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COMPATIBLE TIGHT RIESZ ORDERS II

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R. N. Ball (unpublished) and G. E. Davis and C. D. Fox [**1**] established that if Ω is a doubly homogeneous totally ordered set, the l-group $A(\Omega)$ of all order-preserving permutations of Ω endures a compatible tight Riesz order. Specifically $T = \{g \in A(\Omega)^+ : \text{supp}(g) \text{ is dense in } \Omega\}$ is a compatible tight Riesz order for $A(\Omega)$. Using this fact, I inserted Theorem 3.7 into [**2**; *MR 53* (1977), #13070] at the galley proof stage. (It was also included in *MR 54* (1977), #7350 and [**3**; p. 472].) Theorem 3.7 stated: Let Ω be homogeneous. Then $A(\Omega)$ endures a compatible tight Reisz order if and only if Ω is dense. I stated that it was obvious that if Ω were homogeneous and discrete, $A(\Omega)$ could not endure a compatible tight Riesz order. This "obvious" is neither obvious nor true. My purpose in this note is to prove in a unified way (and without recourse to the machinery developed in [**2**]):

THEOREM. *Let Ω be a homogeneous linearly ordered set. Then $A(\Omega)$ endures a compatible tight Riesz order if and only if Ω is not ordermorphic to* $\mathbf{Z}$.

Let Ω be a homogeneous linearly ordered set (i.e., $A(\Omega)$ is transitive). The set of $A(\Omega)$-congruences on Ω forms a chain (under inclusion), and for each $\alpha, \beta \in \Omega$ with $\alpha \neq \beta$, there exists a (unique) convex $A(\Omega)$-congruence $\mathscr{C}_\gamma$ on Ω that is maximal with respect to $\alpha\mathscr{C}_\gamma \neq \beta\mathscr{C}_\gamma$. Let $\mathscr{C}^\gamma$ be the intersection of all convex $A(\Omega)$-congruences $\mathscr{C}$ on Ω such that $\alpha\mathscr{C}\beta$. Then $\alpha\mathscr{C}^\gamma\beta$ and $\mathscr{C}^\gamma$ covers $\mathscr{C}_\gamma$ (in the set of all convex $A(\Omega)$-congruences on Ω). Let $\gamma = \text{Val}(\alpha, \beta) = (\mathscr{C}_\gamma, \mathscr{C}^\gamma)$, and let Γ be the set of all such γ (as α, β range over Ω with $\alpha \neq \beta$) totally ordered by: $\gamma_1 \leqq \gamma_2$ if and only if $\mathscr{C}_{\gamma_1} \subseteq \mathscr{C}_{\gamma_2}$. Let $\Omega(\gamma) = \alpha\mathscr{C}^\gamma/\mathscr{C}_\gamma$. If $g \in A(\Omega)$ is such that $\alpha\mathscr{C}^\gamma\alpha g$, let $g_{\alpha,\gamma} \in A(\Omega(\gamma))$ be obtained from g by: $(\beta\mathscr{C}_\gamma)g_{\alpha,\gamma} = \beta g\mathscr{C}_\gamma$ $(\beta \in \alpha\mathscr{C}^\gamma)$. Observe that for each $\alpha \in \Omega$,

$$\{g_{\alpha,\gamma} \in A(\Omega(\gamma)) : g \in A(\Omega) \quad \text{and} \quad \alpha\mathscr{C}^\gamma\alpha g\} = A(\Omega(\gamma)).$$

$\{A(\Omega(\gamma)) : \gamma \in \Gamma\}$ is called the *set of o-primitive components of* $A(\Omega)$. For each $\gamma \in \Gamma$, $A(\Omega(\gamma))$ is o-2 transitive (and divisible), isomorphic to $\mathbf{Z}$, or Ohkuma (i.e., $\Omega(\gamma)$ is ordermorphic to a dense subgroup of $\mathbf{R}$—and so has cofinality $\aleph_0$—and $A(\Omega(\gamma))$ is just the right regular representation of $\Omega(\gamma)$). If $\Omega(\gamma)$ is an Ohkuma set, $A(\Omega(\gamma))$ is a dense totally ordered group and hence

$$T = \{g \in A(\Omega(\gamma)) : g > e\} = \{g \in A(\Omega(\gamma))^+ : g \neq e\}$$

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is a *compatible tight Riesz order on* $A(\Omega(\gamma))$. (T is a compatible tight Riesz order on an l-group G provided

1. T is a proper filter on G^+,
2. T is G-invariant $[f \in T$ implies $(\forall g \in G)(g^{-1}fg \in T)]$,
3. $T \cdot T = T$, and
4. $\inf T = e$).

