J. Austral. Math. Soc. (*Series A*) **58** (1995), 143–153

A NOTE ON BEUKERS' INTEGRAL

MASAYOSHI HATA

(Received 18 May 1992)

Communicated by J. H. Loxton

Abstract

The aim of this note is to give a sharp lower bound for rational approximations to $\zeta(2) = \pi^2/6$ by using a specific Beukers' integral. Indeed, we will show that π^2 has an irrationality measure less than 6.3489, which improves the earlier result 7.325 announced by D. V. Chudnovsky and G. V. Chudnovsky.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 11J82.

1. Introduction

After Apéry's remarkable irrationality proofs for $\zeta(2)$ and $\zeta(3)$, Beukers [1] gave elegant proofs using the Legendre polynomials $L_n(x) = (x^n(1-x)^n)^{(n)}/n!$. Our aim in this note is to give a sharp lower bound for rational approximations to $\zeta(2) = \pi^2/6$ using Beukers' double integral

$$\iint_S \frac{F(x)G(y)}{1-xy}\,dx\,dy, \tag{1.1}$$

where S is the unit square $[0, 1] \times [0, 1]$ and $F(x)$, $G(y)$ are non-zero polynomials with integral coefficients. Let D_n be the least common multiple of $\{1, 2, \dots, n\}$ and let $\mathrm{ord}_0(F)$ be the order of the zero point of $F(x)$ at the origin. (We put $\mathrm{ord}_0(F) = 0$ if $F(0) \neq 0$.) The double integral (1.1) is very important in the arithmetical study of $\zeta(2)$ by virtue of the following lemma due to Beukers:

LEMMA 1.1. *We have*

$$\iint_S \frac{F(x)G(y)}{1-xy}\,dx\,dy = a\zeta(2) + b,$$

https://doi.org/10.1017/S1446788700038192 Published online by Cambridge University Press

where

$$a = \frac{1}{2\pi i}\int_C F(z)G\left(\frac{1}{z}\right)\frac{dz}{z}$$

is an integer, C denotes a closed curve enclosing the origin and b is a rational number whose denominator is a divisor of $D_N D_M$ *with* $M = \max\{\deg(F), \deg(G)\}$ *and*

$$N = \min\{\max\{\deg(F), \deg(G) - \mathrm{ord}_0(F)\}, \max\{\deg(G), \deg(F) - \mathrm{ord}_0(G)\}\}.$$

Note that the expression for N in the above lemma comes from [1, Lemma 1]. In fact, the rational part in the right-hand side of the formula

$$\iint_S \frac{(xy)^n}{1-xy}\,dx\,dy = \zeta(2) - \frac{1}{1^2} - \cdots - \frac{1}{n^2}$$

belongs to the set $\mathbb{Z}/D_N^2$ since $n \le \min\{\deg(F), \deg(G)\} \le N$. (Note that always $M \ge N$.) Similarly, in the formula

$$\iint_S \frac{x^n y^m}{1-xy}\,dx\,dy = \frac{1}{|n-m|}\left(\frac{1}{\min\{n,m\}+1} + \cdots + \frac{1}{\max\{n,m\}}\right)$$

for $n \ne m$, the right-hand side belongs to the set $\mathbb{Z}/D_N D_M$, since

$$|n-m| \le \max\{\deg(F) - \mathrm{ord}_0(G), \deg(G) - \mathrm{ord}_0(F)\} \le N$$

and $\max\{n, m\} \le M$, as required.

Beukers has studied the integral (1.1) with $F(x) = L_n(x)$ and $G(y) = (1-y)^n$, which produces a good irrationality measure for π^2:

$$\mu = 1 + \frac{\log(2+\sqrt{5}) + 6/5}{\log(2+\sqrt{5}) - 6/5} = 11.85078\ldots$$

Taking more complicated polynomials in (1.1), the above measure μ has been pared down to 10.02979... by Dvornicich and Viola [4], to 7.552 by Rukhadze [9], to 7.5252 by the author [6], to 7.398537 by Rhin and Viola [8] and to 7.325 by Chudnovsky and Chudnovsky [2]. (The measures 7.552 and 7.325 were announced without proofs.)

