives a solution. First formally

$$(3.6) \qquad u' = \mathbf{W}(t, t)h(t) + \int_\tau^t \mathbf{W}_t(t, \xi)h(\xi)d\xi = \int_\tau^t \mathbf{W}_t(t, \xi)h(\xi)d\xi.$$

Using Proposition 2, equation (3.6) may be justified rigorously if we note in addition that $(t, \xi) \to \mathbf{W}_t(t, \xi)$ is continuous with values in $\mathfrak{L}_s(H)$ for $0 \leqslant \xi \leqslant t \leqslant l$; cf. (**6, 7, 9**). Similarly we obtain

$$(3.7) \qquad u'' = \mathbf{W}_t(t, t)h(t) + \int_\tau^t \mathbf{W}_{tt}(t, \xi)h(\xi)d\xi$$

$$= f(t) + \int_\tau^t \mathbf{W}_{tt}(t, \xi)h(\xi)d\xi,$$

where now we require, say, $h \in \mathfrak{C}^0(D(\Lambda^\gamma))$; thus f is continuous with values in $D(\Lambda^\gamma)$. Note also here that $(t, \xi) \to \mathbf{W}_{tt}(t, \xi)$ is continuous with values in $\mathfrak{L}_s(D(\Lambda^\gamma), H)$ for $0 \leqslant \xi \leqslant t \leqslant l$ (recall that we have been assuming throughout that $a \in C^1[0, l]$ and $b, r, s \in C^0[0, l]$; also $P \geqslant 0$ is stipulated). Therefore if h is as above, the function u satisfies $u \in \mathfrak{C}^2(H)$, $u \in \mathfrak{C}^0(D(\Lambda^{\gamma+\frac{1}{2}}))$, $u \in \mathfrak{C}^1(D(\Lambda^\gamma))$. Note that hypotheses of the form $h \in L^1(D(\Lambda^\gamma))$ may also be envisioned, but we shall not treat this kind of theory here. Now since $\gamma \geqslant \frac{1}{2}$, equation (1.1) will be satisfied by the function constructed above. It should be pointed out that we must have closed Λ^α, Λ^β in order to carry Λ^β, say, under an integral sign (**25**); however, for self-adjoint Λ this is automatic. We may now state

THEOREM 1. *Assume that* $a \in C'[0, l]$; $b, r, s \in C^0[0, l]$; $P \geqslant 0$; $h = f/Q \in \mathfrak{C}^0(D(\Lambda^\gamma))$; $\gamma = \max(\frac{1}{2}, \alpha)$; $2\beta \leqslant 1$; $\mathrm{Re}(z_3\, s(t)) \geqslant 0$. *Then there exists a solution of* (1.1) *given by* (2.7) *with* $u \in \mathfrak{C}^2(H)$, $u \in \mathfrak{C}^0(D(\Lambda^{\gamma+\frac{1}{2}}))$, *and* $u \in \mathfrak{C}^1(D(\Lambda^\gamma))$.

We turn now to uniqueness via the relation (2.5), which when applied to Y yields (**7, 8**)

$$(3.8) \qquad Y_\tau = -Z + \lambda^\alpha z_3\, s(\tau) Y.$$

Hence we shall need to know something about Z. In the first place $Z(t, \tau, z_i, \lambda)$ is the unique solution of (2.1) satisfying $Z(\tau, \tau) = 1$, $Z_t(\tau, \tau) = 0$ (by classical results). Thus as with Y we need only bound Z in some sense. Duplicating our previous estimates (2.8), etc., there results

$$(3.9)\quad |Z_t|^2 + \int_\tau^t 2\,\mathrm{Re}(\lambda^\alpha z_3\, s(\xi))|Z_\xi|^2\, d\xi + \lambda a(t)|Z|^2 - \lambda a(\tau) - \lambda \int_\tau^t a'|Z|^2\, d\xi \leqslant \int_\tau^t |z_1|(|b|^2|Z|^2 + |Z_\xi|^2)d\xi + \int_\tau^t |z_2|(|r|^2\lambda^{2\beta}|Z|^2 + |Z_\xi|^2)d\xi.$$

Under the same assumptions as before it follows that

$$(3.10)\quad \lambda a(t)|Z|^2 + |Z_t|^2 \leqslant \lambda a(\tau) + 2c_1 \int_\tau^t |Z_\xi|^2\, d\xi + \lambda \int_\tau^t P|Z|^2\, d\xi,$$

$$(3.11)\quad |Z_t|^2 + \lambda a(t)|Z|^2 \leqslant \lambda a(\tau)\, E(t, \tau) + \int_\tau^t \lambda P|Z|^2\, E(t, \xi)d\xi.$$