Note that $A(\mathbf{Z}) \cong \mathbf{Z}$ and so, by 3, cannot endure a compatible tight Riesz order. Finally, if $\mathscr{C}$ is a convex $A(\Omega)$-congruence on Ω, let

$$L(\mathscr{C}) = \{g \in A(\Omega) : \alpha\mathscr{C}\alpha g \text{ for all } \alpha \in \Omega\}.$$

$L(\mathscr{C})$ is an l-ideal of $A(\Omega)$. For proofs and further details of these facts, see [**3**].

Throughout this paper, assume that Ω is a homogeneous linearly ordered set.

LEMMA 1. *If $A(\Omega)$ has an o-primitive component that is o-2 transitive, then $A(\Omega)$ endures a compatible tight Riesz order.*

Proof. Let $\gamma \in \Gamma$ be such that $A(\Omega(\gamma))$ is o-2 transitive. Let

$$T_\gamma = \{f \in A(\Omega(\gamma))^+ : \operatorname{supp}(f) \text{ is dense in } \Omega(\gamma)\},$$

a compatible tight Riesz order on $A(\Omega(\gamma))$. Let

$$T' = \{g \in L(\mathscr{C}^\gamma)^+ : (\forall \beta \in \Omega)(g_{\beta,\gamma} \in T_\gamma)\}.$$

T' is an $A(\Omega)$-invariant subset of $A(\Omega)^+$ that satisfies $T' \cdot \mathrm{T}' = \mathrm{T}'$ (since $A(\Omega(\gamma))$ is divisible). Moreover, $f, g \in T'$ implies $f \wedge g \in T'$. Let $\alpha \in \Omega$ and $h \in T_\gamma$. There exists $g_{\alpha,\gamma} \in T_\gamma$ such that $\alpha\mathscr{C}_\gamma$ is fixed by $g_{\alpha,\gamma}$. Let $g \in A(\Omega)^+$ be such that $g_{\beta,\gamma'} = e$ if $\gamma' < \gamma$ and $\beta\mathscr{C}^\gamma\alpha$, and

$$g_{\beta,\gamma} = \begin{cases} g_{\alpha,\gamma} & \text{if } \beta\mathscr{C}^\gamma\alpha. \\ h & \text{otherwise.} \end{cases}$$

Then $g \in T'$ and $\alpha g = \alpha$. Consequently, $\inf T' = e$. Therefore

$$T = \{g \in A(\Omega) : g \geqq f \text{ for some } f \in T'\}$$

is a compatible tight Riesz order on $A(\Omega)$.

LEMMA 2. *If $A(\Omega)$ has a non-maximal o-primitive component that is Ohkuma, then $A(\Omega)$ endures a compatible tight Riesz order.*

Proof. Let $\gamma \in \Gamma$ be such that $A(\Omega(\gamma))$ is Ohkuma. Let

$$T_\gamma = \{f \in A(\Omega(\gamma))^+ : f \neq e\},$$

a compatible tight Riesz order on $A(\Omega(\gamma))$. Let

$$T' = \{g \in L(\mathscr{C}^\gamma)^+ : (\exists\, \sigma, \tau \in \Omega)(\sigma < \tau \text{ and } (\forall \beta \in \Omega)$$
$$[(\beta < \sigma \text{ or } \tau < \beta) \rightarrow (g_{\beta,\gamma} \in T_\gamma)])\}.$$

