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Zero-divisor Graphs of Ore Extensions Over Reversible Rings

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Abstract. Let R be an associative ring with identity. First we prove some results about zero-divisor graphs of reversible rings. Then we study the zero-divisors of the skew power series ring $R[[x;\alpha]]$, whenever R is reversible and α-compatible. Moreover, we compare the diameter and girth of the zero-divisor graphs of $\Gamma(R)$, $\Gamma(R[x;\alpha,\delta])$, and $\Gamma(R[[x;\alpha]])$, when R is reversible and (α,δ)-compatible.

1 Introduction

The zero-divisor graph of a commutative ring R with identity, denoted by $\Gamma(R)$, is the graph associated with R such that its vertex set consists of all its non-zero zero-divisors and that two distinct vertices are joined by an edge if and only if the product of these two vertices is zero. This concept of zero-divisor graphs was initiated by Beck [9] when he studied the coloring problem of a commutative ring. Later, Anderson and Livingston [4] introduced and studied the zero-divisor graph whose vertices are the non-zero zero-divisors of a ring. Redmond [26] studied the zero-divisor graph of a non-commutative ring. Several papers are devoted to studying the relationship between the zero-divisor graph and algebraic properties of rings; see [1, 2, 4–6, 9, 23, 26, 28].

Let R be an arbitrary associative ring with identity. The *zero-divisors* of R, denoted by $Z(R)$, is the set of elements $a \in R$ such that there exists a non-zero element $b \in R$ with $ab = 0$ or $ba = 0$. The zero-divisor graph of R, denoted by $\Gamma(R)$, is the graph with vertices $Z^*(R) = Z(R) - \{0\}$, and for distinct $x, y \in Z^*(R)$, the vertices x and y are adjacent if and only if $xy = 0$ or $yx = 0$.

Axtell, Coykendall, and Stickles [8] examined the preservation of diameter and girth of zero-divisor graphs of commutative rings under extensions to polynomial and power series rings. Lucase [23] continued the study of the diameter of zero-divisor graphs of polynomial and power series rings over commutative rings. Moreover, Anderson and Mulay [5] studied the girth and diameter of commutative rings and investigated the girth and diameter of zero-divisor graphs of polynomial and power series rings over commutative rings. Afkhami, Khashayarmanesh, and Khorsandi [1] compared the girth and diameter of zero-divisor graphs of $R[x;\alpha,\delta]$ and R, when R is a commutative (α,δ)-compatible ring and $R[x;\alpha,\delta]$ is a reversible ring.

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According to Cohn [11] a ring R is called *reversible* if $ab = 0$ implies that $ba = 0$ for $a, b \in R$. Anderson and Camillo [3], observing the rings whose zero products commute, used the term ZC_2 for what is called reversible, while Krempa and Niewieczerzal [20] took the term C_0 for it. Clearly, *reduced* rings (*i.e.,* rings with no non-zero nilpotent elements) and commutative rings are reversible. Kim and Lee [18] studied extensions of reversible rings and showed that polynomial rings over reversible rings need not be reversible. In view of [26, Theorem 2.3] over a reversible ring R, the graph $\Gamma(R)$ is connected with $\operatorname{diam}(\Gamma(R)) \leq 3$, where $\operatorname{diam}(\Gamma(R))$ is the diameter of $\Gamma(R)$.

Another extension of a ring R is the Ore extension. Assume that $\alpha\colon R \to R$ is a ring endomorphism and $\delta\colon R \to R$ is an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$. The Ore extension $R[x; \alpha, \delta]$ of R is the ring obtained by giving the polynomial ring (with indeterminate x) over R with the multiplication $xa := \alpha(a)x + \delta(a)$ for all $a \in R$. In the special case where $\alpha = I_R$ or $\delta = 0$, we denote $R[x; \alpha, \delta]$ by $R[x; \delta]$ and $R[x; \alpha]$, respectively. Also we denote the skew power series ring by $R[[x; \alpha]]$, where $\alpha\colon R \to R$ is an endomorphism. The skew power series ring $R[[x; \alpha]]$ is the ring consisting of all power series of the form $\sum_{i=0}^{\infty} a_i x^i$ ($a_i \in R$), which are multiplied using the distributive law and the Ore commutation rule $xa = \alpha(a)x$, for all $a \in R$.

For two distinct vertices a and b in the graph Γ, the distance between a and b, denoted by $d(a, b)$, is the length of shortest path connecting a and b if such a path exists; otherwise, we put $d(a, b) := \infty$. Recall that the *diameter* of a graph Γ is defined as follows:

$$\operatorname{diam}(\Gamma) := \sup\big\{ d(a, b) \mid a \text{ and } b \text{ are distinct vertices of } \Gamma \big\}.$$

The *girth* of a graph Γ, denoted by $g(\Gamma)$, is the length of the shortest cycle in Γ, provided Γ contains a cycle; otherwise, $g(\Gamma) = \infty$. We will use the notation $g(\Gamma(R))$ to denote the girth of the graph of $Z^*(R)$. A graph is said to be *connected* if there exists a path between any two distinct vertices, and a graph is *complete* if it is connected with diameter one.

For an element $a \in R$, let $\ell_R(a) = \{b \in R | ba = 0\}$ and $r_R(a) = \{b \in R | ab = 0\}$. Note that if R is a reversible ring and $a \in R$, then $\ell_R(a) = r_R(a)$ is an ideal of R, and we denote it by $\operatorname{ann}(a)$. We write $Z_\ell(R)$ and $Z_r(R)$ for the set of all left zero-divisors of R and the set of all right zero-divisors of R, respectively. Clearly, $Z(R) = Z_\ell(R) \cup Z_r(R)$.

