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RESEARCH ARTICLE

Picky elements, subnormalisers, and character correspondences

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Abstract
We gather evidence on a new local-global conjecture of Moretó and Rizo on values of irreducible characters of finite groups. For this we study subnormalisers and picky elements in finite groups of Lie type and determine them in many cases, for unipotent elements as well as for semisimple elements of prime power order. We also discuss subnormalisers of unipotent and semisimple elements in connected as well as in disconnected reductive linear algebraic groups.

1. Introduction

In this paper we gather evidence for a new far-reaching conjecture of Alex Moretó and Noelia Rizo on character values of finite groups. Let G be a finite group. For an element $x \in G$ let $\mathrm{Irr}^x(G)$ denote the set of irreducible complex characters of G that do not vanish at x. Define the *subnormaliser* of a subgroup H of G to be

$$S_G(H) := \{g \in G \mid H \lhd\lhd \langle g, H\rangle\}$$

and for $x \in G$ let

$$\mathrm{Sub}_G(x) := \big\langle S_G(\langle x\rangle)\big\rangle.$$

The following conjecture was put forward by Moretó and Rizo [30]:

Conjecture 1 (Moretó–Rizo). *Let G be a finite group and p a prime. Then for any p-element $x \in G$ there exists a bijection $f_x : \mathrm{Irr}^x(G) \to \mathrm{Irr}^x(\mathrm{Sub}_G(x))$ such that*

(1) $\chi(1)_p = f_x(\chi)(1)_p$, *and*
(2) $\mathbb{Q}(\chi(x)) = \mathbb{Q}(f_x(\chi)(x))$.

Let us comment. (A much more thorough discussion of the conjecture and various extensions of it is given in [30]). First, observe that $\mathrm{Sub}_G(x)$ contains the normaliser of any Sylow p-subgroup P of G containing x. In fact, as we show, it is generated by these. Thus, by the now proven McKay conjecture there exist bijections between $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ and both $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathrm{Sub}_G(x))$, hence also between the latter two. Since p'-degree characters do not vanish on any p-element, this would form part of the required bijection f_x. Now, in addition, Conjecture 1 predicts a bijection on characters in $\mathrm{Irr}^x(G) \setminus \mathrm{Irr}_{p'}(G)$.

By Navarro's extension of the McKay conjecture, there should even exist a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-equivariant bijection $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathrm{Sub}_G(x))$, where $\mathbb{Q}_p$ denotes the field of p-adic numbers. Thus, and in view of the examples we discuss in this paper, we are led to ask whether in the setting of Conjecture 1 there exists a bijection f_x such that moreover for any $\chi \in \mathrm{Irr}^x(G)$ we have

Here for an algebraic number $\alpha \in \overline{\mathbb{Q}}$ we write $\alpha_p := \left|\mathrm{N}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha)\right|_p^{1/[\mathbb{Q}(\alpha):\mathbb{Q}]} \in \mathbb{R}_{\geq 0}$ for its p-adic valuation.

Of course the conjecture is only meaningful when $\mathrm{Sub}_G(x) < G$. An interesting special case is as follows: A p-element $x \in G$ is called *picky* if it belongs to a unique Sylow p-subgroup of G. Note that G has a picky p-element if and only if it does not have a *redundant Sylow p-subgroup* in the sense of [28]. As an example, assume G has *trivial intersection* (TI) Sylow p-subgroups. Then by definition any nonidentity p-element lies in a unique Sylow p-subgroup of G and thus is picky. This is the case, in particular, when G has cyclic Sylow p-subgroups of order p. Now note that x is picky in G if and only if $\mathrm{Sub}_G(x) = \mathbf{N}_G(P)$, for $P \leq G$ a Sylow p-subgroup of G containing x (see Corollary 2.7).

In this paper we undertake to test Conjecture 1 and Properties (3) and (4) above in simple groups. The paper is organised as follows. In Section 2 we collect some basic results on picky elements and subnormalisers. In Section 3 we classify the picky unipotent elements in groups of Lie type and determine the subnormalisers of unipotent elements in most types. Based on this we prove in Section 4 the validity of Conjecture 1 for unipotent elements of various families of groups of Lie type. In Section 5 we classify picky semisimple p-elements in groups of simply connected Lie type for all $p \neq 2$. In the final Section 6 we discuss subnormalisers of unipotent and semisimple elements in connected and disconnected reductive algebraic groups.

2. Basic observations

Let's make some easy observations about picky p-elements. Throughout G is a finite group and p is a prime number.

Lemma 2.1. *Let $P \leq G$ be a Sylow p-subgroup and $x \in P$ picky in G. Then $\mathbf{N}_G(\langle x\rangle) \leq \mathbf{N}_G(P)$.*

Proof. Let $g \in \mathbf{N}_G(\langle x\rangle)$. Then $\langle x\rangle \leq P^g$, so x lies in the Sylow p-subgroups P and P^g. As x is picky in G, we must have $g \in \mathbf{N}_G(P)$. □

Lemma 2.2. *Let $P \leq G$ be a Sylow p-subgroup and $x \in P$. If both P and $\mathbf{C}_G(x)$ are abelian then x is picky.*

Proof. If $x \in P, P^g$ for some $g \in G$ then $P, P^g \leq \mathbf{C}_G(x)$, which has a unique Sylow p-subgroup, being abelian, so $P = P^g$. □

Lemma 2.3. *Let $H \leq G$ with $[G : H]$ prime to p and $x \in H$ a p-element.*

1. *If x is picky in G then x is picky in H.*
2. *If $H \trianglelefteq G$ then x is picky in G if and only if it is picky in H.*

Proof. Part (a) follows as any Sylow p-subgroup of H is one of G, while in (b), any p-element and any Sylow p-subgroup of G is contained in H. □

Lemma 2.4. *Let $N \trianglelefteq G$ where N is either a p-group or central. Then a p-element $x \in G$ is picky if and only if xN is picky in G/N.*

Proof. If N is a p-group, the Sylow p-subgroups of G/N are of the form P/N for P a Sylow p-subgroup of G, from which the assertion follows. If N is central, then by the previous part we may assume it is a p'-group. Let $x \in P_i$ for $P_i \in \mathrm{Syl}_p(G)$ with $P_1 \neq P_2$. As P_i is the unique Sylow p-subgroup of P_iN we also have $P_1N \neq P_2N$ and so xN lies in two distinct Sylow p-subgroups of G/N. The reverse direction is clear. □

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Thus to study picky elements of a simple group S we can instead consider a covering group of S, or an extension of S by a group of p'-automorphisms.

Next, we collect some elementary properties of subnormalisers.

Lemma 2.5. *Let $H \le G$ and $x \in G$ a p-element with $\langle x\rangle \trianglelefteq\trianglelefteq H$. Then $x \in \mathbf{O}_p(H)$.*

Proof. Let $\langle x\rangle \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_r = H$ be a subnormal series. Clearly $x \in \mathbf{O}_p(N_1)$, and since $\mathbf{O}_p(N_i)$ is characteristic in N_i we have $\mathbf{O}_p(N_i) \le \mathbf{O}_p(N_{i+1})$ for all i, whence the claim. □

The following characterisation turns out to be very useful and could also be taken as definition of subnormaliser for p-elements. It shows that the subnormaliser of a p-element x is very closely related to the p-local structure "around x":

Proposition 2.6. *Let $x \in G$ be a p-element. Then $\mathrm{Sub}_G(x)$ is generated by the normalisers of those Sylow p-subgroups of G that contain x.*

Proof. Clearly $\langle x\rangle$ is subnormal in the normaliser of any Sylow p-subgroup containing x, and hence $\mathrm{Sub}_G(x)$ contains all of these normalisers. For the converse, first assume that $\langle x\rangle \trianglelefteq\trianglelefteq H$ for some subgroup $H \le G$. Then $x \in \mathbf{O}_p(H)$ by Lemma 2.5, so x is contained in all Sylow p-subgroups of H. Let P be a Sylow p-subgroup of H. By the Frattini argument, we have $H = \mathbf{O}^{p'}(H)\mathbf{N}_H(P)$, hence H is generated by the normalisers in H of its Sylow p-subgroups, each of which contains x.

Now let $N \le G$ be any subgroup with a normal Sylow p-subgroup Q containing x. In particular, x is subnormal in N. Let $\langle x\rangle \trianglelefteq Q_1 \trianglelefteq \ldots \trianglelefteq Q \trianglelefteq N$ be a subnormal series in N. If Q is not a Sylow p-subgroup of G there is a p-subgroup $P \le G$ with $Q \trianglelefteq P$ and $P > Q$. Since $Q \trianglelefteq P$ then $\langle x\rangle \trianglelefteq Q_1 \trianglelefteq \ldots \trianglelefteq Q \trianglelefteq H$ is a subnormal series in $H := \langle N, P\rangle$. That is, any N as above is contained in a subgroup with a larger Sylow p-subgroup in which $\langle x\rangle$ is still subnormal. Hence, combining with the previous paragraph, any subgroup of G in which x is subnormal is contained in a subgroup generated by (subgroups of) Sylow p-normalisers of G that contain x. Our claim follows. □

See also Proposition 6.4 for an analogue for algebraic groups. This shows (see also [30, Thm 2.9]):

Corollary 2.7. *A p-element $x \in G$ is picky if and only if $\mathrm{Sub}_G(x) = \mathbf{N}_G(P)$ for a Sylow p-subgroup P of G containing x.*

Recall that a p-subgroup $Q \le G$ is *radical* if $Q = \mathbf{O}_p(\mathbf{N}_G(Q))$. Since Sylow p-subgroups are clearly radical, we also obtain (see [5, Prop. 2.1(a)]):

Corollary 2.8. *Let $x \in G$ be a p-element. Then $\mathrm{Sub}_G(x)$ is generated by the normalisers of the radical p-subgroups of G that contain x.*

The subnormaliser of a p-element controls its fusion in a Sylow subgroup; Moretó and Rizo had obtained a different proof of this fact based upon [5, Prop. 2.1].

Lemma 2.9. *Let P be a Sylow p-subgroup of G and $x \in P$. Then $\mathrm{Sub}_G(x)$ is generated by the elements $g \in G$ with $x^g \in P$.*

Proof. Let $x^g \in P$. Then $x \in {}^gP$ whence ${}^gP \le \mathrm{Sub}_G(x)$. Since all Sylow p-subgroups of $\mathrm{Sub}_G(x)$ are conjugate, there is $h \in \mathrm{Sub}_G(x)$ with ${}^gP = P^h$ and so $hg \in \mathbf{N}_G(P) \le \mathrm{Sub}_G(x)$, giving $g \in \mathrm{Sub}_G(x)$.

Conversely, let $R := \langle g \in G \mid x^g \in P\rangle$. Clearly, $\mathbf{N}_G(P) \le R$. Assume $x \in {}^hP$ for some $h \in G$. Then $x, x^h \in P$ and thus $h \in R$. Thus, R contains the normalisers of all Sylow p-subgroups of G containing x and hence $\mathrm{Sub}_G(x)$ by Proposition 2.6. □

Corollary 2.10. *Let $x \in G$ be a p-element. Then x is G-conjugate to $y \in \mathrm{Sub}_G(x)$ if and only if x, y are already $\mathrm{Sub}_G(x)$-conjugate. In fact, $\mathrm{Sub}_G(x)$ is the smallest subgroup of G containing both $\mathbf{C}_G(x)$ and a Sylow p-subgroup of G with this property.*

Proof. If $x, y \in \mathrm{Sub}_G(x)$ are G-conjugate, then up to conjugation in $\mathrm{Sub}_G(x)$ we may assume they lie in a common Sylow p-subgroup P (of $\mathrm{Sub}_G(x)$). Then the first claim follows by Lemma 2.9. For the

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second, let $R \le G$ be a subgroup containing $\mathbf{C}_G(x)$ and a Sylow p-subgroup P of G (which we may assume contains x) with the stated property. Let $g \in G$ with $x^g \in P$ and thus x, x^g are R-conjugate by assumption. That is, there exists $r \in R$ with $x = x^{gr}$, whence $gr \in \mathbf{C}_G(x) \le R$ and so $g \in R$. Lemma 2.9 then shows $\mathrm{Sub}_G(x) \le R$. □

Remark 2.11. It is tempting to ask whether the p-fusion system of the subnormaliser of a p-element of G is determined by the p-fusion system of G. But this is not the case. I'm indebted to Martin van Beek for the following counterexample: the 3-fusion categories of M_{12} and $\mathrm{PSL}_3(3)$ are known to be isomorphic. Yet, as we will show, $\mathrm{PSL}_3(3)$ has a (regular unipotent) picky 3-element, while the corresponding 3-elements in M_{12} have centraliser isomorphic to $3 \times \mathfrak{A}_4$ and subnormaliser M_{12}.