In this note we consider the following integral

$$\epsilon_n = \iint_S \frac{(x(1-x))^{15n}\,(y(1-y))^{14n}}{(1-xy)^{12n+1}}\,dx\,dy \tag{1.2}$$

instead of (1.1), which will produce a better irrationality measure of π^2 in the following sense:

https://doi.org/10.1017/S1446788700038192 Published online by Cambridge University Press

THEOREM 1.2. *There exists a positive integer q_0 such that*

$$\left|\pi^2 - \frac{p}{q}\right| \gg q^{-6.3489}$$

for all $p \in \mathbb{Z}$ and any integer $q \geq q_0$. (In other words, π^2 has an irrationality measure less than 6.3489.)

We thus have immediately

COROLLARY 1.3. *There exists a positive integer q_1 such that*

$$\left|\pi - \frac{p}{q}\sqrt{k}\right| \gg q^{-12.6978}$$

for all $p \in \mathbb{Z}$ and any integer $q \geq q_1$ uniformly in k. ($\pi/\sqrt{k}$ has an irrationality measure less than 12.6978.)

This will be proved using the same techniques as in [6, Corollary 5.2]. However, better irrationality measures for $\pi/\sqrt{k}$ for some particular integral values of k are known; for example, 8.016045 ... and 4.601579 ... for $k = 1$ and $k = 3$ respectively (the author [7]), and 12.11 ... for $k = 640320$ (Chudnovsky and Chudnovsky [3]).

After $12n$-fold partial integration with respect to x, we have, from (1.2),

$$\epsilon_n = \iint_S \frac{U(x)V(y)}{1-xy}\,dx\,dy$$

with $U(x) = (x^{15n}(1-x)^{15n})^{(12n)}/(12n)!$ and $V(y) = y^{2n}(1-y)^{14n}$. Hence it follows from Lemma 1.1 that

$$\epsilon_n = a_n\zeta(2) + b_n \tag{1.3}$$

where $a_n \in \mathbb{Z}$ and $b_n \in \mathbb{Z}/D_{16n}D_{18n}$, since $\mathrm{ord}_0(U) = 3n$ and $\mathrm{ord}_0(V) = 2n$.

In Section 2 we will study the asymptotic behaviour of $|\epsilon_n|$ and $|a_n|$. The most essential part of this note is Sections 3 and 4, in which we will show that the denominator of b_n is comparatively small; more precisely, $b_n \in (d_n/D_{16n}D_{18n})\mathbb{Z}$ for some large product d_n consisting of prime numbers less than $15n$. Then Theorem 1.2 will be proved in Section 5.

2. Estimates of ϵ_n and a_n

From (1.2) it is easily verified that

$$\lim_{n\to\infty} \frac{1}{n}\log|\epsilon_n| = \max_{0<x,y<1} \log\left(\frac{(x(1-x))^{15}(y(1-y))^{14}}{(1-xy)^{12}}\right).$$

Since the maximum α of the right-hand side is attained at $x = (\sqrt{6969} - 27)/96$ and $y = (3\sqrt{6969} - 127)/208$, we get

$$(2.1)\quad \alpha = \log\left(\frac{5^{15}7^{14}(\sqrt{6969}+21)^{12}(5\sqrt{6969}-343)^{15}(99\sqrt{6969}-7519)^{14}}{2^{257}3^{15}13^{28}17^{12}}\right) < -36.0223.$$

We now estimate $|a_n|$. Since

$$a_n = \frac{1}{2\pi i}\int_C U(z)V\left(\frac{1}{z}\right)\frac{dz}{z} = \left(\frac{1}{2\pi i}\right)^2 \int_C\int_{C_z} \frac{(w(1-w))^{15n}}{(w-z)^{12n+1}}\,dw\,\frac{(1-z)^{14n}}{z^{16n+1}}\,dz,$$

where C and C_z are circles centered at $z = 0$ and $w = z$ with radii r and ρ respectively, we have

$$\limsup_{n\to\infty}\frac{1}{n}\log|a_n| \le \min_{r,\rho>0}\log\left(\frac{(1+r)^{14}(r+\rho)^{15}(1+r+\rho)^{15}}{r^{16}\rho^{12}}\right).$$

The minimum β of the right-hand side is attained at $r = (3\sqrt{6969} - 127)/224$ and $\rho = (285 - \sqrt{6969})/336$; therefore we get

(2.2)

$$\beta = \log\left(\frac{5^{15}(\sqrt{6969}+285)^{12}(3\sqrt{6969}+97)^{14}(3\sqrt{6969}+127)^{16}(5\sqrt{6969}+343)^{15}}{2^{269}3^{15}7^{14}13^{28}17^{12}}\right) < 35.4093.$$

3. The denominator of b_n (I)

In this section we investigate the prime factors p of $T_n \equiv b_n D_{16n} D_{18n}$ satisfying $5\sqrt{n} < p < 15n$.

