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ON THE SPECTRUM OF PERIODIC ELLIPTIC OPERATORS

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§ 0. Introduction

It was observed in [Su5] that the spectrum of a *periodic* Schrödinger operator on a Riemannian manifold has *band structure* if the transformation group acting on the manifold satisfies the *Kadison property* (see below for the definition). Here band structure means that the spectrum is a union of mutually disjoint, possibly degenerate closed intervals, such that any compact subset of $\mathbf{R}$ meets only finitely many. The purpose of this paper is to show, by a slightly different method, that this is also true for general periodic elliptic self-adjoint operators.

Let X be a Riemannian manifold of dimension n on which a discrete group Γ acts isometrically, effectively, and properly discontinuously. We assume that the quotient space $\Gamma \backslash X$ (which may have singularities) is compact. Let E be a Γ-equivariant hermitian vector bundle over X, and $D : C^\infty(E) \longrightarrow C^\infty(E)$ a formally self-adjoint elliptic operator which commutes with the Γ-action. For short, we call such a D a *Γ-periodic* operator. It is easy to show (see Section 1) that the symmetric operator D with the domain $C_0^\infty(E)$ is *essentially self-adjoint*, so that D has a unique self-adjoint extension in the Hilbert space $L^2(E)$ of square integrable section of E, which we denote also by D by a slight abuse of notation.

Let $C^*_{\text{red}}(\Gamma, \mathcal{K})$ denote the tensor product of the *reduced group C^*-algebra* of Γ with the algebra $\mathcal{K}$ of compact operators on a separable Hilbert space of infinite dimension, and by tr_Γ the canonical trace on $C^*_{\text{red}}(\Gamma, \mathcal{K})$. We define the *Kadison constant* $C(\Gamma)$ by

$$C(\Gamma) = \inf \{\text{tr}_\Gamma P \,;\, P \text{ is a non-zero projection in } C^*_{\text{red}}(\Gamma, \mathcal{K})\}.$$

By definition, Γ is said to satisfy the Kadison property if $C(\Gamma) > 0$. It is a conjecture proposed by Kadison that, if Γ is torsion free, then $C(\Gamma) = 1$. A

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geometric example of a discrete group with the Kadison property is the fundamental group of a closed Riemann surface (cf. [P]).

THEOREM 1. *If Γ has the Kadison property, then the spectrum of any Γ-periodic elliptic operator has band structure.*

In the case that the operator D is bounded from below, we may establish a quantitative result on the number of intervals in the spectrum.

THEOREM 2. *Suppose that D is a Γ-periodic elliptic operator of order p, and is bounded from below. Let $N(\lambda)$ be the number of components of the spectrum of D which intersect the interval $(-\infty, \lambda]$. If Γ has the Kadison property, then*

$$\limsup_{\lambda\to\infty} N(\lambda)\lambda^{-n/p} \leq C(\Gamma)^{-1}\Gamma(1+n/p)\int_{\Gamma\backslash X} A(x)dx, \ (n = \dim X),$$

where the function $A(x)$ can be evaluated explicitly in terms of the principal symbol $\sigma D(x, \xi)$ of D;

$$A(x) = (2\pi)^{-n-1}\int_{\mathbf{R}^n} d\xi \int_{-\infty-i\gamma}^{\infty-i\gamma} \mathrm{tr}(\sigma D(x, \xi) + i\tau)^{-1} e^{i\tau} d\tau. \tag{0.1}$$

In Section 6, as a byproduct of our argument, we shall establish a property of the *integrated density of states* associated to a periodic elliptic operator.