T' is an $A(\Omega)$-invariant subset of $A(\Omega)^+$ that satisfies $T' \cdot T' = T'$ (since T_γ

has no least element and is totally ordered). Moreover, f, $g \in T'$ implies $f \wedge g \in T'$. Let $\alpha \in \Omega$. Since γ is non-maximal, there exist $\sigma, \tau \in \Omega$ such that $\sigma\mathscr{C}^\gamma < \alpha\mathscr{C}^\gamma < \tau\mathscr{C}^\gamma$. Let $h \in T_\gamma$. Define $g \in A(\Omega)^+$ such that $g_{\beta,\gamma} = h$ for all $\beta \in \Omega$ for which $\beta\mathscr{C}^\gamma \leqq \sigma\mathscr{C}^\gamma$ or $\beta\mathscr{C}^\gamma \geqq \tau\mathscr{C}^\gamma$ and $g_{\beta,\gamma'} = e$ for all $\gamma' \leqq \gamma$ and $\beta \in \Omega$ for which $\sigma\mathscr{C}^\gamma < \beta\mathscr{C}^\gamma < \tau\mathscr{C}^\gamma$. Then $g \in T'$ and $\alpha g = \alpha$. It follows that $\inf T' = e$. Therefore

$$T = \{g \in A(\Omega) : g \geqq f \text{ for some } f \in T'\}$$

is a compatible tight Riesz order on $A(\Omega)$.

Note that T defined above is equal to

$$\{g \in A(\Omega)^+ : (\exists\, \sigma, \tau \in \Omega)(\sigma < \tau \quad \text{and} \quad (\forall \beta)[(\beta < \sigma \quad \text{or} \quad \tau < \beta) \rightarrow \text{Val}\,(\beta, \beta g) \geqq \gamma])\}.$$

LEMMA 3. *If there exists* $\{\gamma_n : n \in \mathbf{Z}^+\} \subseteq \Gamma$ *with* $\gamma_1 > \gamma_2 > \gamma_3 > \ldots$, *then* $A(\Omega)$ *endures a compatible tight Riesz order.*

Proof. In view of Lemmas 1 and 2, we may assume that $A(\Omega(\gamma)) \cong \mathbf{Z}$ for all $\gamma \in \Gamma \backslash \{\gamma_1\}$. Let

$$T = \{g \in A(\Omega)^+ : (\exists\, n \in \mathbf{Z}^+)(\exists\, \sigma, \tau \in \Omega)(\sigma < \tau \quad \text{and} \quad (\forall \beta \in \Omega)[(\beta < \sigma \quad \text{or} \quad \tau < \beta) \rightarrow \text{Val}\,(\beta, \beta g) \geqq \gamma_n])\}.$$

Clearly T is an $A(\Omega)$-invariant subset of $A(\Omega)^+$. Let $T_{\gamma_2} = \{f \in \mathbf{Z}^+ : f \neq e\}$. The proof in Lemma 2 that $\inf T' = e$ shows that $\inf T = e$ (replace γ by γ_2). Finally, let $g \in T$. Let n, σ, τ show this. Define $h \in L(\mathscr{C}^{\gamma_{n+1}})$ so that $h_{\beta,\gamma} = 0$ if

$$\sigma\mathscr{C}^{\gamma_{n+1}} \leqq \beta\mathscr{C}^{\gamma_{n+1}} \leqq \tau\mathscr{C}^{\gamma_{n+1}} \quad \text{and} \quad \gamma \leqq \gamma_{n+1},$$

and $h_{\beta,\gamma_{n+1}} = +1$ if

$$\beta\mathscr{C}^{\gamma_{n+1}} < \sigma\mathscr{C}^{\gamma_{n+1}} \quad \text{or} \quad \beta\mathscr{C}^{\gamma_{n+1}} > \tau\mathscr{C}^{\gamma_{n+1}}.$$

Then if $\beta < \sigma$ or $\tau < \beta$, Val $(\beta, \beta h) = \gamma_{n+1}$; so $h \in T$. Since $h_{\beta,\gamma} = 0$ if $\gamma \leqq \gamma_{n+1}$ and $\sigma\mathscr{C}^{\gamma_{n+1}} \leqq \beta\mathscr{C}^{\gamma_{n+1}} \leqq \tau\mathscr{C}^{\gamma_{n+1}}$ and Val $(\beta, \beta g) \geqq \gamma_n$ if $\beta < \sigma$ or $\tau < \beta$, $gh^{-1} > e$ and Val$(\beta, \beta gh^{-1}) \geqq \gamma_n$ if $\beta < \sigma$ or $\tau < \beta$. Thus $gh^{-1} \in T$ and as $g = gh^{-1} \cdot h$, $T \cdot T = T$. Consequently, T is a compatible tight Riesz order on $A(\Omega)$.