Proposition 2.12. *Let $P \le G$ be a Sylow p-subgroup and $x \in P$. If P is abelian then* $\mathrm{Sub}_G(x) = \langle \mathbf{C}_G(x), \mathbf{N}_G(P) \rangle$.

Proof. By Burnside's theorem, in this case $\mathbf{N}_G(P)$ controls fusion of elements in P, so the claim follows from Corollary 2.10.

Alternatively, let $H := \langle \mathbf{C}_G(x), \mathbf{N}_G(P) \rangle$. Since x is subnormal in both $\mathbf{C}_G(x)$ and $\mathbf{N}_G(P)$ we have $H \le \mathrm{Sub}_G(x)$. On the other hand, as P is abelian, $\mathbf{C}_G(x)$, and hence H, contains all Sylow p-subgroups of G containing x. As H contains $\mathbf{N}_G(P)$, and all Sylow p-subgroups of H are H-conjugate, it even contains the normalisers of all Sylow p-subgroups of G containing x. Hence $\mathrm{Sub}_x(G) \le H$ by Proposition 2.6, and we are done. □

Observe that Proposition 2.6, Lemma 2.9, Corollary 2.10 and Proposition 2.12 hold more generally with x replaced by any p-subgroup H of G and $\mathbf{C}_G(x)$ by $\mathbf{N}_G(H)$, with identical proofs.

Lemma 2.13. *Let $G = H_1 \times \cdots \times H_r$ and $x = (x_1, \ldots, x_r) \in G$. Then* $\mathrm{Sub}_G(x) = \mathrm{Sub}_{H_1}(x_1) \times \cdots \times \mathrm{Sub}_{H_r}(x_r)$. *In particular, x is picky in G if and only if the projection of x into each component H_i is.*

Proof. Clearly, $g = (g_1, \ldots, g_r) \in S_G(x)$ if and only if $g_i \in S_{H_i}(x_i)$ for all i, showing the first claim. Since the Sylow p-subgroups of G are of the form $\prod_{i=1}^r P_i$, with Sylow p-subgroups $P_i \le H_i$, the second assertion follows from the first using Corollary 2.7. □

The next observation allows one to bound subnormalisers from above:

Lemma 2.14. *Let $x \in G$ be a p-element with $x \in P \in \mathrm{Syl}_p(G)$. If $H \le G$ contains $\mathbf{N}_G(P)$, and $x \in H^g$ for $g \in G$ implies $H^g = H$, then $\mathrm{Sub}_G(x) \le H$. If moreover $\langle x \rangle \lhd\lhd H$, then $\mathrm{Sub}_G(x) = H$.*

Proof. Let $g \in G$ be such that $x \in P^g$. Then $x \in P^g \le H^g$, so $H^g = H$ by assumption, and moreover $\mathbf{N}_G(P^g) = \mathbf{N}_G(P)^g \le H^g = H$. Thus, H contains the normalisers of all Sylow p-subgroups containing x, whence the claim by Proposition 2.6. □

The following result extends Lemma 2.4:

Lemma 2.15. *Let $N \trianglelefteq G$, where N is either a p-subgroup or central, and $x \in G$ a p-element. Then $N \le \mathrm{Sub}_G(x)$ and $\mathrm{Sub}_{G/N}(xN) = \mathrm{Sub}_G(x)/N$.*

Proof. If N is central the claim is obvious. Assume N is a p-group. Since $\langle x, N \rangle$ is a p-group by assumption, we have $\langle x \rangle \lhd\lhd \langle x, N \rangle$ and so $N \le \mathrm{Sub}_G(x)$. Suppose $gN \in S_{G/N}(xN)$, hence $\langle xN \rangle \lhd\lhd \langle gN, xN \rangle$. Then $\langle x, N \rangle \lhd\lhd \langle g, x, N \rangle$ and so $\langle x \rangle \lhd\lhd \langle g, x, N \rangle$, yielding $g \in S_G(x)$. The reverse inclusion is analogous. See also [5, Lemma 2.3]. □

It is tempting to define G has *almost normal Sylow p-subgroups* if $\mathrm{Sub}_G(x) = G$ for all p-elements $x \in G$. This is in some sense at the opposite extreme from TI-Sylow p-subgroups, for which any p-element $x \ne 1$ lies in exactly one Sylow p-subgroup, but the two notions coincide when the Sylow p-subgroup is normal. An example with non-normal Sylow p-subgroup for $p = 2$ is the group `SmallGroup(324,37)`, of the form $3^3.\mathfrak{A}_4$. In fact, similar examples exist for all primes p. Nonsolvable examples of almost normal Sylow p-subgroups are given by the simple groups $\mathrm{PSL}_3(7)$, $\mathrm{PSL}_3(13)$, $\mathrm{PSU}_3(5)$ and $\mathrm{PSU}_3(11)$ at $p = 3$.

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3. Unipotent elements in groups of Lie type

Here we investigate picky elements and subnormalisers for unipotent elements of finite groups of Lie type. More precisely, we assume $\mathbf{G}$ is a connected reductive linear algebraic group over an algebraically closed field of characteristic p and $F : \mathbf{G} \to \mathbf{G}$ is a Steinberg map with (finite) group of fixed points $G := \mathbf{G}^F$ (see, e.g., [27]). Observe that unipotent elements of G are exactly the p-elements of G. The case of p'-elements will be discussed in Section 5.

3.1. Unipotent picky elements

Let $T \le B$ be a maximal torus contained in a Borel subgroup of G. Thus B is the group of F-fixed points of an F-stable Borel subgroup $\mathbf{B} \le \mathbf{G}$ with F-stable maximal torus $\mathbf{T}$ where $T = \mathbf{T}^F$. Then $B, N := \mathbf{N}_G(\mathbf{T})$ form a split BN-pair in G with Weyl group $W := N/T$. Let Φ be the root system of this BN-pair, and $\Phi^+, \Delta \subset \Phi$ be the positive respectively simple roots determined by B. Now $U := \mathbf{O}_p(B)$ is a Sylow p-subgroup of G and B is its normaliser. Any element in U can be written uniquely as a product of root elements from the root subgroups U_α, for $\alpha \in \Phi^+$, contained in U (see [27, §23]).

Proposition 3.1. *Let $\mathbf{G}$ be connected reductive with Steinberg map F. Then a unipotent element $x \in G = \mathbf{G}^F$ is picky if and only if, up to conjugation, x is a product of root elements of G in B in which all simple roots do appear.*

Proof. Assume $x \in U = \mathbf{O}_p(B)$ can be written as a product of root elements not involving elements from U_α where $\alpha \in \Delta$. Let $s \in W$ be the corresponding simple reflection and $\dot{s} \in \mathbf{N}_G(\mathbf{T})$ a preimage. Then $x^{\dot{s}} \in B$, that is, x lies in the Borel subgroups B and $B^{\dot{s}}$ which are distinct, as $B^{\dot{s}}$ does not contain root elements for the root α. Hence x lies in two different Sylow p-subgroups of G and thus is not picky.

For the converse, assume x lies in two Sylow p-subgroups, so (up to conjugation) $x \in B$ and $x \in B^g$ for some $g \in G$. Writing g in the Bruhat decomposition $g = u_1 t \dot{w} u_2$ with $u_i \in U$, $t \in T$ and $w \in W$ we find $x \in (B \cap B^w)^{u_2}$, so up to conjugation $x \in B \cap B^w$ for some $1 \ne w \in W$. Now by [4, Prop. 2.5.9] then $x \in U_{w_0 w}$, with $w_0 \in W$ the longest element, and $U_{w_0 w}$ is the product of the root subgroups U_α for $\alpha \in \Phi^+ \cap w(\Phi^+)$ by [4, Prop. 2.5.16]. Since $w \ne 1$ there is some simple root $\alpha \in \Delta$ made negative by w and so the corresponding root element cannot occur in the unique expression of x as a product of root elements. □

We can now classify picky unipotent elements. Note that by Lemmas 2.3 and 2.4 the precise isogeny type of our connected reductive group $\mathbf{G}$ does not matter since $[G : [G, G]]$ and $|\mathbf{Z}(G)|$ are both prime to p. Thus, by virtue of Lemma 2.13 we are reduced to the case when $\mathbf{G}$ is simple, which we now assume.

Theorem 3.2. *Let $\mathbf{G}$ be simple simply connected with Steinberg map $F : \mathbf{G} \to \mathbf{G}$. A unipotent element $x \in G \setminus \{1\}$ is picky if and only if one of the following holds:*

(1) *x is regular unipotent;*
(2) *$G = \mathrm{SU}_{2n+1}(q)$ with $n \ge 1$ and x has Jordan block sizes $(2n, 1)$;*
(3) *$G = {}^2B_2(2^{2f+1})$ with $f \ge 0$ is a Suzuki group;*
(4) *$G = {}^2G_2(3^{2f+1})$ with $f \ge 0$ is a Ree group; or*
(5) *$G = {}^2F_4(2^{2f+1})$ with $f \ge 0$ is a Ree group and $|\mathbf{C}_G(x)| = 2q^6$, for $q^2 = 2^{2f+1}$.*

Proof. By [4, Prop. 5.1.3], a unipotent element $x \in \mathbf{G}$ is regular if and only if it lies in a unique Borel subgroup of $\mathbf{G}$. Assume $x \in G$ lies in two distinct Sylow p-subgroups, so in two Borel subgroups $B_1 \ne B_2$ of G, say $B_i = \mathbf{B}_i^F$ for F-stable Borel subgroups $\mathbf{B}_i$ of $\mathbf{G}$, $i = 1, 2$. Then x lies in $\mathbf{B}_1 \ne \mathbf{B}_2$ and hence is not regular unipotent. So regular unipotent elements are picky.

For the converse, first assume G is untwisted, so F acts trivially on the Weyl group of $\mathbf{G}$ and hence all root subgroups of $\mathbf{G}$ in $\mathbf{B}$ are F-stable. Let $x \in U$ be not regular unipotent. Then, again by [4, Prop. 5.1.3], it can be written as a product of root elements in which at least one simple root α does not occur, so it cannot be picky by Proposition 3.1.

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Now assume F acts nontrivially on W and hence on Δ. Let $x \in U$ be picky. Then by Proposition 3.1 it is a product involving root elements from all root subgroups for simple roots $\alpha \in \Delta$. Now by [27, Ex. 23.10], for example, the root subgroup U_α consists of elements which are products of root elements of $\mathbf{G}$ lying in an F-orbit of simple roots of $\mathbf{G}$, unless α is the image under the orbit map of a pair of roots of $\mathbf{G}$ forming a diagram of type A_2, B_2 or G_2. Thus, if we are not in one of the latter cases, x is a product of root elements of $\mathbf{G}$ involving all simple roots and hence is regular.

It remains to discuss the cases for which the Dynkin diagram has a subgraph of type A_2, B_2 or G_2 with nontrivial F-action. Thus G is one of $\mathrm{SU}_{2n+1}(q)$, ${}^2B_2(q^2)$, ${}^2G_2(q^2)$ or ${}^2F_4(q^2)$. The rank 1 groups $\mathrm{SU}_3(q)$, ${}^2B_2(q^2)$ and ${}^2G_2(q^2)$ have TI Sylow p-subgroups [1, Prop. 2.3], so any of their nontrivial p-elements is picky. For $G = {}^2F_4(q^2)$ the list of unipotent class representatives in [33] reveals that in addition to the regular classes, there are two further cases, u_{13}, u_{14}, involving root elements for both types of simple roots, with centraliser as given in the statement, while all other elements cannot be picky, by Proposition 3.1. (Note that here the elements from the two simple root subgroups are denoted $\alpha_1(t_1)\alpha_2(t_2)$ and $\alpha_3(t)$ respectively.) Finally, for $G = \mathrm{SU}_{2n+1}(q)$ it remains to identify picky elements x that involve elements from all simple root subgroups of G, but not from all simple root subgroups of $\mathbf{G}$. Thus, expressed as a product of root elements for $\mathbf{G}$, x involves root elements from $\mathbf{U}_\alpha$ for all $\alpha \in \{\alpha_1, \dots, \alpha_{n-1}, \alpha_n + \alpha_{n+1}, \alpha_{n+2}, \dots, \alpha_{2n}\}$, if the simple roots $\alpha_1, \dots, \alpha_{2n}$ of $\mathbf{G}$ are numbered along the Dynkin diagram of $\mathbf{G}$, and hence has Jordan block sizes $(2n, 1)$ in the natural representation. □

Note that the elements occurring in (2) and (5) above are *subregular*, that is, their centraliser dimension in $\mathbf{G}$ is only 2 larger than that of regular elements

3.2. Subnormalisers of unipotent elements

We keep the setting at the beginning of this section, so $G = \mathbf{G}^F$ with $\mathbf{G}$ connected reductive and F a Steinberg map.