Hence, in particular,

$$(3.12)\quad a(t)|Z|^2 \leqslant a(\tau)E(t, \tau) + \int_\tau^t P|Z|^2\, E(t, \xi)d\xi.$$

Now using Lemma 1 slightly modified (set $\chi = \int P|Z|^2 E d\xi$; then

$$a(\chi' - 2c_1\chi) \leqslant a(\tau)PE + P\chi$$

and

$$\chi + a(\tau)E \leqslant a(\tau)E\exp(\int (P/a)d\xi),$$

we obtain

$$(3.13)\quad a(t)|Z|^2 \leqslant a(\tau)E(t, \tau)\,\frac{a(t)}{a(\tau)}\,\phi(t, \tau)\psi(t, \tau).$$

Therefore it has been proved that

LEMMA 2. *Under the assumptions of Theorem* 1

$$(3.14)\quad \psi(\tau)\phi(\tau)|Z|^2 \leqslant c_2.$$

This implies that, setting $q = \sqrt{(\psi\phi)}$, $T = qZ$ will determine an operator $\mathbf{T}$ in $\mathfrak{A}''$. Also we observe from (2.5) that

$$(3.15)\quad Z_\tau = Y[\lambda a(\tau) + \lambda^\beta z_2\, r(\tau) + z_1\, b(\tau)].$$

It is easily seen now that the following results hold.

PROPOSITION 3. *Under the above assumptions* $(t, \tau) \to \mathbf{T}(t, \tau) \in \mathfrak{E}^0(\mathfrak{L}_s(H))$ *and also* $|Q(\tau)Z_\tau| \leqslant c_8\,\lambda^{\frac{1}{2}}$.

Now for $\tau > 0$, Y and Z define themselves as perfectly good operators $\mathbf{Y}$ and $\mathbf{Z}$ in $\mathfrak{A}''$. Also using (3.8) and (3.15) we see that $|Y_\tau| \leqslant c_9\,\lambda^{\alpha-\frac{1}{2}}$ if $\alpha \geqslant \frac{1}{2}$ and if $\alpha \leqslant \frac{1}{2}$, $|Y_\tau| \leqslant c_9$; thus for $\alpha \geqslant \frac{1}{2}$ (the case $\alpha \leqslant \frac{1}{2}$ is simple and similar and hence omitted explicitly in our proof)

$$\xi \to \mathbf{Y}(t, \xi) \in \mathfrak{E}^1(\mathfrak{L}_s(D(\Lambda^{\alpha-\frac{1}{2}}), H)) \qquad \text{for } \xi > 0.$$

Similarly, $|z_\tau| \leqslant c_{10}\lambda^{\frac{1}{2}}$ means that $\xi \to \mathbf{Z}(t, \xi) \in \mathfrak{C}^1(\mathfrak{L}_s(D(\Lambda^{\frac{1}{2}}), H))$. Therefore assuming that $\tau > 0$ we suppose u is a solution of (1.1) with $u(\tau)$ and $u'(\tau)$ prescribed, rewrite (1.1) with t replaced by ξ, and "multiply" by $\mathbf{Y}(t, \xi)$ $(0 < \tau \leqslant \xi \leqslant t \leqslant l < \infty)$. This gives formally

$$(3.16)\qquad \mathbf{Y}(t, \xi)u_\xi\Big|_\tau^t - \int_\tau^t [\mathbf{Y}_\xi - \Lambda^\alpha Ss(\xi)\mathbf{Y}]u_\xi d\xi + \int_\tau^t \mathbf{Y}[\Lambda a(\xi) + Bb(\xi) + \Lambda^\beta Rr(\xi)]ud\xi = \int_\tau^t \mathbf{Y}fd\xi.$$

Using now (3.8) and (3.15) **(7, 8)**, we obtain, if $u \in \mathfrak{C}^0(D(\Lambda^{\gamma+\frac{1}{2}}))$, $u \in \mathfrak{C}^2(H)$, and $u \in \mathfrak{C}^1(D(\Lambda^\gamma))$, a rigorous justification of (3.16), and the result

$$(3.17)\qquad u(t) - \mathbf{Z}(t, \tau)u(\tau) - \mathbf{Y}(t, \tau)u_t(\tau) - \int_\tau^t [\mathbf{Y}_\xi - \Lambda^\alpha Ss(\xi)\mathbf{Y} + \mathbf{Z}]u_\xi\, d\xi = \int_\tau^t \mathbf{Y}fd\xi$$

where by (3.8) it is seen that the integral on the left side of (3.17) is zero. When $u(\tau) = u'(\tau) = 0$ we have the result **(24)** that any solution of (1.1) must have the form (2.7) (with $\tau > 0$ as lower limit of integration). If now $\tau = 0$, we first proceed as above for the lower limit $\tau + \epsilon = \epsilon$. From (3.17) it is then seen that

$$(3.18)\qquad u(t) = \mathbf{Z}(t, \epsilon)u(\epsilon) + \mathbf{Y}(t, \epsilon)u_t(\epsilon) + \int_\epsilon^t \mathbf{Y}(t, \xi)f(\xi)d\xi.$$