LEMMA 4. *If* Λ *is ordermorphic to* $\mathbf{Z}$ *or an Ohkuma set, then* $A(\mathbf{Z} \overleftarrow{\times} \Lambda)$ *endures a compatible tight Riesz order.*

Proof. $A(\mathbf{Z} \overleftarrow{\times} \Lambda) =$

$$\{(\{g_\lambda : \lambda \in \Lambda\}, \bar{g}) : \bar{g} \in A(\Lambda) \quad \text{and} \quad (\forall \lambda \in \Lambda)(g_\lambda \in \mathbf{Z})\}.$$

Let $T =$

$$\{(\{g_\lambda : \lambda \in \Lambda\}, \bar{g}) : \bar{g} > 0 \quad \text{or} \quad (\bar{g} = 0, \text{all } g_\lambda \geqq 0 \quad \text{and} \quad (\forall n \in \mathbf{Z}^+)(\exists\, \lambda_n \in \Lambda)(g_\lambda \geqq n \text{ for all } \lambda \geqq \lambda_n))\}.$$

Clearly T is an $A(\mathbf{Z} \overleftarrow{\times} \Lambda)$-invariant filter on $A(\mathbf{Z} \overleftarrow{\times} \Lambda)^+$ and $\inf T = e$. Let $g \in T$. If $\bar{g} > 0$, let $f = (\{f_\lambda : \lambda \in \Lambda\}, 0)$ where

$$f_\lambda = \begin{cases} n & \text{if} \quad \lambda \in [n, n+1) \quad \text{and} \quad n > 0 \\ 0 & \text{if} \quad \lambda < 0. \end{cases}$$

$f \in T$ and $\overline{gf^{-1}} = \bar{g} > 0$. Hence $gf^{-1} \in T$ and $g = gf^{-1} \cdot f$. If $\bar{g} = 0$, let $\{\lambda_n : n \in \mathbf{Z}^+\}$ show that $g \in T$. Let $f = (\{f_\lambda : \lambda \in \Lambda\}, 0)$ where

$$f_\lambda = \begin{cases} g_\lambda/2 & \text{if} \quad g_\lambda \text{ is even} \\ (g_\lambda + 1)/2 & \text{if} \quad g_\lambda \text{ is odd.} \end{cases}$$

Then $f \in T$ and $gf^{-1} \in T$ since f_λ, $(gf^{-1})_\lambda \geqq n$ if $\lambda \geqq \lambda_{2n}$. So $T \cdot T = T$. Consequently, T is a compatible tight Riesz order on $A(\Omega)$.

The technique employed in the above proof is due to N. R. Reilly [**4**, p. 159].

LEMMA 5. *If* $|\Gamma| \geqq 2$ *and* $A(\Omega)$ *has a minimal* 0-*primitive component that is isomorphic to* $\mathbf{Z}$, *then* $A(\Omega)$ *endures a compatible tight Riesz order.*

Proof. By Lemmas 1 and 3, we may assume that no o-primitive component of $A(\Omega)$ is o-2 transitive and that Γ is well-ordered. Let γ_0 be the least element of Γ and γ_1 its successor. Then $\Lambda = \Omega(\gamma_1)$ is an Ohkuma set or $\mathbf{Z}$. By Lemma 4, $A(\mathbf{Z} \overleftarrow{\times} \Lambda)$ endures a compatible tight Riesz order, say T'. Let

$$\bar{T} = \{g \in L(\mathscr{C}^{\gamma_1})^+ : (\forall \alpha \in \Omega)(g_{\alpha,\gamma_1} \in T')\} \quad \text{and}$$
$$T = \{g \in A(\Omega) : g \geqq f \text{ for some } f \in T'\}.$$

Then T is a compatible tight Riesz order on $A(\Omega)$.

By Lemmas 1–5, the theorem follows.

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