Recall that the standard parabolic subgroups of G are exactly the overgroups of the Borel subgroup B, and are in natural bijection with the subsets Γ of Δ [27, Prop. 12.2]; we write P_Γ for the corresponding parabolic subgroup. Let $x \in U$ be unipotent. Then the lattice of standard parabolic subgroups P of G such that $x \in \mathbf{O}_p(P)$ has a unique maximal element with respect to inclusion, which we denote $P(x)$. Indeed, writing x as a product of root elements $u_\alpha \in U_\alpha$, with $\alpha \in \Phi^+$, we have $P(x) = P_\Gamma$ for $\Gamma := \{\alpha \in \Delta \mid u_\alpha = 1\}$. The Chevalley commutator formula then shows that $P(x) = P(x^u)$ for all $u \in U$.

The following is a generalisation of the argument used to prove Proposition 3.1:

Lemma 3.3. *Let $x \in U$ and $C = x^G$ the conjugacy class of x in G. Assume that $P(x) \geq P(y)$ for all $y \in C \cap U$. Then $P = \mathrm{Sub}_G(x)$.*

Proof. Since x is subnormal in $P(x)$ by definition, it suffices to see that any $H \leq G$ such that $\langle x \rangle \lhd\lhd H$ lies in $P(x)$. Let H be such a subgroup. In particular $x \in \mathbf{O}_p(H)$ by Lemma 2.5. By the Borel–Tits theorem [27, Thm 26.5] there is a parabolic subgroup $H' \geq H$ such that $x \in \mathbf{O}_p(H')$, so we may assume H is parabolic. Furthermore, we may replace H by any overgroup H' with $x \in \mathbf{O}_p(H')$. As any such overgroup is again parabolic, we may assume H is parabolic and maximal with respect to $x \in \mathbf{O}_p(H)$.

Let $g \in G$ such that H^g is a standard parabolic subgroup, say $H^g = P_\Gamma$ for $\Gamma \subseteq \Delta$. Then $x^g \in C \cap \mathbf{O}_p(P_\Gamma) \subseteq C \cap U$ and hence $P_\Gamma \leq P(x)$ by assumption. In fact, by our choice of H we have $P_\Gamma = P(x^g)$. Using the Bruhat decomposition we can write $g = u\dot{w}b$ with $u \in U$, $b \in B \leq P_\Gamma$, $w \in W$, so $H^{uw} = P_\Gamma$ and $x^{uw} \in \mathbf{O}_p(P_\Gamma)$. Since also $y := x^u \in \mathbf{O}_p(P_\Gamma)$ we have $y, y^w \in \mathbf{O}_p(P_\Gamma)$. Let $W_\Gamma \leq W$ denote the Weyl group of the standard Levi subgroup of P_Γ. Any $w \notin W_\Gamma$ sends at least one simple root in $\Delta \setminus \Gamma$ into a negative root. On the other hand, since $P_\Gamma = P(x^g)$ all simple roots from $\Delta \setminus \Gamma$ occur in y, whence we can't have ${}^w y \in P_\Gamma$. Thus, $w \in W_\Gamma$, giving $g \in P_\Gamma$ and so $H = {}^g P_\Gamma = P_\Gamma \leq P(x)$. This proves our claim. □

Proposition 3.4. *Let $\mathbf{G}$ be connected reductive with Steinberg map F. Let $x \in G = \mathbf{G}^F$ be unipotent. Then $\mathrm{Sub}_G(x)$ is a parabolic subgroup of G.*

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Proof. By Proposition 2.6, $\mathrm{Sub}_G(x)$ contains a Sylow p-normaliser of G, hence a Borel subgroup. Since all overgroups of B are parabolic subgroups, so is $\mathrm{Sub}_G(x)$. □

Proposition 3.5. *Let $x \in B$ be unipotent. Suppose that x can be written as a product of root elements in which no simple root in $\Delta_1 \subseteq \Delta$ occurs. Then $P_{\Delta_1} \leq \mathrm{Sub}_G(x)$. In particular, if $\Delta_1 = \Delta$ then $\mathrm{Sub}_G(x) = G$.*

Proof. Let $\alpha \in \Delta_1$ be a simple root and $P_\alpha := P_{\{\alpha\}}$ the corresponding standard parabolic subgroup of G. By assumption x lies in the unipotent radical of P_α, so $\langle x \rangle$ is subnormal in P_α. The claim follows since $\langle P_\alpha \mid \alpha \in \Delta_1 \rangle = P_{\Delta_1}$ (see [27, Prop. 12.2]). The final assertion is now obvious. □

Subnormalisers in the simply laced, untwisted case are the easiest to determine:

Proposition 3.6. *Let G be one of $\mathrm{SL}_n(q)$ ($n \geq 2$), $\mathrm{SO}_{2n}^+(q)$ ($n \geq 4$), $E_6(q)$, $E_7(q)$ or $E_8(q)$ and $x \in G$ be unipotent. Then $\mathrm{Sub}_G(x) = G$ if and only if x is not regular.*

Proof. For regular elements this is Theorem 3.2. For nonregular elements, we first consider $G = \mathrm{SL}_n(q)$. For $n = 2$ the only nonregular unipotent element is the identity, for $n = 3$ the nontrivial nonregular unipotent elements have Jordan type $(2, 1)$ and thus are conjugate to a root element for the nonsimple positive root, so $\mathrm{Sub}_G(x) = G$ by Proposition 3.5. Now assume $n \geq 4$. Let $x \in G$ be a nonregular unipotent element. Then x has at least two Jordan blocks, so up to conjugation, x can be written as a product of simple root elements in which at least one simple root α is missing. If this α lies at an end of the Dynkin diagram, then x^{s_α} is a conjugate in B in which the simple root next to the end node is missing. So we may assume $\alpha \neq \alpha_1, \alpha_{n-1}$. Let P_1, P_2 be the end-node standard parabolic subgroups of G, corresponding to $\Delta \setminus \{\alpha_1\}, \Delta \setminus \{\alpha_{n-1}\}$ respectively. Let L_i be the corresponding standard Levi subgroups, so $L_i \cong P_i/U_i$, with $U_i = \mathbf{O}_p(P_i)$ the unipotent radical of P_i. Then for $i \in \{1, 2\}$, the image $\bar{x}$ of x in L_i is a product of root elements not involving the simple root α of L_i, hence not regular, so by induction $\mathrm{Sub}_{L_i}(\bar{x}) = L_i$. (Note that $L_i \cong \mathrm{GL}_{n-1}(q)$, containing $\mathrm{SL}_{n-1}(q)$ as a normal subgroup of p'-index.) As U_i is a p-group, $\mathrm{Sub}_{P_i}(x) = P_i$ for $i = 1, 2$ by Lemma 2.15. Since $\langle P_1, P_2 \rangle = G$ by [27, Prop. 12.2] this achieves the proof.

Now assume G is of one of the other listed types. Let $x \in G$ be nonregular unipotent. By [4, Prop. 5.1.3], up to conjugation, x can be written as a product of root elements in which at least one simple root α is missing. Since the Dynkin diagram of G has three end nodes, there are at least two end node simple roots $\alpha_1, \alpha_2 \in \Delta$ such that $\alpha \in \Delta \setminus \{\alpha_i\}$, $i \in \{1, 2\}$. Let P_i be the corresponding standard parabolic subgroups of G. Since their Levi subgroups are of type D_{n-1}, E_{n-1} or A_3, for which we know the result by induction, respectively by the first part, we can argue exactly as before to see that $\mathrm{Sub}_{P_i}(x) = P_i$ for $i = 1, 2$ and then $\mathrm{Sub}_G(x) \geq \langle P_1, P_2 \rangle = G$. □

3.3. *Nonsimply laced and twisted types*

In order to determine subnormalisers of unipotent elements in nonsimply laced-type groups we first need to deal with the groups of rank 2. Note that for the rank 1 groups $\mathrm{SU}_3(q)$, ${}^2B_2(q^2)$ and ${}^2G_2(q^2)$ the subnormalisers of all nontrivial unipotent elements are Borel subgroups by Theorem 3.2.

From now on, we number the simple roots in Δ, for $\mathbf{G}$ a simple group, as in [27, Tab. 9.1, Tab. 23.2], and we write $P_i := P_{\alpha_i} := P_{\{\alpha_i\}}$ for $\alpha_i \in \Delta$.

Proposition 3.7. *Let G be one of $B_2(q)$, $G_2(q)$ or ${}^3D_4(q)$ with $q = p^f$, or ${}^2F_4(q^2)$ with $q^2 = 2^{2f+1}$, and $x \in G$ unipotent. Then $\mathrm{Sub}_G(x) = G$ unless one of*

(1) *x is picky, where $\mathrm{Sub}_G(x) \sim_G B$;*
(2) *$G = B_2(q)$ with $p \neq 2$ and $|\mathbf{C}_G(x)| = 2q^3(q + 1)$, where $\mathrm{Sub}_G(x) \sim_G P_2$;*
(3) *$G = G_2(q)$ with $p \neq 3$ and $|\mathbf{C}_G(x)| = 3q^4$, where $\mathrm{Sub}_G(x) \sim_G P_1$;*
(4) *$G = {}^3D_4(q)$ and x is in class $D_4(a_1)$ with $|\mathbf{C}_G(x)| = q^6$, where $\mathrm{Sub}_G(x) \sim_G P_2$; or*
(5) *$G = {}^2F_4(q^2)$ and $|\mathbf{C}_G(x)| \in \{3q^{12}, 2q^8, 4q^8\}$, where $\mathrm{Sub}_G(x) \sim_G P_1$.*

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Proof. For all of these groups we know parametrisations of unipotent classes in parabolic subgroups and expressions for class representatives in terms of root elements. The picky elements have subnormaliser conjugate to $\mathbf{N}_G(U) = B$, by Corollary 2.7. We consider the remaining classes.

Let us start with $G = B_2(q)$. Here representatives for the unipotent conjugacy classes are given in [9, 34]. With the criterion in Proposition 3.5 we see that for $p = 2$ there cannot be a class (apart from the regular ones) with $\mathrm{Sub}_G(x) < G$, and for $p > 2$ only the class with representative denoted A_{22} in [34] could possibly have that property. Representatives for the unipotent conjugacy classes of $G_2(q)$ are given in [6, 10, 11], and again by Proposition 3.5 it follows that only the class of $G_2(2^f)$ denoted A_4 in [11] and the class of $G_2(p^f)$, $p \geq 5$, with representative u_4 in [6] could have $\mathrm{Sub}_G(x) < G$. For $G = {}^3D_4(q)$, consulting the tables of class representatives in [12, 16], we see that we only need to consider the classes with representatives $u_\beta(1)u_{2\alpha+\beta}(a)$ and $u_\alpha(1)u_{\alpha+\beta}(1)$. Now note that conjugating $u_\beta(1)u_{2\alpha+\beta}(a)$ with the simple reflection s_α gives an element not involving any simple root element. So we are left with the class $D_4(a_1)$ with representative $u_\alpha(1)u_{\alpha+\beta}(1)$. Finally, for $G = {}^2F_4(q^2)$ by [33, Tab. II] apart from the picky cases identified in Theorem 3.2 only the representatives $u_9, u_{10}, u_{11}, u_{12}$ involve one of the simple root elements.

It remains to determine the subnormaliser for the elements in (2)–(5). By the tables in the cited literature, respectively in [15, 16] for ${}^3D_4(q)$ and [18] for ${}^2F_4(q^2)$, in all four groups we have the following situation: exactly one of the unipotent radicals of the two maximal standard parabolic subgroups contains an element x as considered. Moreover, the centraliser of x in that parabolic subgroup is the same as in G. By counting, in fact each such x lies inside exactly one such maximal parabolic subgroup, say P. Since any parabolic subgroup contains a Borel subgroup, hence a Sylow normaliser, we conclude by Lemma 2.14 that $\mathrm{Sub}_G(x) = P$. □

Observe that the unipotent elements in (2)–(4) above are again subregular.