After k-fold and $(12n-k)$-fold partial integrations with respect to x and y respectively, we obtain, from (1.2),

$$(3.1)\qquad \epsilon_n = \binom{12n}{k}^{-1} \iint_S \frac{F_k(x)G_k(y)}{1-xy}\,dx\,dy$$

with

$$F_k(x) = x^{k-12n}\left(x^{15n}(1-x)^{15n}\right)^{(k)}/k!, \quad G_k(y) = \left(y^{14n-k}(1-y)^{14n}\right)^{(12n-k)}/(12n-k)!$$

https://doi.org/10.1017/S1446788700038192 Published online by Cambridge University Press

for any integer $k \in [0, 12n]$. Note that $F_k(x) \in \mathbb{Z}[x]$ and $G_k(y) \in \mathbb{Z}[y]$ with $\deg(F_k) = 18n$, $\mathrm{ord}_0(F_k) = 3n$, $\deg(G_k) = 16n$ and $\mathrm{ord}_0(G_k) = 2n$. Then, from Lemma 1.1, (1.3) and (3.1),

$$a_{k,n}\zeta(2) + b_{k,n} = \binom{12n}{k}\epsilon_n = \binom{12n}{k}\{a_n\zeta(2) + b_n\}$$

for some $a_{k,n} \in \mathbb{Z}$ and $b_{k,n} \in \mathbb{Z}/D_{16n}D_{18n}$. Thus, putting $T_{k,n} \equiv b_{k,n}D_{16n}D_{18n}$, we obtain

$$T_{k,n} = \binom{12n}{k}T_n \tag{3.2}$$

since $\zeta(2)$ is irrational. For any prime p let $\nu_p(n)$ be the exponent of p in the resolution of n into its prime factors. Hence, from (3.2),

$$\nu_p(T_{k,n}) = [12\omega] - [12\omega - \kappa] + \nu_p(T_n) \tag{3.3}$$

where $\omega = \{n/p\}$ and $\kappa = \{k/p\}$ for any prime $p > 5\sqrt{n}$. (Here $\{x\}$ denotes the fractional part of x.)

On the other hand, it is easily seen from Lemma 1.1 that if p is a common prime factor of all the coefficients of $F_k(x)$, or of $G_k(y)$, then p becomes a divisor of $T_{k,n}$. More precisely, we have

$$\nu_p(T_{k,n}) \geq \min_{0\leq j\leq 15n} \nu_p(A_{j,k,n}) + \min_{0\leq l\leq 14n} \nu_p(B_{l,k,n})$$

where

$$A_{j,k,n} = \binom{15n}{j}\binom{15n+j}{k} \qquad \text{and} \qquad B_{l,k,n} = \binom{14n}{l}\binom{14n-k+l}{12n-k}.$$

Since $\nu_p(A_{j,k,n}) = [15\omega]-[15\omega-\theta]+[15\omega+\theta]-[15\omega+\theta-\kappa] \equiv I(\omega,\theta,\kappa)$, say, and $\nu_p(B_{l,k,n}) = [14\omega]-[14\omega-\theta']+[14\omega-\kappa+\theta']-[12\omega-\kappa]-[2\omega+\theta'] \equiv J(\omega,\theta',\kappa)$, say, for any $p > 5\sqrt{n}$ where $\theta = \{j/p\}$ and $\theta' = \{l/p\}$, we have

$$\nu_p(T_{k,n}) \geq \min_{0\leq\theta<1} I(\omega,\theta,\kappa) + \min_{0\leq\theta'<1} J(\omega,\theta',\kappa) \tag{3.4}$$

for any $k \in [0, 12n]$.