Whereas for (3.17) with $\tau > 0$ it is only necessary to suppose that $u \in \mathfrak{C}^0(D(\Lambda^{\gamma+\frac{1}{2}}))$, etc., we must require more for $\tau = 0$. Thus, if

$$t \to u/q \in \mathfrak{C}^0(D(\Lambda^{\gamma+\frac{1}{2}}))$$

and $t \to u'/Q \in \mathfrak{C}^0(D(\Lambda^\gamma))$, then by hypocontinuity **(7, 8)** it follows that $\mathbf{Z}(t, \epsilon)u(\epsilon) \to 0$ and $\mathbf{Y}(t, \epsilon)u'(\epsilon) \to 0$ as $\epsilon \to 0$. Assuming that

$$h = f/Q \in \mathfrak{C}^0(D(\Lambda^\gamma)),$$

all of the terms in (3.18) have limits as $\epsilon \to 0$ $(\int_\epsilon^t \mathbf{W}(t, \xi)h(\xi)d\xi \to \int_0^t \mathbf{W}hd\xi)$. Hence for $f = 0$ we obtain:

THEOREM 2. *Under the hypotheses of Theorem* 1 *there is only one solution of* (1.1) *with* $u'/Q \in \mathfrak{C}^0(D(\Lambda^\gamma))$, $u \in \mathfrak{C}^2(H)$, $u/q \in \mathfrak{C}^0(D(\Lambda^{\gamma+\frac{1}{2}}))$.

In general the requirements of Theorem 2 are too strong, however. Therefore we shall give another uniqueness result in the case $q > 0$. Let u be a solution of (1.1) with $u \in \mathfrak{C}^0(D(\Lambda^{\gamma+\frac{1}{2}}))$, $u \in \mathfrak{C}^2(H)$, $u \in \mathfrak{C}^1(D(\Lambda^\gamma))$, $u(0) = u'(0) = 0$, and $f = 0$. Then under the hypotheses of Theorem 1

$$\exp\left(-\lambda^\alpha z_3 \int_\tau^t s(\xi)d\xi\right)$$

determines an operator in $\mathfrak{A}''$ which we denote by

$$\exp\left(-\Lambda^{\alpha}S\int_{\tau}^{t}s(\xi)d\xi\right) = \mathbf{L}(t,\tau).$$

By going to **h** under θ, integrating (1.1) partially, and returning then to H, we have

$$u' = -\int_0^t \mathbf{L}(t,\xi)[\Lambda^{\beta}Rr(\xi) + \Lambda a(\xi) + Bb(\xi)]u(\xi)d\xi. \tag{3.19}$$

Hence since $||\Lambda^{\beta}u||$ and $||\Lambda u||$ are bounded by assumption,

$$\begin{aligned} ||u'|| &\leqslant \tilde{c}\int_0^t (a + c_{11}|r| + c_{12}|b|)d\xi \\ &\leqslant \tilde{c}\left(\int_0^t a d\xi\right)^{\frac{1}{2}}\left[\left(\int_0^t a\right)^{\frac{1}{2}} + c_{11}\left(\int_0^t \frac{|r|^2}{a}\right)^{\frac{1}{2}} + c_{12}\left(\int_0^t \frac{|b|^2}{a}\right)^{\frac{1}{2}}\right] \\ &\leqslant c_{13}\left(\int_0^t a d\xi\right)^{\frac{1}{2}} \end{aligned} \tag{3.20}$$

(recall that $q > 0$ means $\int |r|^2/a < \infty$ and $\int |b|^2/a < \infty$). Now

$$(t,\tau) \to [\sqrt{a(\tau)}]\mathbf{Y}(t,\tau)$$

is continuous here with values in $\mathfrak{L}_s(H, D(\Lambda^{\frac{1}{2}}))$, and the term $\mathbf{Y}(t,\epsilon)u'(\epsilon)$ in (3.18) may be written, for example, as

$$\mathbf{Y}(t,\epsilon)u'(\epsilon) = (\sqrt{a(\epsilon)}\,\mathbf{Y}(t,\epsilon))\frac{\left(\int_0^{\epsilon} a d\xi\right)^{\delta}}{\sqrt{a(\epsilon)}}\frac{u'(\epsilon)}{\left(\int_0^{\epsilon} a d\xi\right)^{\delta}}. \tag{3.21}$$

But

$$u' \Big/ \left(\int_0^{\epsilon} a d\xi\right)^{\delta}$$

is continuous for $\delta < \frac{1}{2}$. Hence since the $\mathbf{Z}u$ term in (3.18) tends to zero now (since $q > 0$), we have