Lemma 3.8. *Let $G = \mathrm{SO}_{2n+1}(q)$ ($n \geq 3$) with q odd and $x \in G$ be unipotent. Then $\mathrm{Sub}_G(x) \neq G$ if and only if either x is regular, or if x has Jordan form $(2n-1, 1^2)$ and $|\mathbf{C}_G(x)| = 2q^{n+1}(q+1)$, in which case $\mathrm{Sub}_G(x) = P_n$.*

Proof. Let P be the standard maximal parabolic subgroup of G of type A_{n-1}, that is, the stabiliser of a maximal isotropic subspace in the natural representation, and Q the standard maximal parabolic subgroup of type B_{n-1}. We claim that if (up to conjugation) $\mathrm{Sub}_P(x) = P$ then $\mathrm{Sub}_G(x) = G$. Indeed, assume $\mathrm{Sub}_P(x) = P$ and let $\bar{x}$ be the image of x in the standard Levi subgroup $L \cong \mathrm{GL}_n(q)$ of P. By Proposition 3.6, $\bar{x}$ is not regular, so (up to conjugation) its image y in the Levi factor of a standard parabolic subgroup L_1 of L of type $\mathrm{GL}_{n-1}(q)$ is not regular either. Now, L_1 is a standard Levi subgroup of the standard Levi subgroup M of Q of type $\mathrm{SO}_{2n-1}(q)$, and hence by induction $\mathrm{Sub}_M(y) = M$. (The induction base is given by the case $2n-1 = 5$ from Proposition 3.7.) But then $\mathrm{Sub}_Q(x) = Q$, and since P is maximal in G and Q is distinct from P, we have $\mathrm{Sub}_G(x) = G$.

Now from the explicit class representatives given in [7, §4.1.2] it can be checked that unless x is regular or has type $V_\beta(2n-1) \oplus V_\beta(1) \oplus V(1)$ (in the notation of loc. cit.), there is a conjugate $y \in P$ of x whose image in L is not regular, so has $\mathrm{Sub}_P(y) = P$ and then $\mathrm{Sub}_G(y) = G$ by the first part. Finally assume x has type $V_\beta(2n-1) \oplus V_\beta(1) \oplus V(1)$. Then x has Jordan type $(2n-1, 1^2)$. In particular, writing $x \in U$ as a product of root elements, all u_α, $\alpha \in \Delta \setminus \{\alpha_n\}$, must be nonzero. But then the only standard parabolic subgroups of G that contain x in their radical are B and P_n (of type B_1). The claim now follows with Lemma 2.14. □

Lemma 3.9. *Let $G = F_4(q)$ and $x \in G$ be unipotent. Then $\mathrm{Sub}_G(x) = G$ unless x is regular, or q is odd and x is in class $F_4(a_1)$, with $|\mathbf{C}_G(x)| = 2q^6$ and representative x_{24} in [35, Tab. 5 resp. 6], where* $\mathrm{Sub}_G(x_{24}) = P_{\{3,4\}}$.

Proof. Representatives for the unipotent conjugacy classes of G as well as for a Levi subgroup L of G of type B_3 were determined in [32, 35]. First assume $p = 2$. We claim that only the regular unipotent elements in L have subnormaliser strictly smaller than L. Indeed, seven of the class representatives given

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in [32, Prop. 2.1] do not involve any simple root element, and the other three nonregular ones can be conjugated by a simple reflection to elements x which only involve the 3rd simple root of L. Then x has trivial image in the standard Levi subgroup of L of type B_2, and image involving just one simple root in the standard Levi subgroup of type A_2, so $\mathrm{Sub}_L(x) = L$ by Proposition 3.5.

Now we discuss the unipotent class representatives of G from [32, Thm 2.1]. The ones labelled x_0 through x_{19} do not involve any simple root element, and similarly for x_{27}, x_{28}. For elements $x \in \{x_{22}, \dots, x_{26}\}$ only one of the short simple roots occurs, so the images of x in standard Levi subgroups of type A_2 and B_3 are nonregular and we conclude by Proposition 3.5 and the previous paragraph. Similarly for $x = x_{20}, x_{21}$ the image in a Levi of type C_3 is trivial, and in a Levi of type B_3 is nonregular. By Lemma 3.8 this leaves us with the elements x_{29} and x_{30}, of type $F_4(a_1)$. The given representatives show that these elements are regular in the subsystem subgroup of type B_4. Conjugating $x = x_{29}$ or x_{30} with the simple reflection not belonging to the B_3-subsystem, we obtain an element which only involves simple root elements for the 2nd and 3rd simple root. By the first paragraph, the subnormaliser of that element contains the standard parabolic subgroup of type B_3, but by Proposition 3.6 also the one of type A_2, hence it is all of G.

If p is odd, Proposition 3.7 shows that apart from the regular classes, only the unipotent class of L with representative z_8 (in the notation of [35, Tab. 3]) has proper subnormaliser. From [35, Tab. 5 and 6] we conclude that again at most the classes of G of type $F_4(a_1)$ with representatives x_{23}, x_{24} might have a proper subnormaliser. Conjugating $x = x_{23}, x_{24}$ by the simple reflection not in the B_3-subsystem we obtain elements whose image in L equals z_7, z_8 respectively. As observed above, $\mathrm{Sub}_L(z_7) = L$ and so $\mathrm{Sub}_G(x_{23}) = G$ by the usual argument. On the other hand, we have $\mathrm{Sub}_L(z_8)$ is the standard parabolic subgroup corresponding to the 3rd node of the diagram of B_3, which is also the 3rd node of the diagram of F_4. Thus the set of standard parabolic subgroups of G containing x_{24} in their radical has the unique maximal element $P_{\{3,4\}}$, and so $\mathrm{Sub}_G(x_{24}) = P_{\{3,4\}}$ by Lemma 2.14. □

Lemma 3.10. *Let $G = \mathrm{SU}_{2m}(q)$ and $x \in G$ be unipotent. Then $\mathrm{Sub}_G(x) = G$ unless x is regular, or x has Jordan type $(2m-1, 1)$, when $\mathrm{Sub}_G(x) \sim_G P_m$.*

Proof. There is nothing to show for $m = 1$, so assume $m \geq 2$. Let $x \in G$ be unipotent and denote by $\lambda_1 \geq \lambda_2 \geq \cdots$ the lengths of its Jordan blocks. According to the normal forms given in [7, 4.1.3], x has a conjugate in B in which the entry at position $(i, i+1)$ is zero for at least $2m - \lambda_1$ indices $1 \leq i \leq m$. Assume $2m - \lambda_1 > 1$. Then the image of x in the standard Levi subgroup $\mathrm{GL}_m(q^2)$ is not regular and thus $\mathrm{Sub}_G(x)$ contains the corresponding maximal parabolic subgroup by Propositions 3.5 and 3.6. Furthermore, the image of x in the standard Levi subgroup $\mathrm{GU}_{2m-2}(q)$ also has at least two zero entries directly above the main diagonal, unless $\lambda_1 = \lambda_2$. In the latter case we must have $2m = 4$, and then the image of x in $\mathrm{GU}_{2m-2}(q) = \mathrm{GU}_2(q)$ is trivial. So in either case, by induction $\mathrm{Sub}_G(x)$ also contains this end node parabolic subgroup and thus equals G.

This only leaves the cases when $2m - \lambda_1 \leq 1$, that is, x has Jordan type $(2m)$ or $(2m-1, 1)$. The first of these is the class of regular unipotent elements. If $x \in U$ has a Jordan block of length $2m - 1$, then it must involve elements from all simple root subgroups except possibly for the 'middle' one. In particular, the only standard parabolic subgroups whose unipotent radical could contain x are B and P_m. It is easy to see that $\mathbf{O}_p(P_m)$ contains elements with a Jordan block of length $2m - 1$; then Lemma 2.14 shows $\mathrm{Sub}_G(x) = P_m$. □

The subnormalisers of unipotent elements in $\mathrm{Sp}_{2n}(q)$ $(n \geq 3)$, $\mathrm{SU}_{2m+1}(q)$ $(m \geq 3)$, $\mathrm{SO}_{2n}^-(q)$ $(n \geq 4)$ and ${}^2E_6(q)$ seem more involved and we will not discuss them here. For example, in $\mathrm{Sp}_{2n}(q)$ the number of unipotent classes with proper subnormaliser seems to increase with the rank n.

4. On Conjecture 1 for the defining prime

We now use our results on subnormalisers to verify Conjecture 1 for unipotent elements of groups of Lie type of rank at most 2. We keep the notation and setting from the beginning of Section 3. Of course only

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Table 1. *Character values for* $\mathbf{G}_2(2^f)$ ....

	θ_1, θ_1'	θ_2, θ_2'	θ_3	$\theta_4, \theta_9(\pm 1)$	θ_8	$\chi_2(k)$	$\chi_2'(k)$
#	2	2	1	3	1	$\frac{1}{2}(q-3-\epsilon)$	$\frac{1}{2}(q-1+\epsilon)$
A_4	$\frac{1}{6}q(\epsilon q-1)$	$-\frac{1}{2}q(\epsilon q-1)$	$\frac{1}{3}q(\epsilon q-1)$	$\frac{1}{3}q(\epsilon q+2)$	q	q	$-q$

Table 2. ... *and for* P_1.

	$\theta_3(\pm 1)$	$\theta_2(\pm 1)$	θ_4	$\theta_5, \theta_6(\pm 1)$	$\chi_3(k)$
#	2	2	1	3	$q-1$
A_5	$\frac{1}{6}q(\epsilon q-1)$	$-\frac{1}{2}q(\epsilon q-1)$	$\frac{1}{3}q(\epsilon q-1)$	$\frac{1}{3}q(\epsilon q+2)$	q

the elements $x \in G$ with $\mathrm{Sub}_G(x) < G$ are of interest. We write *Conjecture 1*$^+$ to include Properties (3) and (4) from the introduction. Our first result is for groups of arbitrary rank.

Proposition 4.1. *Assume that p is a good prime for* $\mathbf{G}$ *and* $\mathbf{Z}(\mathbf{G})$ *is connected. Then Conjecture 1*$^+$ *holds for regular unipotent elements of G.*

Proof. Let $x \in B \leq G$ be regular unipotent. Under our assumptions on $\mathbf{G}$ and p, by the theorem of Green–Lehrer–Lusztig [4, Cor. 8.3.6] the irreducible characters of G that do not vanish on x are exactly those of degree prime to p, there are $|\mathbf{Z}(G)|q^l$ of these, where l denotes the semisimple rank of $\mathbf{G}$, and they all take value ± 1 on x. This implies also that x must be rational. Now $\mathrm{Sub}_G(x) = B$ as x is picky by Theorem 3.2. By the proven McKay conjecture for groups of Lie type in defining characteristic [29] there is the same number of irreducible characters of B of p'-degree. Moreover, as x is picky, by Lemma 2.1 it must also be rational in B and $|\mathbf{C}_B(x)| = |\mathbf{C}_G(x)|$. This implies that the $|\mathbf{Z}(G)|q^l$ irreducible characters of B not vanishing on x must also take values ± 1. □

This of course leaves open the case of nonconnected centre, as well as that of bad primes; we'll discuss a few examples for the latter in which complete character tables are known.

Proposition 4.2. *Conjecture 1*$^+$ *holds for* $G_2(q)$, $q = p^f$, *at the prime p.*

Proof. Let $x \in G := G_2(q)$ be a p-element, hence unipotent. There is nothing to prove when $\mathrm{Sub}_G(x) = G$, so we are in one of the cases of Proposition 3.7. If x is regular and p is good for G the claim follows by Proposition 4.1. So we need to consider regular elements for the bad primes $p = 2, 3$, and the subregular unipotent class for any $p \neq 3$.

The character tables for $G = G_2(2^f)$ and a Borel subgroup B of G were determined by Enomoto–Yamada [11]. First let $x \in B$ be regular (there are two such classes) so $\mathrm{Sub}_G(x) = B$. According to Tables I and IV of that paper both $\mathrm{Irr}^x(G)$ and $\mathrm{Irr}^x(B)$ consist of q^2 characters of p'-degree and four characters with p-part of the degree equal to $q/2$. The values of the former on x are ± 1, and $\pm q/2$ for the latter in both groups, and their rationality properties agree (for the p'-characters this follows by [31]), so Conjecture 1$^+$ holds for x.

The character table of $G = G_2(3^f)$ and of its Borel subgroup can be found in [10, Tab. I and VII]. The group G contains three classes of regular unipotent elements x, where again $\mathrm{Sub}_G(x) \sim_G B$. Here, both $\mathrm{Irr}^x(G)$ and $\mathrm{Irr}^x(B)$ consist of q^2 characters of p'-degree and six characters with p-part of the degree equal to $q/3$. The values of the p'-characters on x are all ± 1. The other characters take values $\pm 2q/3$ (four times) and $\pm q/3$ (twice) on one of the classes, on the other two the values are $\pm q/3$ (four times) and $q/3 + \zeta_3^i q$ for $i = 1, 2$, where ζ is a primitive third root of unity, both for G and for B. Again, Conjecture 1$^+$ is seen to hold.