We now distinguish four cases, as follows:

CASE I: We take $k = 12n$ giving $\kappa = \{12\omega\}$. From (3.3), $\nu_p(T_{12n,n}) = \nu_p(T_n)$. Suppose now that $2\{15\omega\} < \{12\omega\}$. Then clearly

$$\begin{aligned} I(\omega,\theta,\{12\omega\}) &= [15\omega] - [15\omega-\theta] + [15\omega+\theta] - [12\omega] - [3\omega+\theta] \\ &\geq 2[15\omega] - [12\omega] - [18\omega] = \{12\omega\} + \{18\omega\} - 2\{15\omega\} \\ &> \{18\omega\} \geq 0 \end{aligned}$$

for any $\theta \in [0, 1)$; hence we have $\nu_p(T_n) \geq 1$ from (3.4), since $I(\omega,\theta,\kappa) \in \mathbb{Z}$.

https://doi.org/10.1017/S1446788700038192 Published online by Cambridge University Press

CASE II: We take $k = 0$ giving $\kappa = 0$. From (3.3), $\nu_p(T_{0,n}) = \nu_p(T_n)$. Suppose now that $2\{14\omega\} < \{12\omega\}$. Then

$$\begin{aligned} J(\omega, \theta', 0) &= [14\omega] - [14\omega - \theta'] + [14\omega + \theta'] - [12\omega] - [2\omega + \theta'] \\ &\geq 2[14\omega] - [12\omega] - [16\omega] = \{12\omega\} + \{16\omega\} - 2\{14\omega\} \\ &> \{16\omega\} \geq 0 \end{aligned}$$

for any $\theta' \in [0, 1)$; hence $\nu_p(T_n) \geq 1$ from (3.4).

CASE III: From (3.3) we have always $\nu_p(T_n) \geq \nu_p(T_{k,n}) - 1$. Suppose that $\{2\omega\} + \{12\omega\} < 1$, $\{2\omega\} + \{14\omega\} < 1$ and $2\{15\omega\} < \{14\omega\}$. Since $p < 14n$, one can take $k = k(p) \equiv 14n - [14n/p]p$; hence $\kappa = \kappa_\omega \equiv \{2\omega\} + \{12\omega\}$. Then, for any $\theta,\ \theta' \in [0, 1)$,

$$\begin{aligned} I(\omega, \theta, \kappa_\omega) &= [15\omega] - [15\omega - \theta] + [15\omega + \theta] - [14\omega] - [\omega + \theta] \\ &\geq 2[15\omega] - [14\omega] - [16\omega] = \{14\omega\} + \{16\omega\} - 2\{15\omega\} \\ &> \{16\omega\} \geq 0 \end{aligned}$$

and

$$\begin{aligned} J(\omega, \theta', \kappa_\omega) &= [14\omega] - [14\omega - \theta'] + 1 + [2\omega] - [2\omega + \theta'] \\ &\geq 1 + [2\omega] + [14\omega] - [16\omega] = 1 + \{16\omega\} - \{2\omega\} - \{14\omega\} \\ &> \{16\omega\} \geq 0. \end{aligned}$$

Hence $\nu_p(T_{k(p),n}) \geq 2$ from (3.4) and so $\nu_p(T_n) \geq 1$.

CASE IV: The same argument as in Case III can be applied to the following expression:

$$\epsilon_n = \binom{12n}{k}^{-1} \iint_S \frac{U_k(x)V_k(y)}{1 - xy}\, dx\, dy \tag{3.5}$$

with

$$U_k(x) = \frac{\left(x^{15n-k}(1-x)^{15n}\right)^{(12n-k)}}{(12n-k)!}, \qquad V_k(y) = \frac{y^{k-12n}\left(y^{14n}(1-y)^{14n}\right)^{(k)}}{k!}$$

for any integer $k \in [0, 12n]$, which comes from k-fold and $(12n - k)$-fold partial integrations with respect to y and x respectively. Then, by taking $k = 15n - [15n/p]p$, it can be seen that $\nu_p(T_n) \geq 1$ for any ω satisfying $\{3\omega\} + \{12\omega\} < 1$, $\{3\omega\} + \{15\omega\} < 1$ and $2\{14\omega\} < \{15\omega\}$. We have thus proved the following

LEMMA 3.1. *Suppose a prime $p \in (5\sqrt{n}, 15n)$ satisfies one of the following four conditions:*

(I) $2\{15\omega\} < \{12\omega\}$;
(II) $2\{14\omega\} < \{12\omega\}$;
(III) $\{2\omega\} + \{12\omega\} < 1$, $\{2\omega\} + \{14\omega\} < 1$ *and* $2\{15\omega\} < \{14\omega\}$;
(IV) $\{3\omega\} + \{12\omega\} < 1$, $\{3\omega\} + \{15\omega\} < 1$ *and* $2\{14\omega\} < \{15\omega\}$.