2 Properties of $\Gamma(R)$

A ring R is called *abelian* if each idempotent element of R is central. Clearly, commutative rings and reduced rings are reversible. Also, reversible rings are abelian by [22, Proposition 1.3] and [27, Lemma 2.7]. But these implications are irreversible as follows: (i) There is a non-commutative non-reduced reversible ring by [3, Example II.5]. (ii) There is a non-reversible abelian ring by [18, Examples 1.5 and 1.10(3)].

Since reversible rings are abelian, one can prove the following result using a method similar to that used in the proof [4, Theorem 2.5].

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Remark 2.1 Let R be a reversible ring. Then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex if and only if either $R \cong \mathbb{Z}_2 \times D$ where D is a domain or $Z(R)$ is an annihilator ideal.

By using Remark 2.1 and a method similar to that used in the proof of [4, Theorem 2.8], one can prove the following result.

Remark 2.2 Let R be a reversible ring. Then $\Gamma(R)$ is complete if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$.

Recall that an ideal $\mathcal{P}$ of R is *completely prime* if $ab \in \mathcal{P}$ implies $a \in \mathcal{P}$ or $b \in \mathcal{P}$ for $a, b \in R$.

Proposition 2.3 *Let R be a reversible ring and $\mathfrak{A} = \{\operatorname{ann}(a) | 0 \neq a \in R\}$. If $\mathcal{P}$ is a maximal element of $\mathfrak{A}$, then $\mathcal{P}$ is a completely prime ideal of R.*

Proof Let $xy \in \mathcal{P} = \operatorname{ann}(a)$ and $x \notin \mathcal{P}$. Then $xa \neq 0$ and hence $\operatorname{ann}(ax) \in \mathfrak{A}$. Since $\mathcal{P} \subseteq \operatorname{ann}(xa)$ and $\mathcal{P}$ is a maximal element of $\mathfrak{A}$, so $\operatorname{ann}(a) = \mathcal{P} = \operatorname{ann}(ax)$. Since $axy = 0$, we have $ay = 0$, which implies that $y \in \mathcal{P}$. Therefore, $\mathcal{P}$ is a completely prime ideal of R. ∎

Proposition 2.4 *Let R be a reversible ring. Then $\Gamma(R)$ is connected and we have $\operatorname{diam}(\Gamma(R)) \leq 3$. Moreover, if $\Gamma(R)$ contains a cycle, then $g(\Gamma(R)) \leq 4$.*

Proof Using a similar method as in the proof of [4, Theorem 2.3], one can show that $\operatorname{diam}(\Gamma(R)) \leq 3$. ∎

Using a similar method as in the proof of [4, Theorem 2.2] one can prove the following theorem.

Theorem 2.5 *Let R be a reversible ring. Then $\Gamma(R)$ is finite if and only if either R is finite or a domain.*

3 Some Properties of Zero-divisors of a Reversible Ring

Lemma 3.1 *Let R be a reversible ring. Then $Z(R)$ is a union of prime ideals.*

Proof Let $S = R - Z(R)$. Then S is an m-system. Let $0 \neq a \in Z(R)$. Then $ab = 0$ for some $0 \neq b \in Z(R)$. Let $I = \operatorname{ann}(b)$. Then $a \in I$ and I is an ideal of R, since R is reversible. Let $\mathfrak{A} = \{J \trianglelefteq R | I \subseteq J, J \cap S = \phi\}$. By Zorn's lemma, $\mathfrak{A}$ has a maximal element, say $\mathcal{P}$. Then $\mathcal{P}$ is a prime ideal of R by [21, Proposition 10.4]. Hence, $Z(R)$ is a union of prime ideals. ∎

Hence, the collection of zero-divisors of a reversible ring R is the set-theoretic union of prime ideals. We write $Z(R) = \bigcup_{i \in \Lambda} \mathcal{P}_i$ with each $\mathcal{P}_i$ prime. We will also assume that these primes are maximal with respect to being contained in $Z(R)$.

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For a reversible ring R, $r_R(a)$ is an ideal of R for each $a \in R$. Hence, by a similar method to the one used in the proof of [17, Theorem 8], one can prove the following result.

Remark 3.2 Let R be a reversible and right or left Noetherian ring. Then $Z(R) = \bigcup_{i \in \Lambda} \mathcal{P}_i$, where Λ is a finite set and each $\mathcal{P}_i$ is the annihilator of a non-zero element of $Z(R)$.

Kaplansky [17, Theorem 81] proved that if R is a commutative ring and $J_1, \dots, J_n$ a finite number of ideals in R and S a subring of R that is contained in the set-theoretic union $J_1 \cup \cdots \cup J_n$ and at least $n-2$ of the J's are prime, then S is contained in some J_k. Here we have the following theorem.