§ 1. Periodic elliptic operators

Let $D: C^\infty(E) \longrightarrow C^\infty(E)$ be a Γ-periodic, formally self-adjoint elliptic operator. We shall prove that D with the domain $C_0^\infty(E)$ is essentially self-adjoint. To show this, it is enough to prove that the *minimal domain* and *maximal domain* of D coincide. In the case that Γ acts *freely* on X, this is established by M. F. Atiyah [A]. The key of his proof is that D has an *almost local* pseudo-differential parametrix Q, which, in the case of free actions, may be constructed by lifting an almost local parametrix on the quotient manifold $\Gamma\backslash X$. Here an operator is said to be almost local if its Schwartz kernel has support close to the diagonal. In the general case, the constrution is carried out in the following way.

Let U_0, U_1, U_2, be open sets in X with $U_0 \subset \bar{U}_0 \subset U_1 \subset \bar{U}_1 \subset U_2$. We assume that U_0 contains the closure of a relatively compact fundamental domain for the Γ-action. One may construct an almost local pseudo-differential operator P and a local smooth operator H such that

$$PDs = s + Hs$$

for $s \in C_0^\infty(E \mid U)$ and

$$P(C_0^\infty(E \mid \bar{U}_1^c)) \cap C_0^\infty(E \mid U_0) = (0)$$

$$H(C_0^\infty(E \mid \bar{U}_1^c)) \cap C_0^\infty(E \mid U_0) = (0).$$

We select $\varphi \in C_0^\infty(U_0)$ with

$$\sum_{\sigma \in \Gamma} \sigma \cdot \varphi \equiv 1. \tag{1.1}$$

Since

$$\varphi PD = \varphi + \varphi H$$

on $C_0^\infty(E)$, putting

$$Q = \sum_{\sigma \in \Gamma} \sigma(\varphi P)\sigma^{-1}$$

$$K_1 = \sum_{\sigma \in \Gamma} \sigma(\varphi H)\sigma^{-1},$$

we have

$$QD = I + K_1.$$

From the way of constructions, the operators Q and K_1 are almost local, and K_1 is smooth. It is straightforward to see that

$$DQ = I + K_2,$$

where

$$K_2 = {}^tK_1 + DK_1\,({}^tQ) - DQ\,({}^tK_1),$$

which is also almost local and smooth.

§ 2. Heat kernel

It should be noted that, to prove Theorem 1, we only have to establish the assertion for positive elliptic operators. In fact, the general case reduces to the positive case in the following manner. Suppose that the spectrum of a self-adjoint elliptic operator D has a cluster at $a \in \mathbf{R}$, in the sense that there exists a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbf{R}$ such that $\lim a_n = a$, and a_m and a_n lie in different components of the resolvent set of D if $m \neq n$. We set, with $k \in \mathbf{N}$,

$$D' = \left(D - \frac{a + a_1}{2}\right)^{2k},$$

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$$a'_n = \left(a_n - \frac{a+a_1}{2}\right)^{2k}.$$

Let ε be a positive number such that ε-neighborhood of a_1 is contained in the resolvent set. Then, for every a_n with $|a - a_n| < \varepsilon$, a'_n is in the resolvent set of D', and a'_m and a'_n lie in different components of the resolvent set of D' for $m \neq n$. This implies that the spectrum of the positive operator D' has a cluster at $(a - a_1)^{2k}/2^{2k}$. Thus it suffices to handle the case of positive D with order greater than the dimension of X. This assumption considerably simplifies our argument.

Let D be a Γ-periodic self-adjoint positive elliptic operator on X of order $p = 2k$ $(> n := \dim X)$. We denote by $\mathfrak{D}(D)$ the domain of D. The heat semigroup, $\exp(-tD)$, is well-defined for $t > 0$. It is our first purpose to give good kernel estimates for $\exp(-tD)$, uniformly on X. We denote by d the distance function on X, by E^* the dual bundle of E, and by $E \boxtimes E^*$ the bundle over $X \times X$ with fiber $E_x \boxtimes E^*_y$ over (x, y). □

PROPOSITION 1. *Let $k(t\,; x,y)$ be the kernel function for $\exp(-tD)$, which is, for $t > 0$, a smooth section of $E \boxtimes E^*$. Then, for $T > 0$ fixed, there are positive constants C_1 and C_2 such that*

$$|k(t\,; x, y)|_{E\boxtimes E^*} \leq C_1 t^{-n/p} \exp\left(-C_2 t^{-1/(p-1)} d(x, y)^{p/(p-1)}\right),$$

uniformly for $t \in (0, T]$ and $(x, y) \in X \times X$.

we start our construction locally near an arbitrary $x_0 \in X$.