THEOREM 3. *Assume u is a solution of* (1.1) *with*

$$u \in \mathfrak{C}^2(H),\ u \in \mathfrak{C}'(D(\Lambda^{\gamma})),\ u \in \mathfrak{C}^0(D(\Lambda^{\gamma+\frac{1}{2}})),$$

and let $\int |r|^2/a < \infty$, $\int |b|^2/a < \infty$ *with* $(\int_0^t a d\xi)^{\delta}/\sqrt{a}$ *continuous for some* $\delta < \frac{1}{2}$. *Then u is unique.*

If $a = t^m$ it is seen that

$$\left(\int_0^t a d\xi\right)^{\delta} \Big/ a^{\frac{1}{2}}$$

is continuous if

$$\delta \geqslant \frac{1}{2}\left(\frac{m}{m+1}\right).$$

Various other criteria for uniqueness can easily be envisioned. We note that our problem gives rise to a turning-point situation at $t = 0$; cf. **(17)**. However, this will not be exploited here.

4. We shall now examine the condition $P \geqslant 0$ and compare the present results with **(15)** in a special case (assume $\lambda_0 \geqslant 1$). First recall that $P = a' + c_1(|r|^2 + |b|^2/\lambda)$; and in order to have $P \geqslant 0$ for all λ, we must have $a' + c_1|r|^2 \geqslant 0$ (conversely this is a sufficient condition). This gives a bound for a', viz.

$$a' \geqslant -c_1|r|^2. \tag{4.1}$$

Thus a is not required to be monotone. Also since no condition is imposed on $\int_\tau^l(|r|^2/a)d\xi$ as to growth, it is possible for a' to oscillate while going to zero faster than $|r|^2$. For example let $a = t^m$, $r = t^n$; then (if $2n - m \neq -1$)

$$\int_\tau^l \frac{|r|^2}{a}\,d\xi = O\left(\frac{1}{\tau^{m-2n-1}}\right). \tag{4.2}$$

Now roughly if $a = O(t^m)$ with $a' = O(t^{m-1})$, then to ensure (4.1) with oscillation, we shall want $-t^{m-1} \geqslant -\hat{c}t^{2n}$ which will hold (for t small) if $m - 1 - 2n > 0$ (and sometimes when $m - 1 = 2n$). Thus the case of non-monotone a seems to be associated with the case of $\int_\tau^l(|r|^2/a)d\xi \to \infty$. The case $\int_\tau^l|r|^2/a < \infty$ corresponds roughly to the situation of **(15)**, where it is assumed that if $a = O(t^m)$ then $r = O(t^{\frac{1}{2}m-1}\beta(t))$ with $\beta \to 0$. In our case if $\int_\tau^l|r|^2/a < \infty$, then we require $n > \frac{1}{2}m - \frac{1}{2}$. This is a stronger hypothesis than that of **(15)**; but our solution is stronger. The case $\int_\tau^l|r|^2/a \to \infty$ seems to involve a new situation (cf. **(5)** where non-monotone a are also allowed) as indicated below (assume $\int|b|^2/a < \infty$)

$$\int|r|^2/a < \infty \sim n > \tfrac{1}{2}m - \tfrac{1}{2} \sim f/t^{\frac{1}{2}m} \text{ continuous}, \tag{4.3}$$

$$\int|r|^2/a \to \infty \sim n < \tfrac{1}{2}m - \tfrac{1}{2} \sim f/(t^{\frac{1}{2}m}\sqrt{\phi}) \text{ continuous}; \tag{4.4}$$

here

$$\phi = \exp\left(-c_1 \int_t^l \frac{|r|^2}{a}\,d\xi\right).$$

In (4.3) no oscillation in a is allowed (essentially) whereas (4.4) permits a to be wilder in the nature of its oscillations (recall that $a \geqslant 0$ always and $a > 0$ for $t > 0$). However, if f arises from an initial-value problem, then for example $f \sim a = O(t^m)$ and then in (4.4) $t^{\frac{1}{2}m}/\sqrt{\phi}$ is required to be continuous. But

$$\phi \sim \exp(-c_1 t^{-(m-2n-1)}) \qquad (m > 2n - 1),$$

which vanishes faster than $t^{\frac{1}{2}m}$. Suppose on the other hand that $2n - m = -1$; then $\int|r|^2/a = O(-\log t)$ and $\phi \sim t^k$ for some k. Therefore if $k \leqslant m$, $f/\sqrt{(a\phi)}$ is continuous and hence some initial-value problems seem to admit oscillation in a. Some further discussion of examples is given in **(26)**.

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Rutgers University, University of Illinois, and University of Saskatchewan, Regina Campus