Theorem 3.3 *Let R be a reversible ring and $Z(R) = \bigcup_{i \in \Lambda} \mathcal{P}_i$. If Λ is a finite set and I an ideal of R that is contained in $Z(R)$, then $I \subseteq \mathcal{P}_k$, for some k.*

Proof Suppose that $Z(R) = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_n$ and I is an ideal of R contained in $Z(R)$. We use induction on n to show that $I \subseteq \mathcal{P}_i$, for some $1 \le i \le n$. If $n = 2$, then clearly $I \subseteq \mathcal{P}_1$ or $I \subseteq \mathcal{P}_2$. Let $n \ge 3$ and for every k, $I \nsubseteq \mathcal{P}_k$. Since $\mathcal{P}_k$ is a maximal prime ideal contained in $Z(R)$, hence $\mathcal{P}_k + I$ contains a regular element s_k for all k. Thus, $s_k = x_k + a_k$ for some $x_k \in \mathcal{P}_k$ and $a_k \in I$. Then

$$s_1 s_2 \cdots s_n = (x_1 + a_1)(x_2 + a_2) \cdots (x_n + a_n) = x_1 x_2 \cdots x_n + \alpha,$$

for some $\alpha \in I$. Since $I \subseteq Z(R) = \bigcup_{i=1}^{n} \mathcal{P}_i$, there exists $1 \le j \le n$ such that $\alpha \in \mathcal{P}_j$. But since $x_1 x_2 \cdots x_n \in \bigcap_{i=1}^{n} \mathcal{P}_i$, this means that $s_1 s_2 \cdots s_n = x_1 x_2 \cdots x_n + \alpha \in \mathcal{P}_j$, which is a contradiction. Therefore, $I \subseteq \mathcal{P}_k$, for some $1 \le k \le n$. ■

Note that Remark 3.2 shows that any left or right Noetherian ring satisfies the hypothesis of Theorem 3.3.

Corollary 3.4 *Let R be a reversible and left or right Noetherian ring. Let $\mathcal{P}$ be a prime ideal of R maximal with respect to being contained in $Z(R)$. Then $\mathcal{P}$ is completely prime and $\mathcal{P} = \operatorname{ann}(a)$, for some $a \in R$.*

Proof This follows from Remark 3.2 and Theorem 3.3. ■

By a slight modification of the proof of [8, Corollary 3.5], in conjunction with Theorem 3.3, we have the following result.

Corollary 3.5 *Let R be a reversible ring with $\operatorname{diam}(\Gamma(R)) \le 2$ and $Z(R) = \bigcup_{i \in \Lambda} \mathcal{P}_i$. If Λ is a finite set, then $|\Lambda| \le 2$.*

Proposition 3.6 *Let R be a reversible ring with $\operatorname{diam}(\Gamma(R)) = 2$. Let $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$ such that $\mathcal{P}_1$ and $\mathcal{P}_2$ are distinct maximal primes in $Z(R)$. Then*

(i) $\mathcal{P}_1 \cap \mathcal{P}_2 = \{0\}$ *(in particular, for all $x \in \mathcal{P}_1$ and $y \in \mathcal{P}_2$, $xy = 0$);*
(ii) *$\mathcal{P}_1$ and $\mathcal{P}_2$ are completely prime ideals of R.*

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Proof (i) This can be proved using a method similar to that used to prove [8, Proposition 3.6].

(ii) Since $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$, hence $\mathcal{P}_1 = \operatorname{ann}(x)$ and $\mathcal{P}_2 = \operatorname{ann}(y)$, for each $0 \neq x \in \mathcal{P}_2$ and $0 \neq y \in \mathcal{P}_1$. Let $ab \in \mathcal{P}_1$ and $a \notin \mathcal{P}_1$. Then $xa \neq 0$ for some $0 \neq x \in \mathcal{P}_2$. Hence $b \in \operatorname{ann}(xa) = \operatorname{ann}(x) = \mathcal{P}_1$. ∎

4 Diameter and Girth of $\Gamma(R)$, $\Gamma(R[[x;\alpha]])$ and $\Gamma(R[x;\alpha,\delta])$

According to Krempa [19], an endomorphism α of a ring R is said to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is said to be α-*rigid* if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced by Hong, Kim and Kwak [16]. Properties of α-rigid rings have been studied in Krempa [19], Hirano [15], and Hong, Kim, and Kwak [16].

Assume that $\alpha: R \to R$ is a ring endomorphism and $\delta: R \to R$ is an α-derivation of R. Following [14], we say that R is α-*compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. Moreover, R is said to be δ-*compatible* if for each $a, b \in R$, $ab = 0$ implies that $a\delta(b) = 0$. If R is both α-compatible and δ-compatible, we say that R is (α, δ)-compatible. In this case, clearly the endomorphism α is injective. In [14, Lemma 2.2], the authors proved that R is α-rigid if and only if R is α-compatible and reduced.

Lemma 4.1 ([14, Lemmas 2.1 and 2.3]) *Let R be an (α, δ)-compatible ring. Then we have the following:*

(i) *If $ab = 0$, then $a\alpha^n(b) = \alpha^n(a)b = 0$ for any positive integer n.*
(ii) *If $\alpha^k(a)b = 0$ for some positive integer k, then $ab = 0$.*
(iii) *If $ab = 0$, then $\alpha^n(a)\delta^m(b) = 0 = \delta^m(a)\alpha^n(b)$ for any positive integers m, n.*
(iv) *If $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha,\delta]$ and $r \in R$, then $f(x)r = 0$ if and only if $a_ir = 0$ for each i.*

Let R be an α-compatible ring and $f(x) = \sum_{i=0}^{\infty} a_ix^i \in R[[x;\alpha]]$ and $r \in R$. Then by using Lemma 4.1 one can show that $f(x)r = 0$ if and only if $a_ir = 0$ for each i.