Now let $x \in G = G_2(2^f)$ be in the subregular unipotent class from Proposition 3.7(3) where $\mathrm{Sub}_G(x) \sim_{P_1} P_1$, the first maximal standard parabolic subgroup. In Tables 1 and 2 we have extracted

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Table 3. Some character values for $^2G_2(q^2)$ and for B on unipotent classes.

$^2G_2(q^2)$	1	X	T	YT^i	B	1	X	T	YT^i
$\overline{\xi_5}, \xi_7$	$\frac{1}{2}\bar{q}\Phi_1\Phi_6''$	$-\frac{1}{2}(q^2+\bar{q})$	$\bar{q}b$	$\bar{q}\zeta_3^i$	$\psi_3, \overline{\psi_5}$	$\frac{1}{2}\bar{q}\Phi_1$	$\frac{1}{2}\bar{q}\Phi_1$	$\bar{q}b$	$\bar{q}\zeta_3^i$
$\overline{\xi_6}, \xi_8$	$\frac{1}{2}\bar{q}\Phi_1\Phi_6'$	$\frac{1}{2}(q^2-\bar{q})$	$\bar{q}b$	$\bar{q}\zeta_3^i$	$\psi_4, \overline{\psi_6}$	$\frac{1}{2}\bar{q}\Phi_1$	$\frac{1}{2}\bar{q}\Phi_1$	$\bar{q}b$	$\bar{q}\zeta_3^i$
$\overline{\xi_9}, \xi_{10}$	$\bar{q}(q^4-1)$	$-\bar{q}$	$2\bar{q}b$	$-\bar{q}\zeta_3^i$	$\psi_1, \overline{\psi_2}$	$\bar{q}\Phi_1$	$\bar{q}\Phi_1$	$2\bar{q}b$	$-\bar{q}\zeta_3^i$
ξ_4	$q^2\Phi_6$	q^2	.	.	ψ	$q^2\Phi_1$	$-q^2$	.	.

from [11] the values of all irreducible characters of G and P_1 (denoted P_a in [11]) *of even degree* that do not vanish on x. (The class representative is denoted A_4 in G and A_5 in P_1 in [11].) Here $\epsilon \in \{\pm 1\}$ is such that $q \equiv \epsilon \pmod 3$. Visibly, there is a map as required in Conjecture 1 which even preserves values up to sign. For the characters of odd degree, such a map additionally preserving character fields over $\mathbb{Q}_p$ exists by [31], so it satisfies Conjecture 1^+. For $p \geq 5$ the character table of $G_2(p^f)$ was found by Chang and Ree [6], the one for P_2 by Yamada [40], and the same sets of values arise. □

Proposition 4.3. *Conjecture 1^+ holds for $^3D_4(q)$, $q = p^f$, at the prime p.*

Proof. Let $x \in G := {}^3D_4(q)$ be unipotent. By Proposition 3.7 we know $\mathrm{Sub}_G(x) = G$ unless x is regular or subregular. For regular elements the claim follows by Proposition 4.1 for odd q as G can be constructed as the F-fixed points of a simple group $\mathbf{G}$ of adjoint type D_4 whose only bad prime is $p = 2$.

For $p = 2$, the character table of G was computed in [8] and the one of a Borel subgroup in [16, Tab. A.6]. For both groups there are q^4 characters of odd degree, all taking value ± 1 on regular unipotent elements, and four characters of degree divisible by q^3, taking values $\pm q^2/2$.

Now we consider the elements x in the subregular unipotent class $D_4(a_1)$ of G with $\mathrm{Sub}_G(x) = P_2$, the second maximal standard parabolic subgroup. Here, by the tables in [8, 15, 16] both G and P_2 possess q^3 irreducible characters of degree divisible by p (in fact, precisely by q) not vanishing on x, and all of them take value $\pm q$ on x. Conjecture 1^+ now follows, using [31] for the character fields of p'-characters. □

The Suzuki groups $^2B_2(2^{2f+1})$ possess TI Sylow 2-subgroups. Moretó and Rizo checked their conjectures for the 2-elements of this group [30]. A further interesting case is given by the Ree groups $^2G_2(3^{2f+1})$ whose Sylow 3-subgroups are also TI, so all nonidentity 3-elements are picky.

Proposition 4.4. *Conjecture 1^+ holds for $^2G_2(q^2)$, $q^2 = 3^{2f+1}$, at $p = 3$.*

Proof. The character table of $G := {}^2G_2(q^2)$ was found by Ward [39] while that of a Sylow 3-normaliser was computed by van der Waall [38, p. 173]. In Table 3 we reproduce the values of those characters in $\mathrm{Irr}(G)$ respectively $\mathrm{Irr}(B)$ of degree divisible by p not vanishing on some nontrivial unipotent element.

Here $\bar{q} := q/\sqrt{3}$, $\Phi_1 := q^2 - 1$, $\Phi_6' := q^2 - 3\bar{q} + 1$, $\Phi_6'' := q^2 + 3\bar{q} + 1$, $\Phi_6 := \Phi_6'\Phi_6'' = q^4 - q^2 + 1$, $b := (-1 + \sqrt{-3}\bar{q})/2$, $\zeta_3 := (-1 + \sqrt{-3})/2$ and $i \in \{1, 2\}$.

A quick check shows that Conjecture 1 holds for all (picky) classes, and furthermore a bijection preserving character fields over $\mathbb{Q}_p$ exists using [20]. □

The next example is quite interesting as there exists an element x with $\mathrm{Sub}_G(x) < G$ for which $\mathrm{Irr}^x(G)$ contains characters of six different heights and yet all p-parts are preserved under a suitable bijection.

Proposition 4.5. *Conjecture 1^+ holds for $^2F_4(q^2)$, $q^2 = 2^{2f+1}$, at $p = 2$.*

Proof. We need to consider the regular classes, the picky subregular classes from Theorem 3.2(5) as well as the classes with proper subnormaliser in Proposition 3.7(5). The values of unipotent characters of G are given in [22], the values of all other characters at least on unipotent elements can be found in [13]. The character table of a Borel subgroup of G was computed in [17]. For x in one of the four regular unipotent classes or the two subregular picky classes, $\mathrm{Irr}^x(G)$ and $\mathrm{Irr}^x(B)$ both consists of q^4 characters of odd degree, $2q^2$ characters whose degree has 2-part $\sqrt{2}q/2$, and eight characters whose

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Table 4. 2-Parts of character values for ${}^2F_4(q^2)$ ....

	#	1	u_9	u_{10}	$u_{11} = u_{12}^{-1}$	u_{13-14}	u_{15-18}
$\chi_{2,3,23,24}$	$2q^2$	$\bar q$	$\bar q$	$\bar q$	$\bar q$	$\bar q$	$\bar q$
$\chi_{4,27,30,33}$	q^2	q^2	q^2	q^2	q^2	.	.
$\chi_{5,6,8,9}$	4	$\bar q^4$	$\bar q^4$	$\bar q^4$	$\bar q^4$	$\bar q^3$	$\bar q^2$
χ_{11-14}	4	$\bar q^4$	$\bar q^4$	$\bar q^4$	$(\bar q^4 \pm 2i\bar q^3)_2$	$\bar q^3$	$\bar q^2$
$\chi_{7,10}$	2	$2\bar q^4$	$2\bar q^4$	$2\bar q^4$	$2\bar q^4$	.	.
χ_{15-17}	3	q^4	$2q^4$	.	.	.	.

Table 5. ... and for P_1.

	#	1	$c_{1,33}$	$c_{1,34}$	$c_{1,35} = c_{1,36}^{-1}$	$c_{1,37-38}$	$c_{1,39-42}$
$\chi_{7\text{-}8}(k), \chi_{9-10}$	$2q^2$	$\bar q$	$\bar q$	$\bar q$	$\bar q$	$\bar q$	$\bar q$
$\chi_2(k), \chi_{11}$	q^2	q^2	q^2	q^2	q^2	.	.
χ_{21-24}	4	$\bar q^4$	$\bar q^4$	$\bar q^4$	$\bar q^4$	$\bar q^3$	$\bar q^2$
χ_{14-17}	4	$\bar q^4$	$\bar q^4$	$\bar q^4$	$(\bar q^4 \pm 2i\bar q^3)_2$	$\bar q^3$	$\bar q^2$
$\chi_{13,25}$	2	$2\bar q^4$	$2\bar q^4$	$2\bar q^4$	$2\bar q^4$	.	.
χ_{18-20}	3	q^4	$2q^4$	.	.	.	.

degree has 2-part $q^4/4$. These take values ± 1, $q\sqrt{-2}/2$, and $\pm q^2/2$, $\pm\sqrt{-1}q^2/2$ on regular elements, and values ± 1, $q\sqrt{2}/2$ respectively $\pm\sqrt{2}q^3/4$ on the subregular elements, both for G and B. A bijection preserving character fields over $\mathbb{Q}_p$ was given in [20].

The parabolic subgroup P_1 also possesses q^4 characters of odd degree by [18]. For elements x which are either picky or lie in the classes in Proposition 3.7(5), the p-parts of the values of characters in $\mathrm{Irr}^x(G)$ and $\mathrm{Irr}^x(P_1)$ of even degree are given in Tables 4 and 5, ordered by increasing p-part. Here, we have set $\bar q := q/\sqrt{2}$ and $i := \sqrt{-1}$. The 2-part of $\bar q^4 \pm 2i\bar q^3$ occurring in either table is actually $2\bar q^3$ for $q^2 \neq 8$ and $2\sqrt{2}\bar q^3$ for $q^2 = 8$.

The tables show that there is a bijection as in Conjecture 1, and it can be checked it moreover preserves character fields over $\mathbb{Q}_p$. □

Proposition 4.6. *Conjecture 1⁺ holds for* $\mathrm{Sp}_4(q)$, $q = p^f$, *at the prime* p.

Proof. Here, Proposition 4.1 does not apply as for odd p the centre of the algebraic group $\mathbf{G} = \mathrm{Sp}_4$ is disconnected, and the prime $p = 2$ is bad for $\mathbf{G}$. For $p = 2$ the character tables of G and B were calculated by Enomoto [9, Tab. I and IV]. There are q^2 irreducible characters of odd degree, taking value ± 1 on a regular unipotent element x, and four characters of degree divisible by $q/2$ taking values $\pm q/2$, both for G and for B.

The character table of $G = \mathrm{Sp}_4(q)$ for odd q was determined by Srinivasan [37] and for B by Yamada [40, Tab. I-2]. The characters not vanishing on a regular unipotent element $x \in G$ for both G and B are all of p'-degree, and $q(q-1)$ of them take values ± 1 on x, the other $4q$ take values $\pm(1 \pm \sqrt{q^*})/2$, where $q^* := (-1)^{(q-1)/2}q$.

Finally, for p odd and x the subregular element as in Proposition 3.7(2) we have $\mathrm{Sub}_G(x) = P_1$. The character table of P_1 can be found in [40, II-2], where the corresponding class is denoted $A_3(1)$. The characters of degree divisible by p take values $\pm q$ ($q+3$ times) and $\pm 2q$ ($(q-1)/2$ times) on x, for both G and P_1. □

5. Picky semisimple elements in groups of Lie type

We now turn to the investigation of semisimple elements of groups of Lie type, that is to say, elements of (prime power) order different from the defining characteristic. We stay in the setting from the beginning of Section 3, with $\mathbf{G}$ connected reductive and $G = \mathbf{G}^F$ a finite groups of Lie type, and let $\ell \neq p$ be a prime. To investigate the existence of picky ℓ-elements in simple groups S, by Lemma 2.4, we may consider $\mathbf{G}$ of simply connected type such that $G/\mathbf{Z}(G) \cong S$.

5.1. The case of abelian Sylow ℓ-subgroups

We first assume that F is a Frobenius map with respect to an $\mathbb{F}_q$-structure, so G is not a Suzuki or Ree group. Recall then (see, e.g., [14, §3.5]) that for any integer $d \geq 1$, $(\mathbf{G}, F)$ has Sylow d-tori, that is, F-stable tori $\mathbf{S}$ of $\mathbf{G}$ whose order polynomial is the maximal Φ_d-power dividing the order polynomial of $(\mathbf{G}, F)$. Then $W_d := \mathbf{N}_G(\mathbf{S})/\mathbf{C}_G(\mathbf{S})$ is the *relative Weyl group* of the Sylow d-torus. We write $e_\ell(q)$ for the order of q modulo ℓ, respectively modulo 4 if $\ell = 2$.