LEMMA 1. *For any relatively compact connected open neighborhood U of x_0 and for any $t > 0$, there is a smooth operator, $F_t : L^2(E) \longrightarrow L^2(E)$, with a kernel function $f_t \in C^\infty(E \boxtimes E^*)$, such that the following is true.*

1) *f_t depends smoothly on t, and $f_t(x, y) = 0$ for $x, y \notin U$.*

2) *For $T > 0$ fixed, there are constants $C_3 = C_3(T)$ and $C_4 = C_4(T)$ such that*

$$|f_t(x, y)|_{E\boxtimes E^*} \leq C_3 t^{-n/p} \exp\left(-C_4 d(x, y)^{p/(p-1)} t^{-1/(p-1)}\right), \tag{2.1}$$

uniformly in $0 < t \leq T$, $(x, y) \in U \times U$.

3) *For $\varphi \in C_0^\infty(X)$, with $\varphi = 1$ in a neighborhood of U, the function $t \longmapsto \varphi F_t$ is continuous on $[0, T]$ in the strong topology of bounded operators of $L^2(E)$, and $\lim_{t\to 0} F_t s = s$ in $L^2(E)$, for all $s \in L^2(E)$ with $\operatorname{supp} s \subset U$.*

4) *For all $s \in L^2(E)$, the function*

$$(0,T) \longrightarrow \mathfrak{D}(D)$$
$$t \longmapsto \varphi F_t s$$

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is differentiable, and if we define

$$R_t s := (\partial_t + D)\varphi F_t s,$$

then R_t *has a kernel in* $C^\infty(E \boxtimes E^*)$, *say* r_t, *with smooth dependence on* $t > 0$. *Moreover, we have the estimate*

$$(2.2) \qquad |r_t(x, y)|_{E\boxtimes E^*} \leq C_5 \exp\left(-C_6 d(x, y)^{p/(p-1)} t^{-1/(p-1)}\right),$$

uniformly in $t \in [0, T]$ *and* $(x,y) \in U \times U$.

5) R_t *and* $D\varphi F_t$ *are bounded in* $L^2(E)$, *and*

$$(2.3) \qquad \| \varphi F_t \|_{L^2(E)} + \| R_t \|_{L^2(E)} \leq C_7$$

for $t \in [0,T]$ *and*

$$\| D\varphi F_t \|_{L^2(E)} \leq C_8(\varepsilon)$$

for $t \in [\varepsilon, T]$, $\varepsilon > 0$.

The proof follows from [G].

To construct a global parametrix, we now choose an open connected, relatively compact set U and $\varphi \in C_0^\infty(U)$ satisfying (1.1). Then it is readily seen that

$$\lim_{t\to 0} \sigma(\varphi F_t)\sigma^{-1} s = s \text{ in } L^2(E),$$

for all $s \in L^2(E)$ with supp $s \subset \sigma U$, and that

$$(\partial_t + D)\sigma(\varphi F_t)\sigma^{-1} = \sigma R_t \sigma^{-1}.$$

Moreover, since the kernel functions of $\sigma(\varphi F_t)\sigma^{-1}$ and $\sigma R_t \sigma^{-1}$ are $\varphi(\sigma^{-1}x)\sigma f_t(\sigma^{-1}x, \sigma^{-1}y)\sigma^{-1}$ and $\sigma r_t(\sigma^{-1}x, \sigma^{-1}y)\sigma^{-1}$ respectively, they satisfy the estimates (2.1) (2.2) .