Note that polynomial rings over reversible rings need not be reversible in general by [18, Example 2.1]. Hence, power series rings over reversible rings need not be reversible in general.

Proposition 4.2 *Let R be a reversible and α-compatible ring. If R is Noetherian with* $\operatorname{diam}(\Gamma(R)) = 2$ *and α is surjective, then* $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 2$.

Proof By Corollary 3.5, either $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$ is the union of precisely two maximal prime ideals of $Z(R)$, or $Z(R) = \mathcal{P}$ is a prime ideal.

Assume that $Z(R) = \mathcal{P}$ is a prime ideal. Since R is reversible and right Noetherian, $\mathcal{P} = \operatorname{ann}(a)$ for some $a \in R$, by Corollary 3.4. By Lemma 4.1, $\alpha(\mathcal{P}) \subseteq \mathcal{P}$, which implies that $\mathcal{P}[[x;\alpha]]$ is an ideal of $R[[x;\alpha]]$. We show that $Z(R[[x;\alpha]]) = \mathcal{P}[[x;\alpha]]$.

Since $R[[x;\alpha]]$ is a Noetherian ring,

$$Z\big(R[[x;\alpha]]\big)=\Big[\bigcup_{\lambda\in\Lambda_1} r_{R[[x;\alpha]]}(f_\lambda(x))\Big]\cup\Big[\bigcup_{\lambda\in\Lambda_2}\ell_{R[[x;\alpha]]}(g_\lambda(x))\Big],$$

where for each $\lambda\in\Lambda_1$, $r_{R[[x;\alpha]]}(f_\lambda(x))$ is a maximal right ideal contained in $Z_r(R[[x;\alpha]])$ and for each $\lambda\in\Lambda_2$, $\ell_{R[[x;\alpha]]}(g_\lambda(x))$ is a maximal left ideal contained in $Z_\ell(R[[x;\alpha]])$. Let $f_\lambda(x)=\sum_{i=0}^\infty a_ix^i$ and $g(x)=\sum_{j=0}^\infty b_jx^j\in r_{R[[x;\alpha]]}(f_\lambda(x))$ such that $b_0\neq 0$. Then

$$a_0b_0=0, \tag{4.1}$$

$$a_0b_1+a_1\alpha(b_0)=0, \tag{4.2}$$

$$a_0b_2+a_1\alpha(b_1)+a_2\alpha^2(b_0)=0, \tag{4.3}$$

$$\vdots$$

Multiplying equation (4.2) by b_0 on the left-hand side and using Lemma 4.1 and the reversibility of R, we have $a_1b_0^2=0=b_0^2a_1$. Multiplying equation (4.3) by b_0^2 on the left-hand side and using Lemma 4.1 and the reversibility of R, we have $a_2b_0^3=0=b_0^3a_2$. By a similar argument one can show that $b_0^na_{n-1}=0=a_{n-1}b_0^n$, for each $n\geq 2$. Since $\operatorname{ann}(b_0)\subseteq\operatorname{ann}(b_0^2)\subseteq\operatorname{ann}(b_0^3)\subseteq\operatorname{ann}(b_0^4)\subseteq\cdots$ and R is right Noetherian, there exists $k>0$ such that $\operatorname{ann}(b_0^k)=\operatorname{ann}(b_0^t)$, for each $t\geq k$. Hence, $b_0^ka_i=0=a_ib_0^k$, for each i, which implies that $b_0^kf_\lambda(x)=0$. We can assume that k is the smallest positive integer such that $b_0^kf_\lambda(x)=0$. If $k>1$, then $b_0^{k-1}f_\lambda(x)\neq 0$. Since $r_{R[[x;\alpha]]}(f_\lambda(x))\subseteq r_{R[[x;\alpha]]}(b_0^{k-1}f_\lambda(x))$, we have

$$r_{R[[x]]}(f_\lambda(x))=r_{R[[x;\alpha]]}(b_0^{k-1}f_\lambda(x)),$$

since $r_{R[[x;\alpha]]}(f_\lambda(x))$ is a maximal right ideal contained in $Z_r(R[[x;\alpha]])$. Since R is reversible and α-compatible and $b_0^kf_\lambda(x)=0$, we have $b_0^{k-1}f_\lambda(x)b_0=0$, and so $f_\lambda(x)b_0=0$, which is a contradiction. Therefore, $k=1$ and so $f_\lambda(x)b_0=0=b_0f_\lambda(x)$. By a similar argument one can show that $f_\lambda(x)b_j=0$ for each $j\geq 0$. Hence, all coefficients of $g(x)$ and $f_\lambda(x)$ are zero-divisors, and so $f_\lambda(x),g(x)\in\mathcal{P}[[x;\alpha]]$, which implies that $Z_r(R[[x;\alpha]])\subseteq\mathcal{P}[[x;\alpha]]$. By a similar argument one can show that $Z_\ell(R[[x;\alpha]])\subseteq\mathcal{P}[[x;\alpha]]$, which implies that $Z(R[[x;\alpha]])\subseteq\mathcal{P}[[x;\alpha]]$. Since $\mathcal{P}=\operatorname{ann}(a)$, we have $\mathcal{P}[[x;\alpha]]\subseteq Z(R[[x;\alpha]])$, which implies that $Z(R[[x;\alpha]])=\mathcal{P}[[x;\alpha]]=r_{R[[x;\alpha]]}(a)$. Therefore, $\operatorname{diam}(\Gamma(R[[x;\alpha]]))=2$.