Lemma 5.1. *In the above setting, if ℓ divides $\Phi_d(q)$ for a unique cyclotomic factor Φ_d occurring in the order polynomial of $(\mathbf{G}, F)$, then Sylow ℓ-subgroups of G are abelian. If $\mathbf{G}$ is simple, the converse holds.*

This is shown in [26, Prop. 2.2]. Note that we do not claim that the simple groups $S = G/\mathbf{Z}(G)$ satisfy this dichotomy; counterexamples occur for $\mathrm{SL}_2(q)$ at $\ell = 2$ and $\mathrm{SL}_3(\pm q)$ at $\ell = 3$. The following is an analogue of Proposition 2.6:

Proposition 5.2. *In the above setting, assume that ℓ divides a unique cyclotomic factor $\Phi_d(q)$ in the order polynomial of $(\mathbf{G}, F)$. Let $x \in G$ be an ℓ-element. Then $\mathrm{Sub}_G(x)$ is generated by the F-fixed points of the normalisers of the Sylow d-tori of $\mathbf{G}$ containing x.*

Proof. Let $x \in G$ be an ℓ-element, and P a Sylow ℓ-subgroup of G containing x. Then P is abelian by Lemma 5.1 and lies in a Sylow d-torus $\mathbf{S}$ of $\mathbf{G}$. Conversely, if $\mathbf{S}$ is a Sylow d-torus containing P, then $\mathbf{S} \leq \mathbf{C}_\mathbf{G}(P)$, hence even $\mathbf{S} \leq \mathbf{H} := \mathbf{C}^\circ_\mathbf{G}(P)$. By inspection of the order formulae [27, Tab. 24.1] our condition on ℓ implies that ℓ does not divide $|W|$. Thus, ℓ is not a torsion prime for $\mathbf{G}$ and then neither for $\mathbf{H}$, so $P \leq \mathbf{Z}^\circ(\mathbf{H}) =: \mathbf{S}'$, a torus. Since ℓ divides a unique $\Phi_d(q)$ this implies that $\mathbf{S} \leq \mathbf{S}'$, that is, $\mathbf{S}$ is the Sylow d-torus of $\mathbf{Z}^\circ(\mathbf{C}^\circ_\mathbf{G}(P))$ and thus uniquely determined by P. Since P is characteristic in $\mathbf{S}^F$ we have $N_G(P) = N_G(\mathbf{S}^F)$, and our claim follows by Proposition 2.6. □

Theorem 5.3. *In the above setting, assume that ℓ divides a unique cyclotomic factor $\Phi_d(q)$ in the order polynomial of $(\mathbf{G}, F)$. Then an ℓ-element $x \in G$ is picky if and only if $\mathbf{C}_G(x) = \mathbf{C}_G(\mathbf{S})$ where $\mathbf{S} \leq \mathbf{G}$ is a Sylow d-torus with $x \in \mathbf{S}^F$.*

Proof. By Lemma 5.1 we are in the situation of Proposition 5.2. Hence $x \in G$ lies in a unique Sylow ℓ-subgroup of G if and only if it lies in a unique Sylow d-torus $\mathbf{S}$ of $\mathbf{G}$, if and only if $\mathbf{S}$ is the unique Sylow d-torus of $\mathbf{C}_\mathbf{G}(x)$, if and only if $\mathbf{S} \leq \mathbf{Z}(\mathbf{C}_\mathbf{G}(x))$, if and only if $\mathbf{C}_\mathbf{G}(x) \leq \mathbf{C}_\mathbf{G}(\mathbf{S})$. As $x \in \mathbf{S}$ we also have $\mathbf{C}_\mathbf{G}(\mathbf{S}) \leq \mathbf{C}_\mathbf{G}(x)$, so $\mathbf{C}_\mathbf{G}(\mathbf{S}) = \mathbf{C}_\mathbf{G}(x)$. This implies of course $\mathbf{C}_G(\mathbf{S}) = \mathbf{C}_G(x)$. Conversely, assume $\mathbf{C}_G(\mathbf{S}) = \mathbf{C}_G(x)$ but $\mathbf{C}_\mathbf{G}(\mathbf{S}) < \mathbf{C}_\mathbf{G}(x)$. Since $\mathbf{C}_\mathbf{G}(\mathbf{S})$ is a Levi subgroup, hence connected, this means that either $\mathbf{C}_\mathbf{G}(x)$ is disconnected, or of strictly larger dimension than $\mathbf{C}_\mathbf{G}(\mathbf{S})$. The first is not possible as ℓ is not a torsion prime. In the second case, as $\mathbf{C}_\mathbf{G}(\mathbf{S})$ contains a maximal torus of $\mathbf{G}$, the maximal unipotent subgroups of $\mathbf{C}_\mathbf{G}(x)$ must have larger dimension than those of $\mathbf{C}_\mathbf{G}(\mathbf{S})$. But then $|\mathbf{C}_G(x)|_p > |\mathbf{C}_G(\mathbf{S})|_p$, a contradiction. In conclusion, $x \in G$ is picky if and only if $\mathbf{C}_G(\mathbf{S}) = \mathbf{C}_G(x)$, for $\mathbf{S}$ the (unique) Sylow d-torus containing x. □

Example 5.4. In the setting of Theorem 5.3, assume $d = 1$. Then $x \in G$ is picky if and only if x is regular. Indeed, in this case the centraliser of a Sylow 1-torus is a maximal torus (namely, a maximally split torus) of $\mathbf{G}$, so the condition becomes: $\mathbf{C}_G(x)$ is a torus, which means x is regular. More generally this characterisation continues to hold whenever d is a regular number for (W, F) (in the sense of Springer).

The precise determination of subnormalisers in the abelian Sylow case seems to require a classification of the overgroups of normalisers of Sylow d-tori for simple algebraic groups, a nontrivial and interesting problem, as the following example shows (see also Corollary 6.9 in the algebraic group case):

Example 5.5. Let $\mathbf{G}$ be of type F_4 and $\ell > 3$ dividing $\Phi_3(q)$. Let $x \in G = \mathbf{G}^F$ be an ℓ-element with centraliser $A_2(q).\Phi_3$. There is a subgroup ${}^3D_4(q).3$ of G containing the normalisers of all Sylow

d-tori of $\mathbf{C}_G(x)$, and hence also $\mathrm{Sub}_G(x)$ by Proposition 5.2. But it is not immediately obvious that the subnormaliser could not be smaller.

5.2. Picky elements in the nonabelian case

Proposition 5.6. *Let $\mathbf{S} \le \mathbf{G}$ be a Sylow e-torus of $(\mathbf{G}, F)$ for some $e \ge 1$. Then the order polynomial of $\mathbf{C}_\mathbf{G}(\mathbf{S})$ is not divisible by cyclotomic polynomials $\Phi_{e\ell^i}$ with $i \ge 1$ and $2 < \ell$, except when $\mathbf{G}^F = {}^3D_4(q)$, $e \le 2$ and $\ell^i = 3$.*

Proof. The centraliser $\mathbf{C} := \mathbf{C}_\mathbf{G}(\mathbf{S})$ is an F-stable Levi subgroup of $\mathbf{G}$. Its structure is described in [14, Exmp. 3.5.14 and 3.5.15] for $\mathbf{G}$ of classical type. Note that since $\mathbf{S}$ is a Sylow e-torus, it has the form $\mathbf{TH}$ with $\mathbf{T}$ a torus whose order polynomial only involves factors Φ_d with $d \le 2e$, and $\mathbf{H}$ a semisimple group of rank less than e. The claim can easily be checked by inspection of the order formulas (e.g., in [27, Tab. 24.1]). For groups of exceptional type, Table 3.3 in [14] shows that $\mathbf{C}$ is itself a torus, with the required property, unless either $e = 4$ and $\mathbf{G}$ has type E_7, but the claim still holds in that case, or $\mathbf{G}^F = {}^3D_4(q)$ and $\ell = 3$ (where the claim fails). □

Proposition 5.7. *Assume $\mathbf{G}$ is simple. Let $e = e_\ell(q)$ and assume that ℓ divides $|W_e|$. If W_e has a normal Sylow ℓ-subgroup then $e \in \{1, 2\}$, and either $\ell = 2$ and $W_e = \mathfrak{S}_2$ or $W(B_2)$, or $\ell = 3$ and $W = \mathfrak{S}_3$ or $W(G_2)$.*

Proof. The relative Weyl groups are described in [14, Exmp. 3.5.29] for $\mathbf{G}$ of classical type. Namely, they are either symmetric groups $\mathfrak{S}_n$, in which case $e \in \{1, 2\}$, wreath products $C_d \wr \mathfrak{S}_n$ with $d \in \{e, 2e\}$, or certain subgroups of index 2 in the latter. Note that $\ell > e = e_\ell(q)$ if $\ell \ne 2$. The claim follows in this case from the known normal structure of $\mathfrak{S}_n$. For $\mathbf{G}$ of exceptional type, the occurring relative Weyl groups are given in [3, Tab. 3]. No further examples occur. □

Lemma 5.8. *Assume $\mathbf{G}$ is simple and Φ_e divides the order polynomial of $(\mathbf{G}, F)$. Assume the order of the relative Weyl group W_e of a Sylow e-torus of G is divisible by ℓ, where $\ell > 2$. Then the normaliser of a Sylow ℓ-subgroup of W_e contains no elements of order ℓr for primes $r \equiv 1 \pmod{\ell}$.*

Proof. If $\mathbf{G}$ is of classical type, then $W_e/\mathbf{O}_2(W_e)$ is a symmetric group $\mathfrak{S}_n$ (see [14, 3.5.29]), and in $\mathfrak{S}_n$ the Sylow ℓ-normalisers are easily seen not to contain elements of prime order $r > \ell$. For $\mathbf{G}$ of exceptional type, it suffices to consider $W(E_8)$ since all relative Weyl groups are subquotients of this. Now the only element order ℓr for primes $2 < \ell < r$ occurring in $W(E_8)$ is 15, but here $r \not\equiv 1 \pmod{\ell}$. □

Theorem 5.9. *In the above setting assume that $\ell > 3$ and that Sylow ℓ-subgroups of G are nonabelian. Then G possesses no picky ℓ-elements.*

Proof. Let $P \le G$ be a Sylow ℓ-subgroup of G and set $e = e_\ell(q)$. By [25, Thm 5.16] there is a Sylow e-torus $\mathbf{S}$ of $(\mathbf{G}, F)$ such that the normaliser $\mathbf{N}_G(\mathbf{S})$ contains the normaliser $\mathbf{N}_G(P)$ of P. Now $\mathbf{L} := \mathbf{C}_\mathbf{G}(\mathbf{S})$ is an F-stable Levi subgroup of $\mathbf{G}$. An application of [25, Prop. 5.3] to $[\mathbf{L}, \mathbf{L}]$ shows that $[\mathbf{L}, \mathbf{L}]^F$ is an ℓ'-group. Now $\mathbf{L}/[\mathbf{L}, \mathbf{L}]$ is a torus, hence abelian, so as P is nonabelian by assumption, ℓ must divide the order of the relative Weyl group $W_e = \mathbf{N}_G(\mathbf{S})/\mathbf{C}_G(\mathbf{S})$.