Then p must divide T_n.

4. The denominator of b_n (II)

In this section we investigate prime factors $p \in (5\sqrt{n}, 15n)$ of T_n satisfying $p^2 | T_n$. Suppose now that $\{2\omega\} + \{12\omega\} \geq 1$. In the expression (3.1), one can take $k = k(p) \equiv 14n - [14n/p]p$; thus $\kappa = \tilde{\kappa}_\omega \equiv \{2\omega\} + \{12\omega\} - 1$. Hence, from (3.3), $\nu_p(T_{k(p),n}) = \nu_p(T_n)$. Suppose further that $\{2\omega\} + \{14\omega\} < 1$ and $2\{15\omega\} < \{14\omega\}$. Then, for any $\theta,\ \theta' \in [0, 1)$,

$$\begin{aligned} I(\omega, \theta, \tilde{\kappa}_\omega) &= [15\omega] - [15\omega - \theta] + [15\omega + \theta] - 1 - [2\omega] - [12\omega] - [\omega + \theta] \\ &\geq 2[15\omega] - [14\omega] - [16\omega] = \{14\omega\} + \{16\omega\} - 2\{15\omega\} \\ &> \{16\omega\} \geq 0 \end{aligned}$$

and

$$\begin{aligned} J(\omega, \theta', \tilde{\kappa}_\omega) &= [14\omega] - [14\omega - \theta'] + [2\omega] + 1 - [2\omega + \theta'] \\ &\geq 1 + [2\omega] + [14\omega] - [16\omega] = 1 + \{16\omega\} - \{2\omega\} - \{14\omega\} \\ &> \{16\omega\} \geq 0. \end{aligned}$$

Therefore $\nu_p(T_{k(p),n}) \geq 2$ from (3.4); hence $\nu_p(T_n) \geq 2$.

A similar argument to that above can be applied to the expression (3.5). By taking $k = 15n - [15n/p]p$, we can show that $\nu_p(T_n) \geq 2$ if ω satisfies $\{3\omega\} + \{12\omega\} \geq 1$, $\{3\omega\} + \{15\omega\} < 1$ and $2\{14\omega\} < \{15\omega\}$.

We have thus proved the following

LEMMA 4.1. *Suppose a prime $p \in (5\sqrt{n}, 15n)$ satisfies one of the following two conditions:*

(V) $\{2\omega\} + \{12\omega\} \geq 1$, $\{2\omega\} + \{14\omega\} < 1$ *and* $2\{15\omega\} < \{14\omega\}$;
(VI) $\{3\omega\} + \{12\omega\} \geq 1$, $\{3\omega\} + \{15\omega\} < 1$ *and* $2\{14\omega\} < \{15\omega\}$.

Then p^2 must divide T_n.

5. Proof of Theorem 1.2

Let Ω_1 and Ω_2 be the sets of $\omega \in [0, 1)$ satisfying one of the conditions stated in Lemma 3.1 (I, II, III or IV) and in Lemma 4.1 (V or VI) respectively. Then it is easily verified that the set Ω_1 consists of the following intervals:

$$\left[\frac{1}{15},\frac{1}{12}\right),\ \left[\frac{2}{15},\frac{1}{6}\right),\ \left[\frac{1}{5},\frac{1}{4}\right),\ \left[\frac{4}{15},\frac{5}{18}\right),\ \left[\frac{2}{7},\frac{5}{16}\right),\ \left[\frac{5}{14},\frac{5}{13}\right),\ \left[\frac{2}{5},\frac{5}{12}\right),$$
$$\left[\frac{3}{7},\frac{4}{9}\right),\ \left[\frac{7}{15},\frac{1}{2}\right),\ \left[\frac{8}{15},\frac{9}{16}\right),\ \left[\frac{4}{7},\frac{7}{12}\right),\ \left[\frac{3}{5},\frac{5}{8}\right),\ \left[\frac{9}{14},\frac{11}{16}\right),\ \left[\frac{5}{7},\frac{3}{4}\right),$$
$$\left[\frac{11}{14},\frac{5}{6}\right),\ \left[\frac{6}{7},\frac{8}{9}\right),\quad \text{and} \quad \left[\frac{13}{14},\frac{17}{18}\right).$$