Now assume that $Z(R)=\mathcal{P}_1\cup\mathcal{P}_2$ is the union of precisely two maximal primes in $Z(R)$. Since by Proposition 3.6, $\mathcal{P}_1$ and $\mathcal{P}_2$ are completely prime and $\mathcal{P}_1\cap\mathcal{P}_2=0$, R is reduced. Thus, R is α-rigid, by [14, Lemma 2.2]. Therefore $R[[x;\alpha]]$ is a reduced ring by [16, Proposition 17]. Now by using [16, Proposition 17] one can show that $Z(R[[x;\alpha]])=\mathcal{P}_1[[x;\alpha]]\cup\mathcal{P}_2[[x;\alpha]]$, which implies that $\operatorname{diam}(\Gamma(R[[x;\alpha]]))=2$. ∎

Corollary 4.3 *Let R be a reversible and Noetherian ring. If $\operatorname{diam}(\Gamma(R))=2$, then $\operatorname{diam}(\Gamma(R[[x]]))=2$.*

Lemma 4.4 *Let R be a reversible and α-compatible ring and let $f=\sum_{i=0}^\infty a_ix^i\in R[[x;\alpha]]$. If for some natural number k, a_k is regular in R while a_i is nilpotent for $0\leq i\leq k-1$, then f is regular in $R[[x;\alpha]]$.*

Proof Assume that $fg = 0$ for some non-zero $g \in R[[x;\alpha]]$. We can assume that $g = \sum_{j=0}^{\infty} b_j x^j$ and $a_i g \neq 0$, for each $0 \le i \le k-1$. Since a_0 is nilpotent and $a_0 g \neq 0$, there exists $t_0 \ge 1$ such that $a_0^{t_0} g \neq 0$ and $a_0^{t_0+1} g = 0$. Hence, $g a_0^{t_0} \neq 0$ and $g a_0^{t_0+1} = 0$, since R is reversible and α-compatible. Let $f_0 = \sum_{i=1}^{\infty} a_i x^i$ and $g_0 = g a_0^{t_0}$. Since $g a_0^{t_0+1} = 0$ and R is reversible and α-compatible, we have $f_0 g_0 = 0$. By continuing this process we can find non-negative integers $t_1, \dots, t_{k-1}$ such that $g a_0^{t_0} a_1^{t_1} \cdots a_{k-1}^{t_{k-1}} \neq 0$ and $a_i (g a_0^{t_0} a_1^{t_1} \cdots a_{k-1}^{t_{k-1}}) = 0 = (g a_0^{t_0} a_1^{t_1} \cdots a_{k-1}^{t_{k-1}}) a_i$, for each $0 \le i \le k-1$. Hence,

$$0 = f g a_0^{t_0} a_1^{t_1} \cdots a_{k-1}^{t_{k-1}} = \Big(\sum_{i=k}^{\infty} a_i x^i \Big) (g a_0^{t_0} a_1^{t_1} \cdots a_{k-1}^{t_{k-1}}).$$

Since a_k is a regular element of R, we have $g a_0^{t_0} a_1^{t_1} \cdots a_{k-1}^{t_{k-1}} = 0$, which is a contradiction. Therefore, f is regular in $R[[x;\alpha]]$. ∎

Theorem 4.5 *Let R be a reversible and α-compatible ring in which each zero-divisor is nilpotent and let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x;\alpha]]$. If some a_i is regular in R, then $f(x)$ is regular in $R[[x;\alpha]]$.*

Proof This follows from Lemma 4.4. ∎

The following corollary is a generalization of [12, Theorem 3], when R is a reversible ring.

Corollary 4.6 *Let R be a reversible ring in which each zero-divisor is nilpotent and let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$. If some a_i is regular in R, then $f(x)$ is regular in $R[[x]]$.*

According to [10], a ring R is called *semi-commutative* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Clearly, reversible rings are semi-commutative, but this implication is irreversible by [18, Examples 1.5 and 1.10(3)]. If R is a semi-commutative ring, then by [13, Lemma 2.5] the set of all nilpotent elements of R is an ideal.

Corollary 4.7 *Let R be a reversible and α-compatible ring in which each zero-divisor is nilpotent. If the set of nilpotent elements of R is nilpotent, then in $R[[x;\alpha]]$ each zero-divisor is nilpotent.*

Proof Let N be the set of nilpotent elements of R. Since N is nilpotent, $N^k = 0$ for some $k \ge 2$. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x;\alpha]]$ be a zero-divisor. By Theorem 4.5, $a_i \in N$ for each $i \ge 0$. Clearly, for each $n \ge 0$, the coefficient of x^n in $(f(x))^k$ is a sum of such elements $a_{i_1} \alpha^{i_1}(a_{i_2}) \cdots \alpha^{i_1+i_2+\cdots+i_{k-1}}(a_{i_k})$, where $i_1 + \cdots + i_k = n$. Hence, by Lemma 4.1, $(f(x))^k = 0$. ∎

Proposition 4.8 *Let R be a reversible and (α,δ)-compatible ring for which* $\operatorname{diam}(\Gamma(R)) = 2$. *If $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$ is the union of precisely two maximal primes in $Z(R)$, then $Z(R[x;\alpha,\delta]) = \mathcal{P}_1[x;\alpha,\delta] \cup \mathcal{P}_2[x;\alpha,\delta]$ and* $\operatorname{diam}(\Gamma(R[x;\alpha,\delta])) = 2$.