Now the global parametrix is defined by

$$(2.4) \qquad \mathcal{T}_t s := \sum_{\sigma\in\Gamma} \sigma(\varphi F_t)\sigma^{-1} s.$$

We set

$$\mathcal{R}_t s := \sum_{\sigma\in\Gamma} \sigma(\varphi F_t)\sigma^{-1} s.$$

LEMMA 2. 1) *The operators* $\mathcal{T}_t$ *and* $\mathcal{R}_t$ *are continuous in* $L^2(E)$ *and*

$$\| \mathcal{T}_t \|_{L^2(E)} + \| \mathcal{R}_t \|_{L^2(E)} \leq C_9,$$

uniformly in $t \in [0, T]$.

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2) *For* $s \in L^2(E)$, *the functions* $\mathcal{T}_t s$ *and* $\mathcal{R}_t s$ *are continuous in* $t \in [0,T]$ *with*

$$\lim_{t\to 0} \mathcal{T}_t s = s.$$

3) $\mathcal{T}_t s$ *is differentiable in* $t \in [0, T]$, *has values in* $\mathcal{D}(D)$, *and satisfies the equation*

$$(\partial_t + D)\mathcal{T}_t s = \mathcal{R}_t s. \tag{2.5}$$

Proof. Before the proof we remark that there is a constant C_{10} such that, with φ as above,

$$\sum_{\sigma\in\Gamma} \| \varphi_\sigma s \|_{L^2(E)} \le C_{10} \| s \|_{L^2(E)}, \text{ for } s \in L^2(E), \tag{2.6}$$

where $\varphi_\sigma(x) = \varphi(\sigma x)$.

1) follows easily from Lemma 1, 5) and (2.6).

The estimate in 1) shows that the series (2.4) is uniformly convergent in $[0,T]$. By Lemma 1, 3), we get 2).

It follows from Lemma 1, 4) that each term in the series (2.4) is differentiable as a function $[0,T] \longrightarrow L^2(E)$, and has values in $\mathcal{D}(D)$. From Lemma 1, 4) and (2.6) we derive the estimate

$$\begin{aligned} \| D\mathcal{T}_t s \|_{L^2(E)} &\le \sum_{\sigma\in\Gamma} \| D\varphi F_t \sigma^{-1} s \|_{L^2(E)} \\ &\le C_8(\varepsilon) \sum_{\sigma\in\Gamma} \| \varphi_\sigma s \|_{L^2(E)} \\ &\le C_8(\varepsilon) C_{10} \| s \|_{L^2(E)}, \end{aligned}$$

uniformly in $[\varepsilon, T]$ and $s \in L^2(E)$, for all $\varepsilon \in (0,T)$. Now the assertion 3) follows easily from Lemma 1, 4) and 5). □

Proof of Proposition 1. We apply the abstract theory of evolution equations in Banach spaces to (2.5). It is readily checked, using Lemma 2, that the assumptions of Ch. I, Theorem 6.1 in [*K*] are satisfied. Thus we find for $t \in [0,T]$ the operator equation

$$\exp(-tD) = \mathcal{T}_t - \int_0^t \exp(-(t-u)D)\mathcal{R}_u du. \tag{2.7}$$

Then we define again for $t \in [0,T]$,

$$(\mathcal{T} *^0 \mathcal{R})_t := \mathcal{T}_t,$$

$$(\mathcal{T} *^{j+1} \mathcal{R}) := \int_0^t (\mathcal{T} *^j \mathcal{R})_{t-u} \mathcal{R}_u du,\ j \in \mathbf{Z}_+.$$

This clearly defines continuous families of bounded operators in $L^2(E)$, and from

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Lemma 2, 1) we deduce inductively the norm estimates

$$\|(\mathcal{T} * {}^j\mathcal{R})_t\| \leq C_9^{j+1} t^j / j!, \tag{2.8}$$

On the other hand, iterating (2.7) gives for $N \in \mathbf{Z}_+$

$$\exp(-tD) = \sum_{j=0}^{N} (-1)^j (\mathcal{T} * {}^j\mathcal{R})_t + (-1)^{N+1} (\exp(-\cdot D) * {}^{N+1}\mathcal{R})_t,$$

which leads, with (2.8) and a similar estimate for the remainder, to the Neumann series

$$\exp(-tD) = \sum_{j=0}^{\infty} (-1)^j (\mathcal{T} * {}^j\mathcal{R})_t, \tag{2.9}$$

which is uniformly convergent for $t \in [0, T]$.