Let $x \in G$ be an ℓ-element. First assume $x \in \mathbf{S}_\ell^F$. By Proposition 5.7, since $\ell \ge 5$, W_e has more than one Sylow ℓ-subgroup, so x lies in two different Sylow ℓ-subgroups of $\mathbf{N}_G(\mathbf{S})$ and hence of G. Now assume x does not lie in any Sylow e-torus, and let $\mathbf{T}$ be an F-stable torus of $\mathbf{G}$ containing x. Then the order polynomial of $\mathbf{T}$ must be divisible by a cyclotomic polynomial $\Phi_{e\ell^i}$ for some $i \ge 1$ and hence $|\mathbf{C}_G(x)|$ is divisible by $\Phi_{e\ell^i}(q)$. Let r be a Zsigmondy primitive prime divisor of $\Phi_{e\ell^i}(q)$, which exists as $e\ell^i$ is divisible by a prime at least 5. On the other hand, by Proposition 5.6, the order polynomial of $\mathbf{C}_\mathbf{G}(\mathbf{S})$ is not divisible by $\Phi_{e\ell^i}$, so $|\mathbf{C}_G(\mathbf{S})|$ is prime to r. Thus W_e must contain an element of order ℓr. Now note that $r \equiv 1 \pmod{\ell}$. Thus Lemma 5.8 together with Lemma 2.1 show that x cannot be picky in $\mathbf{N}_G(\mathbf{S})$ and so neither in G. □

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Theorem 5.10. *In the above setting assume that Sylow* 3*-subgroups of* G *are nonabelian. Then* G *possesses a picky* 3*-element* x *if and only if one of:*

1. $G = \mathrm{SL}_3(4), \mathrm{SU}_3(8)$ *or* $G_2(8)$*;*
2. $G = \mathrm{SU}_n(2)$ *with* $4 \leq n \leq 8$*;*
3. $G = \mathrm{Sp}_{2n}(2)$ *with* $3 \leq n \leq 5$*;*
4. $G = \mathrm{SO}_{2n}^+(2)$ *with* $4 \leq n \leq 5$*;*
5. $G = \mathrm{SO}_{2n}^-(2)$ *with* $4 \leq n \leq 6$*; or*
6. $G = G_2(2) \cong \mathrm{SU}_3(3).2$*,* ${}^3D_4(2)$*, or* $F_4(2)$*.*

Proof. The proof of Theorem 5.9 goes through for the prime $\ell = 3$ unless either we are in one of the exceptions of [25, Thm 5.16], in an exception of Proposition 5.7, or if $\Phi_{e3^i}(q)$ has no primitive prime divisor for some $i \geq 1$, where again $e = e_3(q)$. We discuss these in turn. The exceptions from [25, Thm 5.16] are $G = \mathrm{SL}_3(\epsilon q)$ with $\epsilon q \equiv 4, 7 \pmod 9$ and $G = G_2(q)$ with $q \equiv 2, 4, 5, 7 \pmod 9$. For $G = \mathrm{SL}_3(\epsilon q)$, by Lemma 2.4 we may consider $S = \mathrm{PSL}_3(\epsilon q)$ instead. The Sylow 3-subgroups of S are elementary abelian of order 9, and the centraliser of an element of order 3 has structure $((q-\epsilon) \times (q-\epsilon)/3).3$. Since the Sylow 3-normalisers have the form $3^{1+2}.Q_8$ for the appropriate congruences, there cannot exist any picky 3-elements for $q > 4$ by Lemma 2.1.

Now let $G = G_2(q)$ with $q \equiv 2, 4, 5, 7 \pmod 9$ and let $\epsilon \in \{\pm 1\}$ with $q \equiv \epsilon \pmod 3$. The centraliser $C := \mathbf{C}_G(t) \cong \mathrm{SL}_3(\epsilon q)$ of a 3-central element $t \in G$ contains a Sylow 3-subgroup of G. Thus t lies in several Sylow 3-subgroups of C (and thus of G) unless $q = 2$, hence cannot be picky. There is one further class of nontrivial 3-elements of G, centralised by an A_1-subgroup (see [6, 11]), but the normaliser in G of a Sylow 3-subgroup (of order 27) does not contain such a subgroup unless again $q = 2$. In the latter case, $G' = \mathrm{SU}_3(3)$ for which all nontrivial 3-elements are picky by Theorem 3.2, yielding conclusion (3).

Next, the exceptions of Proposition 5.7 occur precisely for the groups $\mathrm{SL}_3(\epsilon q)$, $G_2(q)$ and ${}^3D_4(q)$, since these are the only groups with a relative Weyl group of a Sylow e-torus isomorphic to $\mathfrak{S}_3$ or $W(G_2)$. If $G = \mathrm{SL}_3(\epsilon q)$ we have $3|(q-\epsilon)$ since we assume Sylow 3-subgroups of G are nonabelian, and even $9|(q-\epsilon)$ by the previous paragraph. Then, by [25, Thm 5.16] the normaliser of a Sylow 3-subgroup P of G lies inside the normaliser $\mathbf{N}_G(\mathbf{S})$ of a Sylow e-torus $\mathbf{S}$. If x is a 3-element not lying in a conjugate of $\mathbf{S}_3^F$, then we may conclude as in the proof of Theorem 5.9 that x is not picky. If $x \in \mathbf{S}_3^F$ is not regular, its centraliser involves an A_1-type group, hence is not contained in $\mathbf{N}_G(\mathbf{S}) = \mathbf{S}^F.\mathfrak{S}_3$ and again x is not picky by Lemma 2.1. Now assume $x \in \mathbf{S}_3^F$ is regular. If $\mathbf{N}_G(\mathbf{S})$ has more than one Sylow 3-subgroup, again x is not picky. If $\mathbf{N}_G(\mathbf{S})$ has a unique Sylow 3-subgroup, that is, the Sylow 3-subgroup is normal, then the elements of order three in W_e need to centralise the $3'$-part of $\mathbf{S}^F$ which by inspection forces $\mathbf{S}^F$ to be a 3-group. Thus $q-\epsilon$ is a power of 3 which together with $q \equiv \epsilon \pmod 9$ forces $q = 8, \epsilon = -1$. In the latter case, all regular $x \in \mathbf{S}_3^F$ are picky by direct computation, as claimed in (1).

Next, we consider $G = G_2(q)$ with $q \equiv \pm 1 \pmod 9$ (the other congruences were already discussed above). The argument is now entirely analogous to the one given for $\mathrm{SL}_3(\epsilon q)$, and only $G_2(8)$ gives rise to picky elements, listed in (1).

Next, if $G = {}^3D_4(q)$ with $q \equiv \epsilon \pmod 3$, $\epsilon \in \{\pm 1\}$, then the normaliser of a Sylow 3-subgroup is contained in a torus normaliser of the form $N = (q^3-\epsilon)(q-\epsilon).W(G_2)$ (see the discussion in the proof of [25, Thm 5.14]). Now elements of order 3 in $W(G_2)$ act like a field automorphism on the torus of order $q^3 - \epsilon$. In particular, as soon as $q^2 + \epsilon q + 1$ has a primitive prime divisor (necessarily distinct from 3), a Sylow 3-subgroup is not normal in N and we may conclude as before. This leaves $q = 2$, so $G = {}^3D_4(2)$. Here, by explicit computation, the elements of order 9 with centraliser $A_1(q).(q^3+1)$ are picky.

Finally, if there exists no primitive prime divisor for $\Phi_{e3^i}(q)$ with $i \geq 1$, then $i = 1$, $e = q = 2$. If $x \in G$ is a picky 3-element, then it has order at most 9, as elements of order 27 have centraliser order divisible by $\Phi_{18}(2) = 3^3 19$ in contradiction to Lemma 5.8. We now discuss the various types. For G of classical type, all elements of order 3 lie in a Sylow 2-torus of G and thus in at least two Sylow 3-subgroups, arguing as in the proof of Theorem 5.9. So in particular, $\mathbf{C}_{\mathbf{G}}(x)$ must contain a torus of rational type Φ_6. In $\mathrm{SL}_n(q)$ any centraliser order divisible by $q^3 + 1$ is in fact divisible by $(q^6-1)/(q-1)$, hence by 7 when $q = 2$, contradicting Lemma 5.8. In the remaining groups of classical

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type, the primitive 9th roots of unity can occur at most once as an eigenvalue of x, as otherwise the centraliser contains a subgroup $\mathrm{SU}_2(8)$ and hence elements of order 7. Let V be the submodule of the natural G-module spanned by the eigenspaces for the eigenvalues of x of order at most 3 and H the induced classical group on V. Then H has the same type as G, respectively type $\mathrm{SO}^{-\epsilon}$ if G has type SO^{ϵ} (since in that case by what we showed x has exactly all six primitive 9th roots of unity as eigenvalues and the induced orthogonal group on the corresponding 6-dimensional sum of eigenspaces is of minus type). Then $\mathrm{Sub}_G(x)$ will contain the subnormaliser of $x|_V$ (of order 3), and by direct computation the latter is all of H if $H = \mathrm{SU}_3(2), \mathrm{Sp}_6(2), \mathrm{SO}_8^+(2)$ or $\mathrm{SO}_6^-(2)$. As all of these have order divisible by a prime larger than 3, we see that x cannot be picky in $\mathrm{SU}_n(2), \mathrm{Sp}_{2n}(2), \mathrm{SO}_{2n}^+(2), \mathrm{SO}_{2n}^-(2)$ when $n \geq 9, 6, 6, 7$ respectively. By direct computation, there do exist picky elements of order 9 in all remaining cases.

Similarly, the claim for $G_2(2), {}^3D_4(2), F_4(2)$ and $E_6(2)$ follows by explicit computation in GAP. The maximal subgroup $H = Fi_{22}$ of $S = {}^2E_6(2)$ contains a Sylow 3-subgroup of S. By direct computation, H possesses one class of picky 3-elements, but this is fused in S with one of the nonpicky classes of H, so S has no picky 3-element. The group $G = E_7(2)$ has six classes of elements of order 9, with centralisers

$$\Phi_2^2\Phi_6.{}^2A_2(2)A_1(2),\ \Phi_2.{}^2A_2(8),\ \Phi_2\Phi_6.{}^2A_3(2)A_1(2),$$

$$\Phi_2\Phi_6.{}^2A_4(2),\ \Phi_2\Phi_6.{}^3D_4(2),\ \Phi_2\Phi_6.A_1(8)A_1(2)$$

(private communication of Frank Lübeck), while all elements of order 3 lie in a Sylow 2-torus by [21]. Now the normaliser of a Sylow 3-subgroup of G is contained in the normaliser of a Sylow 2-torus and thus has the form $3^7.N$ where N is the normaliser of a Sylow 3-subgroup of the Weyl group $W(E_7)$, whose 2-part is 2^3. Since all of the above centralisers have order divisible by at least 2^4, G cannot have picky 3-elements by Lemma 2.1. Finally, $G = E_8(2)$ has four classes of elements of order 9, with centralisers

$$\Phi_2.{}^2A_2(8)A_1(2),\ \Phi_2\Phi_6.{}^2A_4(2)A_1(2),\ \Phi_2\Phi_6.{}^2D_5(2),\ \Phi_2\Phi_6.{}^3D_4(2)A_1(2)$$

(again computed by Frank Lübeck), and all elements of order 3 have centralisers of semisimple rank at least 7 by [21]. Since the 2-part of the normaliser of a Sylow 3-subgroup of G is just 2^4, we may argue as before to see that G has no picky 3-elements. □

The case $\ell = 2$ is considerably more tricky. It will by all appearance be even messier than for $\ell = 3$, involving Fermat and Mersenne primes, for example. We will consider it in forthcoming work with M. Schaeffer Fry.

5.3. Subnormalisers in Suzuki and Ree groups

We now assume that $\mathbf{G}$ is of type B_2, G_2 or F_4 in characteristic $p = 2, 3$, or 2, respectively, and F is a Steinberg endomorphism of $\mathbf{G}$ such that $G = \mathbf{G}^F$ is a Suzuki or Ree group. As before, ℓ denotes a prime distinct from the defining characteristic p of $\mathbf{G}$. We obtain the analogue of Theorems 5.3 and 5.9:

Theorem 5.11. *In the above setting, let $x \in G$ be an ℓ-element. Then x is picky if and only if $\ell > 3$ and x is regular, while $\mathrm{Sub}_G(x) = G$ otherwise.*

Proof. First assume that $\ell > 3$ and x is regular. Then ℓ does not divide the order of the Weyl group of $\mathbf{G}$ and hence the Sylow ℓ-subgroups of G are abelian (see [2, Cor. 3.13, p. 259]). We may now proceed as in the proof of Theorem 5.3, replacing cyclotomic polynomials over $\mathbb{Q}$ by suitable cyclotomic polynomials over $\mathbb{Q}(\sqrt{p})$, with corresponding Sylow tori, for which the Sylow theorems continue to hold, see [14, 3.5.3, 3.5.4], to conclude that x is picky.

We now discuss the remaining cases. All semisimple elements $x \neq 1$ of ${}^2B_2(q^2)$ are regular and have order prime to 6, so there is nothing to prove in this case. In $G = {}^2G_2(q^2)$ the only ℓ-elements $x \neq 1$ that are either nonregular or have $\ell \leq 3$ are the involutions, with $\mathbf{C}_G(x) = \langle x \rangle \times \mathrm{PSL}_2(q^2)$. Such x are

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also contained in a Sylow 2-normaliser, of structure $2^3.7.3$, not lying in the maximal subgroup $\mathbf{C}_G(x)$. Thus $\mathrm{Sub}_G(x) = G$ as claimed.

Let $G = {}^2F_4(q^2)$. The conjugacy classes of semisimple elements are given in [33, Tab. IV]. First assume $\ell > 3$. The nonregular representatives $x \neq 1$ are t_i with $i \in \{1, 2, 4, 5, 7, 8\}$, where t_1, t_2, t_5 and t_8 only occur if $q^2 \geq 8$. The noncyclic Sylow Φ-tori are $T(j)$ for $j \in \{1, 6, 7, 8\}$ in [33, (3.1)], with relative Weyl groups of order 16, 96, 96, 48 respectively. Comparison with the list of maximal subgroups of G in [23, Main Thm] shows that none of them can contain both $\mathbf{C}_G(x)$ and $\mathbf{N}_G(\mathbf{T})$ for $\mathbf{T}$ a Sylow Φ-torus of $\mathbf{G}$ containing x.