Proof Since by Proposition 3.6, $\mathcal{P}_1$ and $\mathcal{P}_2$ are completely prime and $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$, R is reduced. Thus, R is α-rigid, by [14, Lemma 2.2]. Therefore, $R[x;\alpha,\delta]$ is a reduced ring by [16, Proposition 6]. Let $0 \neq b \in \mathcal{P}_1$ and $0 \neq a \in \mathcal{P}_2$. Then $\operatorname{ann}(a) = \mathcal{P}_1$

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and $\operatorname{ann}(b) = \mathcal{P}_2$ by Proposition 3.6. By Lemma 4.1, $\alpha(\mathcal{P}_i) \subseteq \mathcal{P}_i$ and $\delta(\mathcal{P}_i) \subseteq \mathcal{P}_i$, for $i = 1, 2$. Thus, $\mathcal{P}_i[x;\alpha,\delta]$ is an ideal of $R[x;\alpha,\delta]$, for $i = 1,2$. Let $f(x) \in Z(R[x;\alpha,\delta])$. Then $f(x)g(x) = 0$, for some $0 \neq g(x) \in R[x;\alpha,\delta]$. Hence, $f(x)c = 0$, where c is the leading coefficient of $g(x)$ by [16, Proposition 6]. Then $f(x) \in \mathcal{P}_1[x;\alpha,\delta]$ or $f(x) \in \mathcal{P}_2[x;\alpha,\delta]$, which implies that $Z(R[x;\alpha,\delta]) \subseteq \mathcal{P}_1[x;\alpha,\delta] \cup \mathcal{P}_2[x;\alpha,\delta]$. Since $\mathcal{P}_1\mathcal{P}_2 = 0 = \mathcal{P}_2\mathcal{P}_1$, we have $\mathcal{P}_1[x;\alpha,\delta] \cup \mathcal{P}_2[x;\alpha,\delta] \subseteq Z(R[x;\alpha,\delta])$, by Lemma 4.1. Therefore, $Z(R[x;\alpha,\delta]) = \mathcal{P}_1[x;\alpha,\delta] \cup \mathcal{P}_2[x;\alpha,\delta]$, which implies that $\operatorname{diam}(\Gamma(R[x;\alpha,\delta])) = 2$. ■

It is often taught in an elementary algebra course that if R is a commutative ring and $f(x)$ is a zero-divisor in $R[x]$, then there is a non-zero element $r \in R$ with $f(x)r = 0$. This was first proved by McCoy [24, Theorem 2]. Based on this result, Nielsen [25] called a ring R *right McCoy* when the equation $f(x)g(x) = 0$ implies $f(x)c = 0$ for some non-zero $c \in R$, where $f(x), g(x)$ are non-zero polynomials in $R[x]$. Left McCoy rings are defined similarly. If a ring is both left and right McCoy, then it is called a *McCoy ring*. Afkhami et al. [1, Theorem 2.4] proved that if R is a reversible and (α,δ)-compatible ring and $f(x)g(x) = 0$ for some $f(x), g(x) \in R[x;\alpha,\delta]$, then there exist non-zero $a, b \in R$ such that $f(x)a = 0 = bg(x)$.

Proposition 4.9 *Let R be a reversible and (α,δ)-compatible ring. If $Z(R) = \mathcal{P}$ is a prime ideal and R is a right or left Noetherian ring with* $\operatorname{diam}(\Gamma(R)) = 2$, *then* $Z(R[x;\alpha,\delta]) = \mathcal{P}[x;\alpha,\delta]$ *and* $\operatorname{diam}(\Gamma(R[x;\alpha,\delta])) = 2$.

Proof Since R is right Noetherian and $Z(R) = \mathcal{P}$, $\mathcal{P} = \operatorname{ann}(a)$ for some $a \in R$ by Corollary 3.4. By Lemma 4.1, $\alpha(\mathcal{P}) \subseteq \mathcal{P}$ and $\delta(\mathcal{P}) \subseteq \mathcal{P}$, implying that $\mathcal{P}[x;\alpha,\delta]$ is an ideal of $R[x;\alpha,\delta]$ and $\mathcal{P}[x;\alpha,\delta] \subseteq Z(R[x;\alpha,\delta])$. Let $f(x)$ be a zero-divisor of $R[x;\alpha,\delta]$. Since R is reversible and (α,δ)-compatible, there exists $0 \neq b \in R$ such that $f(x)b = 0 = bf(x)$, implying that $f(x) \in \mathcal{P}[x;\alpha,\delta]$. Therefore, $Z(R[x;\alpha,\delta]) = \mathcal{P}[x;\alpha,\delta]$.

Now, let $f(x), g(x)$ be zero-divisors of $R[x;\alpha,\delta]$. If $f(x)g(x) = 0$ or $g(x)f(x) = 0$, we are done. If $f(x)g(x) \neq 0 \neq g(x)f(x)$, then neither $f(x)$ nor $g(x)$ is a, and so a is a mutual annihilator of $f(x)$ and $g(x)$. Therefore, $\operatorname{diam}(\Gamma(R[x;\alpha,\delta])) = 2$. ■

Corollary 4.10 *Let R be a reversible and (α,δ)-compatible ring. If R is a right or left Noetherian ring with* $\operatorname{diam}(\Gamma(R)) = 2$, *then* $\operatorname{diam}(\Gamma(R[x;\alpha,\delta])) = 2$.