We claim that $(\mathcal{T} * {}^j\mathcal{R})_t$ has kernel in $C(E \boxtimes E^*)$, for $t \in [0, T]$, with the estimate

$$|(\mathcal{T} * {}^j\mathcal{R})_t(x, y)|_{E \boxtimes E^*} \leq L^{j+1} C_3^{j+1} \left(j! \binom{n/p}{j}\right)^{-1} (-1)^j \cdot \\ \cdot t^{j-n/p} \exp\left(-C_4 d(x, y)^{p/(p-1)} t^{-1/(p-1)}\right), \tag{2.10}$$

where

$$L = \#\{\sigma \in \Gamma ; \sigma(\operatorname{supp} \varphi) \cap (\operatorname{supp} \varphi) \neq \phi\}.$$

This is clear for $j = 0$, by the above discussion. If (2.10) is proved for some $j \geq 0$, we find

$$|(\mathcal{T} * {}^{j+1}\mathcal{R})_t(x, y)|_{E \boxtimes E^*} \\ \leq \int_0^t \int_X |(\mathcal{T} * {}^j\mathcal{R})_{t-u}(x, y)|_{E \boxtimes E^*} \sum_\sigma |\sigma R_u \sigma^{-1}(z, y)|_{E \boxtimes E^*} dz du. \tag{2.11}$$

Now there are at most L terms in the sum which are different from zero, for y fixed, and for each term the volume of the support in z is bounded by vol(supp φ), which is assumed to be one for simplicity. Hence (2.11) is obvious from the induction hypothesis, an elementary inequality for exponentials (cf. Lemma 1.4.2. in [G]), and (2.2).

Having proved (2.10), we can use the series for the kernels to obtain the estimate stated in Proposition 1.

The arguments give above also lead to the following result (cf. [G, Theorem 1.6.1]).

Proposition 2. *tr $k(t, x, x) \sim t^{-n/p} A(x)$ as $t \downarrow 0$, where $A(x)$ is the function defined by* (0.1).

§ 3. Group C^*-algebras

We adopt the terminology employed in [Su5].

Let Γ be a discrete group and let $C^*_{\mathrm{red}}(\Gamma)$ be the reduced group C^*-algebra of Γ. We set $C^*_{\mathrm{red}}(\Gamma, \mathcal{K}) = C^*_{\mathrm{red}}(\Gamma) \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators of some separable Hilbert space, say V. We can identify $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$ with a subalgebra of

$$W^*(\Gamma, \mathcal{L}) = \{A : L^2(\Gamma, V) \to L^2(\Gamma, V)\ ;\ A \text{ a bounded linear operator with } A\sigma = \sigma A \text{ for all } \sigma \in \Gamma\},$$

where we regard $L^2(\Gamma, V)$ as a Γ-module via the right regular representation of Γ on $L^2(\Gamma)$ tensored with the identity on V.