Similarly the set Ω_2 consists of the following intervals:

$$\left[\frac{1}{14},\frac{1}{13}\right),\ \left[\frac{1}{7},\frac{2}{13}\right),\ \left[\frac{3}{14},\frac{2}{9}\right),\ \left[\frac{11}{15},\frac{3}{4}\right),\ \left[\frac{4}{5},\frac{13}{16}\right),\ \left[\frac{13}{15},\frac{7}{8}\right),\ \text{and}\ \left[\frac{14}{15},\frac{15}{16}\right).$$

Note that $\Omega_2 \subset \Omega_1$. Putting

$$\Delta_i = \prod_{\substack{p:\,\mathrm{prime} \\ 5\sqrt{n}<p<15n \\ \{n/p\}\in\Omega_i}} p$$

for $i = 1, 2$, it follows from Lemmas 3.1 and 4.1 that the integer $d_n \equiv \Delta_1\Delta_2$ must divide T_n; that is,

$$b_n \in \frac{d_n}{D_{16n}D_{18n}}\mathbb{Z}. \tag{5.1}$$

Note that d_n is a divisor of $D_{16n}D_{18n}$. Then, as in [6, 7], it follows from the prime number theorem that

$$\lim_{n\to\infty}\frac{1}{n}\log d_n = \int_{\Omega_1} d\psi(x) + \int_{\Omega_2} d\psi(x) \equiv \gamma, \quad \text{say},$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. The above Stieltjes integrals can be expressed as sums of values of some elementary functions, by virtue of Gauss' formula. (See, for example, Erdélyi *et al.* [5, p. 19].) Numerically one obtains

$$\gamma > 9.22875. \tag{5.2}$$

Now, putting $K_n = D_{16n}D_{18n}/d_n \in \mathbb{Z}$, we have, from (1.3) and (5.1),

$$K_n a_n \zeta(2) - K_n\epsilon_n = -K_n b_n \in \mathbb{Z}.$$

Then we get

$$\limsup_{n\to\infty}\frac{1}{n}\log|K_n a_n| \le 34 - \gamma + \beta \equiv \sigma, \quad \text{say},$$

and

$$\lim_{n\to\infty}\frac{1}{n}\log|K_n\epsilon_n| = 34 - \gamma + \alpha \equiv -\tau, \quad \text{say},$$

Therefore it follows from (2.1), (2.2) and (5.2) that π^2 has an irrationality measure

$$1+\frac{\sigma}{\tau}=1-\frac{34-\gamma+\beta}{34-\gamma+\alpha}<\frac{35.4093+36.0223}{9.22875-34+36.0223}<6.3489.$$

This completes the proof.

Addendum

Using the birational diffeomorphism $\tau : (0,1)^2 \to (0,1)^2$ defined by

$$\tau(x,y)=\left(1-xy, \frac{1-x}{1-xy}\right),$$

it follows that the double integral (1.2) can also be written as

$$(*) \qquad \epsilon_n=\iint\limits_S \frac{x^{17n}(1-x)^{14n}y^{15n}(1-y)^{14n}}{(1-xy)^{13n+1}}\,dx\,dy.$$

The transformation τ, satisfying $\tau=\tau^{-1}$, was essentially used in Rhin and Viola [8], although their transformation generates a cyclic group of order 5. The above expression (*) gives us further information about the denominator of $b_n \in \mathbb{Q}$ so that we can obtain

THEOREM. *There exists a positive integer q_0 such that*

$$\left|\pi^2-\frac{p}{q}\right| \gg q^{-5.687}$$

for all $p \in \mathbb{Z}$ and any integer $q \geq q_0$.

PROOF. After k-fold and $(13n-k)$-fold partial integration with respect to x and y respectively in (*), it can be seen that

$$\nu_p(T_n) \geq [13\omega-\kappa]-[13\omega]+\min_{0\leq\theta<1}\tilde{I}(\omega,\theta,\kappa)+\min_{0\leq\theta'<1}\tilde{J}(\omega,\theta',\kappa)$$

with

$$\tilde{I}(\omega,\theta,\kappa)=[14\omega]-[14\omega-\theta]+[17\omega+\theta]-[17\omega+\theta-\kappa],$$
$$\tilde{J}(\omega,\theta',\kappa)=[14\omega]-[14\omega-\theta']+[15\omega-\kappa+\theta']-[13\omega-\kappa]-[2\omega+\theta']$$

for any integer $k \in [0,13n]$ and for any prime $p>6\sqrt{n}$, where $\omega=\{n/p\}$, $\kappa=\{k/p\}$. Using this estimate for $\nu_p(T_n)$, Lemmas 3.1 and 4.1 can now be improved as follows.