Proof This follows from Corollary 3.5 and Propositions 4.8 and 4.9. ■

The following example shows that there is a commutative (α,δ)-compatible ring R such that $R[x;\alpha,\delta]$ is not reversible. Hence, Corollary 4.10 does not follow from [1, Theorems 3.2 and 3.4].

Example 4.11 ([7, Example 11]) Let $R = \mathbb{Z}_2[t]/(t^2)$ with the derivation δ such $\delta(\overline{t}) = 1$, where $\overline{t} = t + (t^2)$ in R and $\mathbb{Z}_2[t]$ is the polynomial ring over the field $\mathbb{Z}_2$ of two elements. Let $\alpha = I_R$. Clearly, R is a commutative (α,δ)-compatible ring. Armendariz et al. [7] showed that $R[x;\delta] \cong M_2(\mathbb{Z}_2)[y]$, where $M_2(\mathbb{Z}_2)[y]$ is the

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polynomial ring over 2×2 matrix ring over $\mathbb{Z}_2$. Since $M_2(\mathbb{Z}_2)$ is not reversible, neither is $R[x;\delta]$.

Now, by using Lemma 4.4 and Remark 2.2 and a method similar to that used in the proof of [8, Proposition 3.12], one can prove the following proposition.

Proposition 4.12 *Let R be a reversible and (α,δ)-compatible ring. If $\Gamma(R)$ is not complete and $(Z(R))^n = 0$, for some integer $n \geq 2$, then*

$$\operatorname{diam}(\Gamma(R[[x;\alpha]])) = \operatorname{diam}(\Gamma(R[x;\alpha,\delta])) = \operatorname{diam}(\Gamma(R)) = 2.$$

Theorem 4.13 *Let R be a reversible and (α,δ)-compatible ring that is not isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$. Then the following are equivalent:*

(i) $\Gamma(R[[x;\alpha]])$ *is complete;*
(ii) $\Gamma(R[x;\alpha,\delta])$ *is complete;*
(iii) $\Gamma(R)$ *is complete.*

Proof Clearly, (i) ⇒ (iii) and (ii) ⇒ (iii). For (iii) ⇒ (i), since $R \ncong \mathbb{Z}_2\times\mathbb{Z}_2$, we have $xy = 0$ for each $x, y \in Z^*(R)$, by Remark 2.2. Therefore, $\Gamma(R)$ complete implies $(Z(R))^2 = 0$. Let $f, g \in Z^*(R[[x;\alpha]])$. By Lemma 4.4, all coefficients of f and g are zero-divisors in R. Since $\Gamma(R)$ is complete and R is α-compatible, we have $fg = 0$, and hence $\Gamma(R[[x;\alpha]])$ is complete.

(iii) ⇒ (ii). Since $R \ncong \mathbb{Z}_2\times\mathbb{Z}_2$, we have $ab = 0$ for each $a, b \in Z^*(R)$ by Remark 2.2. Therefore, $\Gamma(R)$ complete implies $(Z(R))^2 = 0$. Let $f, g \in Z^*(R[x;\alpha,\delta])$. Since R is reversible and (α,δ)-compatible, there exist $0 \neq a, b \in R$ such that $f(x)b = 0$ and $g(x)a = 0$, implying that all coefficients of f and g are zero-divisors in R. Since $\Gamma(R)$ is complete and R is (α,δ)-compatible, we have $fg = 0$, and hence $\Gamma(R[x;\alpha,\delta])$ is complete. ■

Theorem 4.14 *Let $R \ncong \mathbb{Z}_2\times\mathbb{Z}_2$ be a reversible and (α,δ)-compatible ring. If α is surjective and R is a Noetherian ring with non-trivial zero-divisors, then the following are equivalent:*

(i) $\operatorname{diam}(\Gamma(R)) = 2$;
(ii) $\operatorname{diam}(\Gamma(R[x;\alpha,\delta])) = 2$;
(iii) $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 2$;
(iv) *$Z(R)$ is either the union of two primes with intersection $\{0\}$, or $Z(R)$ is prime and $(Z(R))^2 \neq 0$.*

Proof (i)⇒(ii) was proved in Corollary 4.10.

(i)⇒(iii) was proved in Proposition 4.2.

(i)⇒(iv) follows from Corollaries 3.4 and 3.5 and Proposition 3.6.

We will show that (ii)⇒(i), (iii)⇒(i), and (iv)⇒(i). For (ii)⇒(i) and (iii)⇒(i), assume that $\operatorname{diam}(\Gamma(R)) \neq 2$. By Theorem 4.13, if $\operatorname{diam}(\Gamma(R)) = 1$, then $\operatorname{diam}(\Gamma(R[x;\alpha,\delta])) = 1$, since $R \ncong \mathbb{Z}_2\times\mathbb{Z}_2$.