Let $A \in W^*(\Gamma, \mathcal{L})$. We define the *Fourier coefficient* $\widehat{A}(\sigma)$ at σ to be a bounded operator of V given by

$$\widehat{A}(\sigma)v = (A\delta_1^v)(\sigma),$$

where

$$\delta_1^v(\sigma) = \begin{cases} v & \text{if } \sigma = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Recall that

$$\mathrm{tr}_\Gamma A = \mathrm{tr}\, \widehat{A}(1).$$

Let $C_0^*(\Gamma, \mathcal{K})$ be the set of $A \in W^*(\Gamma, \mathcal{L})$ with $\widehat{A}(\sigma) \in \mathcal{K}$ for all $\sigma \in \mathcal{K}$ and $\widehat{A}(\sigma) = 0$ for all but finitely many $\sigma \in \Gamma$. We may identify $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$ with the completion of $C_0^*(\Gamma, \mathcal{K})$ with respect to the operator norm,

LEMMA 3. *Let $A \in W^*(\Gamma, \mathcal{L})$. Then*

$$\|A\| \leq \sum_{\sigma \in \Gamma} \|\widehat{A}(\sigma)\|.$$

Proof.

$$\begin{aligned}
\|A\varphi\|^2 &= \sum_\sigma \Big\| \sum_\mu \widehat{A}(\sigma\mu^{-1})\varphi(\mu) \Big\|^2 \\
&= \sum_\sigma \sum_{\mu_1} \sum_{\mu_2} \langle \widehat{A}(\sigma\mu_1^{-1})\varphi(\mu_1), \widehat{A}(\sigma\mu_2^{-1})\varphi(\mu_2) \rangle \\
&= \sum_\sigma \sum_{\theta_1} \sum_{\theta_2} \langle \widehat{A}(\theta_1)\varphi(\theta_1^{-1}\sigma), \widehat{A}(\theta_2)\varphi(\theta_2^{-1}\sigma) \rangle \\
&= \sum_{\theta_1} \sum_{\theta_2} \Big(\sum_\sigma \| \langle \widehat{A}(\theta_1)\varphi(\theta_1^{-1}\sigma) \|^2 \Big)^{1/2} \Big(\sum_\sigma \| \langle \widehat{A}(\theta_2)\varphi(\theta_2^{-1}\sigma) \|^2 \Big)^{1/2}
\end{aligned}$$

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$$\leq \sum_{\theta_1} \sum_{\theta_2} \| \hat{A}(\theta_1) \| \| \hat{A}(\theta_2) \| \left(\sum_{\sigma} \| \varphi(\sigma) \|^2 \right)$$
$$= \left(\sum_{\theta} \| \hat{A}(\theta) \| \right) \| \varphi \|^2. \qquad \square$$

COROLLARY. *If* $\hat{A}(\sigma) \in \mathcal{K}$ *for every* $\sigma \in \Gamma$ *and if*

$$\sum_{\sigma \in \Gamma} \| \hat{A}(\sigma) \| < \infty,$$

then $A \in C^*_{\text{red}}(\Gamma, \mathcal{K})$.

Proof. Let $K_1 \subset K_2 \subset K_3 \subset \cdots$ be a sequence of finite subsets in Γ with $\bigcup K_i = \Gamma$. For an integer $N > 0$, let $A_N \in W^*(\Gamma, \mathcal{L})$ be defined by

$$\hat{A}_N(\sigma) = \begin{cases} \hat{A}(\sigma), & \sigma \in K_N, \\ 0, & \sigma \notin K_N. \end{cases}$$

Then $A_N \in C_0^*(\Gamma, \mathcal{K})$, and $\| A - A_N \| \to 0$ as $N \to \infty$, so that $A \in C^*_{\text{red}}(\Gamma, \mathcal{K})$. $\square$

§ 4. Proof of Theorem 1

Let D be a positive Γ-periodic elliptic operator on X of order p. Fix a relatively compact fundamental domain $\mathcal{F}$ in X for the Γ-action, and identify $L^2(E \mid \mathcal{F})$ with V. Then we may identify $L^2(E)$ with $L^2(\Gamma, V)$, and $W^*(\Gamma, \mathcal{L})$ with the ring of Γ-equivariant bounded operators of $L^2(E)$.