LEMMA 1. *Suppose a prime $p \in (6\sqrt{n}, 15n)$ satisfies one of the conditions* (I), (II), (III), (IV), (VII) *or* (VIII), *the latter two being:*

(VII) $\{14\omega\} + \{17\omega\} < \{13\omega\}$,

(VIII) $\{2\omega\} + \{13\omega\} < 1$, $\{2\omega\} + \{14\omega\} < 1$ *and* $\{14\omega\} + \{17\omega\} < \{2\omega\} + \{13\omega\}$.

Then p must divide T_n.

LEMMA 2. *Suppose a prime $p \in (6\sqrt{n}, 15n)$ satisfies one of the conditions* (V), (VI) *or*

(IX) $\{2\omega\} + \{14\omega\} < 1$ *and* $1 + \{14\omega\} + \{17\omega\} < \{2\omega\} + \{13\omega\}$.

Then p^2 must divide T_n.

The above lemmas can be easily verified as above by taking $k = 13n$ in Case (VII) and $k = 15n - [15n/p]p$ in Cases (VIII) and (IX). Thus the sets Ω_1 and Ω_2 can be replaced by

$$\tilde{\Omega}_1 = \Omega_1 \cup \left[\frac{1}{17}, \frac{1}{16}\right)^* \cup \left[\frac{2}{17}, \frac{1}{8}\right) \cup \left[\frac{3}{17}, \frac{3}{16}\right) \cup \left[\frac{9}{17}, \frac{8}{15}\right) \cup \left[\frac{10}{17}, \frac{3}{5}\right) \cup \left[\frac{13}{17}, \frac{10}{13}\right),$$

$$\tilde{\Omega}_2 = \Omega_2 \cup \left[\frac{5}{17}, \frac{4}{13}\right) \cup \left[\frac{5}{14}, \frac{3}{8}\right) \cup \left[\frac{3}{7}, \frac{7}{16}\right)$$

respectively. Here the contribution of the interval $[\frac{1}{17}, \frac{1}{16})^*$ to the corresponding Stieltjes integral must be treated differently as follows:

$$\int_{[\frac{1}{17}, \frac{1}{16})^*} d\psi(x) = \sum_{n=1}^{\infty} \frac{1}{(16n+1)(17n+1)} = \psi\left(\frac{1}{16}\right) - \psi\left(\frac{1}{17}\right) - 1,$$

since $p < 15n$. Consequently we obtain a better exponent $\tilde{\gamma} > 10.5383$. Therefore π^2 has an irrationality measure less than

$$\frac{35.4093 + 36.0223}{10.5383 - 34 + 36.0223} < 5.687,$$

which completes the proof.

References

[1] F. Beukers, 'A note on the irrationality of $\zeta(2)$ and $\zeta(3)$', *Bull. London Math. Soc.* **11** (1979), 268–272.

[2] D. V. Chudnovsky and G. V. Chudnovsky, *Padé and rational approximations to systems of functions and their arithmetic applications* Lecture Notes in Mathematics 1052 (Springer, Berlin, 1984) pp. 37–84.

[3] ———, 'Transcendental methods and theta-functions', in: Proc. Sympos. Pure Math. 49 (Amer. Math. Soc., Providence, 1989), pp. 167–232.

[4] R. Dvornicich and C. Viola, 'Some remarks on Beukers' integrals', *Colloq. Math. Soc. János Bolyai* **51** (1987), 637–657.

[5] A. Erdélyi *et al.*, *Higher transcendental functions*, volume 1 (McGraw-Hill, New York, 1953).

[6] M. Hata, 'Legendre type polynomials and irrationality measures', *J. Reine Angew. Math.* **407** (1990), 99–125.

[7] ———, 'Rational approximations to π and some other numbers', *Acta Arithmetica* **63** (1993), 335–349.

[8] G. Rhin and C. Viola, 'On the irrationality measure of $\zeta(2)$', *Ann. Inst. Fourier (Grenoble)* **43** (1993), 85–109.

[9] E. A. Rukhadze, 'A lower bound for the approximation of ln 2 by rational numbers', *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **6** (1987), 25–29 in Russian.

Institute of Mathematics
Yoshida College
Kyoto University

https://doi.org/10.1017/S1446788700038192 Published online by Cambridge University Press