(iv)⇒(i). One can prove this using Proposition 3.6 and a method similar to that used in the proof of [8, Theorem 3.11 ((5)→ (1))]. ■

Lemma 4.15 *Let R be a reversible ring and $n > 0$. If f, g are non-zero elements of $R[x_1, \cdots, x_n]$ and $fg = 0$, then there exist non-zero $a, b \in R$ such that $fa = 0 = bg$.*

Proof That $n = 1$ follows from [25, Theorem 2]. It is enough we prove it for $n = 2$. Suppose that $n = 2$ and $f(x_2), g(x_2) \in Z(R[x_1][x_2])$ such that $f(x_2)g(x_2) = 0$. Write $f(x_2) = f_0 + f_1x_2 + \cdots + f_mx_2^m, g(x_2) = g_0 + g_1x_2 + \cdots + g_nx_2^n$, where $f_i, g_j \in R[x_1]$ for each i, j. Let $k = \deg(f_0) + \cdots + \deg(f_m) + \deg(g_0) + \cdots + \deg(g_n)$, where the degree is as polynomials in x_1 and the degree of the zero polynomial is taken to be 0. Then $f(x_1^k) = f_0 + f_1x_1^k + \cdots + f_mx_1^{km}, g(x_1^k) = g_0 + g_1x_1^k + \cdots + g_nx^{nk} \in R[x_1]$, and the set of coefficients of the f_i's (resp., g_j's) equals the set of coefficients of $f(x_1^k)$ (resp., $g(x_1^k)$). Since $f(x_2)g(x_2) = 0$ and x_1 commutes with elements of R, we have $f(x_1^k)g(x_1^k) = 0$. Hence, there exist non-zero elements $a, b \in R$ such that $f(x_1^k)a = 0 = bg(x_1^k)$, implying that $f(x_2)a = 0 = bg(x_2)$. ∎

Note that since polynomial rings over reversible rings need not be reversible in general by [18, Example 2.1], Lemma 4.15 does not follow from [25, Theorem 2] for $n \geq 2$.

Corollary 4.16 *Let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a reversible and Noetherian ring with non-trivial zero-divisors. The following conditions are equivalent:*

(i) $\operatorname{diam}(\Gamma(R)) = 2$;
(ii) $\operatorname{diam}(\Gamma(R[x])) = 2$;
(iii) $\operatorname{diam}(\Gamma(R[x_1, \ldots, x_n])) = 2$ *for all* $n > 0$;
(iv) $\operatorname{diam}(\Gamma(R[[x]])) = 2$;
(v) *$Z(R)$ is either the union of two primes with intersection $\{0\}$, or $Z(R)$ is prime and $(Z(R))^2 \neq 0$.*

Proof By Theorem 4.14, (i), (ii), (iv), and (v) are equivalent.

(iii)⇒(ii) is trivial.

(ii)⇒(iii). It is enough we prove for $n = 2$. Suppose that $n = 2$ and $f(x_2), g(x_2) \in Z(R[x_1][x_2])$. If $f(x_2)g(x_2) = 0$ or $g(x_2)f(x_2) = 0$, then $d(f, g) = 1$. So suppose that $f(x_2)g(x_2) \neq 0 \neq g(x_2)f(x_2)$. Write $f(x_2) = f_0 + f_1x_2 + \cdots + f_mx_2^m, g(x_2) = g_0 + g_1x_2 + \cdots + g_nx_2^n$, where $f_i, g_j \in R[x_1]$ for each i, j. Let $k = \deg(f_0) + \cdots + \deg(f_m) + \deg(g_0) + \cdots + \deg(g_n)$. Then by the proof of Lemma 4.15, $f(x_1^k), g(x_1^k) \in Z(R[x_1])$ and $f(x_1^k)g(x_1^k) \neq 0 \neq g(x_1^k)f(x_1^k)$. Since $\operatorname{diam}(\Gamma(R[x_1])) = 2$, there exists $h \in R[x_1]$, which annihilates $f(x_1^k)$ and $g(x_1^k)$. Hence, h annihilates $f(x_2)$ and $g(x_2)$, implying that $d(f, g) = 2$. ∎

Proposition 4.17 *Let R be a reversible and (α, δ)-compatible ring. If $f, g \in Z^*(R[x; \alpha, \delta])$ are distinct non-constant polynomials with $fg = 0$, then there exist $a, b \in Z^*(R)$ such that $a - f - g - b - a$ is a cycle in $\Gamma(R[x; \alpha, \delta])$, or $b - f - g - b$ is a cycle in $\Gamma(R[x; \alpha, \delta])$.*

Proof If $f, g \in Z^*(R[x; \alpha, \delta])$, then there exist $a, b \in Z^*(R)$ such that $af = fa = 0 = bg = gb$. Now, using a method similar to that used in the proof of [8, Proposition 4.1] completes the proof. ∎

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Corollary 4.18 *Let R be a reversible and (α,δ)-compatible ring and let $f \in Z^*(R[x;\alpha,\delta])$ a non-constant polynomial. Then there exists a cycle of length 3 or 4 in $\Gamma(R[x;\alpha,\delta])$ with f as one vertex and some $a \in Z^*(R)$ as another.*

The following theorem is a generalization of [8, Theorem 4.3], when R is a reversible ring.

Theorem 4.19 *Let R be a reversible and α-compatible ring. Then*

$$g(\Gamma(R)) \geq g(\Gamma(R[x;\alpha])) \geq g(\Gamma(R[[x;\alpha]])).$$

In addition, if R is a reduced ring and $\Gamma(R)$ contains a cycle, then

$$g(\Gamma(R)) = g(\Gamma(R[x;\alpha])) = g(\Gamma(R[[x;\alpha]])).$$

Proof Using Corollary 4.18 and a method similar to that used in the proof of [8, Theorem 4.3] completes the proof. ∎

Corollary 4.20 *Let R be an α-rigid ring and let $g(\Gamma(R[x;\alpha,\delta])) = 3$. Then $g(\Gamma(R)) = 3$.*

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