We denote by $\| \sigma \|$ the word length of σ associated with a fixed finite set of generators of Γ. It is easy to see that there exist positive constants C_{11} and C_{12} such that

$$\| \sigma \| \leq C_{11} \min_{x, y \in \mathcal{F}} d(\sigma x, y) + C_{12}. \tag{4.1}$$

Since Γ is a homomorphic image of a free group of finite rank, and a free group has exponential growth with respect to word length, we may conclude that exist constants C_{13} and C_{14} such that

$$^{\#}\{\sigma \in \Gamma \,;\, \| \sigma \| \leq R\} \leq C_{13} \exp (C_{14} R). \tag{4.2}$$

LEMMA 4. $\exp(-D) \in C^*_{\text{red}}(\Gamma, \mathcal{K})$.

Proof. Let $A := \exp(-D)$, and let $k(x, y)$ be the kernel function for A. The operator $\hat{A}(\sigma)$ acting on $L^2(E \mid \mathcal{F})$ has the kernel function $\sigma^{-1} k(\sigma x, y)$. In view of Proposition 1 and (4.1), we have

$$\| \hat{A}(\sigma) \| \leq \text{vol}\,(\mathcal{F}) \sup_{x, y \in \mathcal{F}} | k(\sigma x, y) | \leq C_{15} \exp(- C_{16} \| \sigma \|^{\alpha}),$$

where we set $\alpha = p/(p-1) > 1$. From this and (4.2), it follows that

$$\sum_{\sigma} \| \hat{A}(\sigma) \| < \infty,$$

and hence $A \in C^*_{\text{red}}(\Gamma, \mathcal{K})$ by Corollary to Lemma 3. □

The rest of proof is done in a standard manner (see [Su5]). Namely, if two real numbers a, b with $a < b$ lie in the resolvent set of D, then, given a positive ε, one may find a polynomial $p(x)$ with

$$\| (E(b) - E(a)) - p(\exp(-D)) \| < \varepsilon,$$

where $\{E(\lambda)\}_{-\infty<\lambda<\infty}$ denotes the spectral resolution for D; i.e.

$$D = \int_{-\infty}^{+\infty} \lambda \, dE(\lambda).$$

This implies that $E(b) - E(a) \in C^*_{\text{red}}(\Gamma, \mathcal{K})$, and thus leads to Theorem 1 (see also the discussion in the next section).

§ 5. Integrated densities of states

Let D be a self-adjoint Γ-periodic elliptic operator *bounded from below,* and let $\{E(\lambda)\}_{\lambda\in\mathbf{R}}$ be the spectral resolution for D. Note that $E(\lambda)$ is of Γ-trace class [Su5; Lemma 2]. We set

$$\varphi(\lambda) := \varphi_D(\lambda) := \operatorname{tr}_\Gamma E(\lambda).$$

The function φ is what we call the integrated density of states (cf. [Su6]). We readily check that

$$\varphi_{D-a}(\lambda) = \varphi_D(\lambda + a), \tag{5.1}$$

and if D is positive, then

$$\varphi_{D^m}(\lambda) = \varphi_D(\lambda^{1/m}). \tag{5.2}$$

To prove Theorem 2, let $\lambda > 0$, and let $a_1 < a_2 < \cdots < a_n$ be a sequence in the resolvent set such that $a_i < \lambda$ and $E(a_{i+1}) - E(a_i)$ is a nontrivial projection for all i. Since $\sum (E(a_{i+1}) - E(a_i)) \le E(\lambda)$, one has

$$(n-1)C(\Gamma) \le \varphi(\lambda).$$

It remains to show the following.

PROPOSITION 3.

$$\varphi(\lambda) \sim \lambda^{n/p} \Gamma(1 + n/p) \int_{\Gamma\backslash X} A(x) \, dx \quad as \quad \lambda \uparrow \infty.$$

Proof. In view of (5.1) and (5.2), we may assume that D is positive and the order p is greater than the dimension of X.
Since

$$\exp(-tD) = \int e^{-t\lambda} dE(\lambda),$$

by taking the Γ-trace of both sides we obtain

$$\mathrm{tr}_\Gamma \exp(-tD) = \int e^{-t\lambda} d\varphi(\lambda).$$

By using Proposition 2 in § 2, we find

$$\mathrm{tr}_\Gamma \exp(-tD) = \int_{\mathscr{F}} \mathrm{tr}\, k(t, x, x) dx \sim t^{-n/p} \int_{\mathscr{F}} A(x) dx$$

as $t \downarrow 0$, from which the assertion follows by a well-known Tauberian theorem. □

§ 6. Limit formulae for densities of states

Let

$$\Gamma \supset G_1 \supset G_2 \supset \cdots$$

be a sequence of normal subgroups in Γ with G_i acting freely on X, so that $X_i = G_i \backslash X$ is a Riemannian manifold on which the factor group $\Gamma_i = \Gamma / G_i$ acts as a proper discontinuous isometry group. The quotient space $\Gamma_i \backslash X_i$ is naturally identified with $\Gamma \backslash X$. Let $\pi : X \longrightarrow \Gamma \backslash X$ and $\pi_i : X \longrightarrow X_i$ denote the projection maps. One may select a relatively compact fundamental domain $\mathscr{F} \subset X$ for the Γ-action such that $\pi_i : \mathscr{F} \longrightarrow \pi_1(\mathscr{F})$ is a diffeomorphism. It is easily seen that $\pi_i(\mathscr{F})$ is a fundamental domain in X_i for the Γ_i-action.

We may push down the elliptic operator D onto X_i, which is a Γ_i-periodic elliptic operator. We denote it by D_i, and put

$$\varphi_i(\lambda) = \mathrm{tr}_{\Gamma_i} E_i(\lambda),$$

where

$$D_i = \int \lambda\, dE_i(\lambda)$$

is the spectral resolution of D_i.

THEOREM 3. *Let D be a Γ-periodic self-adjoint elliptic operator bounded from below. Suppose that $\bigcap G_i = \{1\}$. Then $\lim_{i \to \infty} \varphi_i(\lambda) = \varphi(\lambda)$ at continuity points of φ.*

Proof. Let $k_i(t; p, q)$ be the kernel function for $\exp(-tD_i)$. Let $p = \pi_i(x)$

and $q = \pi_i(y)$. Then

$$k_i(t\,;p,\,q) = \sum_{g \in G_i} gk(t\,;g^{-1}x,\,y),$$

so that we obtain, using Proposition 1 again,

$$\begin{aligned}\int e^{-\lambda t}\,d\varphi_i(\lambda) &= \mathrm{tr}_{\Gamma_i}\exp(-\,tD_i)\\ &= \int_{\pi_i(\mathcal{F})} \mathrm{tr}\,k_i(t\,;p,\,p)dp\\ &= \sum_{g \in G_i}\int_{\mathcal{F}} \mathrm{tr}\,gk(t\,;g^{-1}x,\,x)dx\\ &= \int_{\mathcal{F}} \mathrm{tr}\,k(t\,;x,\,x)dx + \sum_{\substack{g \neq 1\\ g \in G_i}}\int_{\mathcal{F}} \mathrm{tr}\,gk(t\,;g^{-1}x,\,x)dx\\ &\longrightarrow \mathrm{tr}_{\Gamma}(\exp(-\,tD)) = \int e^{-\lambda t}\,d\varphi(\lambda)\end{aligned}$$

as $i \longrightarrow \infty$. This leads to our assertion (cf. [Su]) . □

Remark. If each G_i has finite index in Γ, then

$$\varphi_i(\lambda) = (\#\Gamma_i)^{-1}\#\{\lambda_k\,;\lambda_k \leq \lambda\},$$

where $\lambda_1 \leq \lambda_2 \leq \cdots$ stands for the eigenvalues of the elliptic operator D_i on the *compact* manifold X_i. Theorem 3 is a generalization of a result in [Su6].

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