So finally assume $\ell = 3$. Any 3-element $x \in G$ is contained in a Sylow Φ_4-torus $\mathbf{T}$ of $\mathbf{G}$, where $\mathbf{N}_G(\mathbf{T}) = (q^2+1)^2.G_{12}$ with the primitive complex reflection group G_{12} of order 48. As $\mathbf{N}_G(\mathbf{T})$ is a maximal subgroup of G by [23, Main Thm], either $\mathrm{Sub}_G(x) = \mathbf{N}_G(\mathbf{T})$ or $\mathrm{Sub}_G(x) = G$. The structure of a Sylow 3-normaliser N is discussed in [25, Thm 8.4]. If $q^2 \equiv 8 \pmod 9$ then N is contained in the torus normaliser $(q^2+1)^2.G_{12}$. Now G_{12} has non-normal Sylow 3-subgroups and hence arguing as in the proof of Theorem 5.9 there cannot exist picky 3-elements, whence $\mathrm{Sub}_G(x) = G$ by what we said above. On the other hand, if $q^2 \equiv 2, 5 \pmod 9$ a Sylow 3-normaliser is isomorphic to $\mathrm{SU}_3(2).2$, not contained in any conjugate of $\mathbf{N}_G(\mathbf{T})$ by [25, Thm 8.4(b)], so again we have $\mathrm{Sub}_G(x) = G$. □

By explicit computation, the subnormaliser of an element of order 3 in the Tits group ${}^2F_4(2)'$ is a maximal subgroup $\mathrm{PSL}_3(3).2$ (not invariant under the outer automorphism of order 2).

6. Subnormalisers in algebraic groups

The concept of subnormaliser of course also makes sense in the algebraic group setting. All algebraic groups considered are over an algebraically closed field of characteristic $p \geq 0$. If $\mathbf{G}$ is connected reductive, $\mathbf{T}$ denotes a maximal torus, Φ the root system, $\mathbf{U}_\alpha$ for $\alpha \in \Phi$ the root subgroups of $\mathbf{G}$ with respect to $\mathbf{T}$, and $\mathbf{B} \geq \mathbf{T}$ a Borel subgroup of $\mathbf{G}$. Not so surprisingly the situation turns out to be much simpler than for the finite reductive groups.

6.1. Subnormalisers of unipotent elements

We first determine subnormalisers for unipotent elements.

Lemma 6.1. *Let* $\mathbf{U}$ *be a unipotent algebraic group and* $x \in \mathbf{U}$. *Then* $\langle x \rangle \trianglelefteq\trianglelefteq \mathbf{U}$.

Proof. By [27, Prop. 2.9] we may embed $\mathbf{U}$ into the (unipotent) group of upper uni-triangular matrices of GL_n for a suitable n, so without loss we may assume $\mathbf{U}$ is connected. By [27, Cor. 2.10] the unipotent group $\mathbf{U}$ is nilpotent, so by [19, Prop. 17.4], for any proper closed subgroup $\mathbf{H} < \mathbf{U}$ we have $\dim \mathbf{H} < \dim \mathbf{N}_{\mathbf{U}}(\mathbf{H})$. Thus by induction on the dimension, any closed subgroup is subnormal in $\mathbf{U}$. As $\langle x \rangle$ is normal in its abelian (see [19, Lemma 15.1.C]) closure $\overline{\langle x \rangle}$ this achieves the proof. □

For H a subgroup of a linear algebraic group we let $\widehat{R_u}(H)$ denote the largest normal unipotent subgroup of H, a characteristic subgroup.

Lemma 6.2. *Let* $\mathbf{G}$ *be a linear algebraic group,* $H \leq \mathbf{G}$ *and* $x \in \mathbf{G}$ *unipotent. If* $\langle x \rangle \trianglelefteq\trianglelefteq H$ *then* $x \in \widehat{R_u}(H)$. *In particular, if* H *is maximal (with respect to inclusion) with* $\langle x \rangle \trianglelefteq\trianglelefteq H$ *then* $H = \mathbf{N}_{\mathbf{G}}(\widehat{R_u}(H))$.

Proof. Let $\langle x \rangle \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_r = H$ be a subnormal series. Set $U := \widehat{R_u}(H)$ and $\tilde{H}_i := H_i U$. Since x normalises U, the group $\langle x, U \rangle$ is unipotent. Then $\langle x, U \rangle \trianglelefteq \tilde{H}_1$ implies $\langle x, U \rangle \leq \widehat{R_u}(\tilde{H}_1) =: U_1$. As $\tilde{H}_1 \trianglelefteq \tilde{H}_2$ we now have $U_1 \trianglelefteq \tilde{H}_2$ and thus $U_1 \leq \widehat{R_u}(\tilde{H}_2) =: U_2$. By induction this yields $\langle x, U \rangle \leq U_1 \leq \cdots \leq U_r = \widehat{R_u}(\tilde{H}_r)$, whence $x \in \widehat{R_u}(H)$.

Now the Zariski closure $\overline{\widehat{R_u}(H)}$ is unipotent [27, Prop. 2.9], giving $\langle x \rangle \trianglelefteq\trianglelefteq \overline{\widehat{R_u}(H)}$ by Lemma 6.1 and then of course $\langle x \rangle \trianglelefteq\trianglelefteq \widehat{R_u}(H)$. Hence, $\langle x \rangle \trianglelefteq\trianglelefteq \mathbf{N}_{\mathbf{G}}(\widehat{R_u}(H)) \geq H$ and the last assertion follows. □

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Lemma 6.3. *Let* $\mathbf{G}$ *be connected reductive. Then the closed unipotent subgroups* $\mathbf{U}$ *of* $\mathbf{G}$ *with* $\mathbf{U} = \widehat{R_u}(\mathbf{N_G}(\mathbf{U}))$ *are exactly the unipotent radicals of the parabolic subgroups of* $\mathbf{G}$.

Proof. Let $\mathbf{U} \leq \mathbf{G}$ be a closed unipotent subgroup with $\mathbf{U} = \widehat{R_u}(\mathbf{N_G}(\mathbf{U}))$. By [27, Cor. 17.15], $\mathbf{U}$ lies in some Borel subgroup of $\mathbf{G}$ and then by [27, Thm 17.10] there is a parabolic subgroup $\mathbf{P} \leq \mathbf{G}$ with $\mathbf{U} \leq R_u(\mathbf{P})$ and $\mathbf{N_G}(\mathbf{U}) \leq \mathbf{P}$. In particular, $\mathbf{N_G}(\mathbf{U})$ normalises $R_u(\mathbf{P})$. Thus, $V := \mathbf{N}_{R_u(\mathbf{P})}(\mathbf{U})$ is contained in and normalised by $\mathbf{N_G}(\mathbf{U})$, so contained in $\widehat{R_u}(\mathbf{N_G}(\mathbf{U})) = \mathbf{U}$, whence $V = \mathbf{U}$. By [19, Prop. 17.4] this forces $\mathbf{U} = R_u(\mathbf{P})$.

Conversely, if $\mathbf{P} \leq \mathbf{G}$ is parabolic, hence connected, then $\mathbf{P}/R_u(\mathbf{P})$ is connected reductive, and so has no nontrivial normal unipotent subgroups (since any such would have to be finite, hence central, but all central elements of connected reductive groups are semisimple, for example, by [27, Cor. 8.13(b)]), whence $\widehat{R_u}(\mathbf{P}) = R_u(\mathbf{P})$. □

We obtain the algebraic group analogue of Proposition 2.6:

Proposition 6.4. *Let* $\mathbf{G}$ *be connected and* $x \in \mathbf{G}$ *unipotent. Then* $\mathrm{Sub}_\mathbf{G}(x)$ *is generated by the Borel subgroups of* $\mathbf{G}$ *containing* x.

Proof. By Lemma 6.1, $\mathrm{Sub}_\mathbf{G}(x)$ contains all Borel subgroups of $\mathbf{G}$ containing x. For the converse, assume $\langle x\rangle \lhd\lhd H$ for some subgroup $H \leq \mathbf{G}$, where without loss of generality H is maximal with respect to inclusion. By Lemma 6.2 then $H = \mathbf{N_G}(\widehat{R_u}(H))$ and $x \in Q := \widehat{R_u}(H)$. Now Q lies in the unipotent radical $\mathbf{U}$ of some Borel subgroup of $\mathbf{G}$ ([27, Cor. 17.15]). If $Q < \mathbf{U}$ then $Q_1 := \mathbf{N_U}(Q) > Q$, and $\langle x\rangle \lhd\lhd \langle H, Q_1\rangle > H$, a contradiction, so in fact $Q = \mathbf{U}$ and $H = \mathbf{N_G}(Q) = \mathbf{N_G}(\mathbf{U})$ is closed (see [27, Ex. 10.18]). Then $\mathbf{N}_H(\mathbf{U})^\circ$ is a Borel subgroup of H, and thus H° is generated by the H-conjugates of $\mathbf{N}_H(\mathbf{U})^\circ$ (see [27, Thm 6.10]), all of which contain Q and hence x. Thus, H is generated by the normalisers (in H) of the maximal unipotent subgroups (of H and hence of G) it contains. The claim follows. □

Theorem 6.5. *Let* $\mathbf{G}$ *be a simple algebraic group in characteristic* $p > 0$ *and* $x \in \mathbf{G}$ *be unipotent. Then* $\mathrm{Sub}_\mathbf{G}(x) = \mathbf{G}$ *if and only if* x *is not regular.*

Proof. Any regular x is picky by [4, Prop. 5.1.3]. Now let $x \in \mathbf{G}$ be nonregular unipotent. There is nothing to prove if $\mathbf{G}$ is of type A_1. If $\mathbf{G}$ is of rank 2, the class representatives for the corresponding finite groups given in [37, 9, 6, 10, 11] show that any nonregular unipotent element has a $\mathbf{G}$-conjugate in $\mathbf{B}$ not involving any simple root element, and thus $\mathrm{Sub}_\mathbf{G}(x) = \mathbf{G}$ by the analogue of Proposition 3.5. Now assume $\mathbf{G}$ has rank at least 3. Then a suitable conjugate of x in $\mathbf{B}$ can be written as a product of root elements in which at least one simple root α of $\mathbf{G}$ does not occur. If α is an end node, then the observation for rank 2 shows that x is conjugate to an element in which neither α nor the simple root connected to it occurs. Thus, we may assume α is not an end node. We can now complete the proof as in Proposition 3.6 using Proposition 6.4. □

6.2. An extension to disconnected groups

In this subsection we consider a slightly different situation. Let now $\mathbf{G}$ be the extension of a connected reductive algebraic group $\mathbf{G}^\circ$ in characteristic p by a graph automorphism σ of order p. We are interested in unipotent elements in nontrivial cosets of $\mathbf{G}^\circ$ in $\mathbf{G}$. Every coset of $\mathbf{G}^\circ$ in $\mathbf{G}$ contains a unique class of *regular unipotent elements* satisfying analogous properties to the case of connected groups (see [36, Prop. II.10.2]).

Let $\mathbf{B} \leq \mathbf{G}$ be the normaliser in $\mathbf{G}$ of a σ-stable Borel subgroup $\mathbf{B}^\circ$ of $\mathbf{G}^\circ$, with σ-stable maximal torus $\mathbf{T}^\circ$ and unipotent radical $\mathbf{U} = \mathbf{U}^\circ$. Let $W = \mathbf{N}_{\mathbf{G}^\circ}(\mathbf{T}^\circ)/\mathbf{T}^\circ$ be the Weyl group of $\mathbf{G}^\circ$; it is normalised by σ, and σ acts on its set of roots Φ with respect to $\mathbf{T}^\circ$, permuting its set Δ of simple roots. We let $\bar\Delta$ denote the set of σ-orbits in Δ, and we write $\bar S$ for the set of simple reflections of W^σ constructed as in [27, Lemma 23.3].

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Proposition 6.6. *Let $\mathbf{G}^\circ$ be of type A_2, A_3 or A_4 in characteristic 2 and $\mathbf{G}$ the extension with the nontrivial graph automorphism σ of order 2. Let $x \in \mathbf{G}^\circ\sigma$ be unipotent. Then $\mathrm{Sub}_{\mathbf{G}}(x) = \mathbf{G}$ unless x is regular, when $\mathrm{Sub}_{\mathbf{G}}(x) = \mathbf{C}_{\mathbf{B}}(\sigma)$.*

Proof. Since $\mathbf{U}$ is a maximal unipotent subgroup in $\mathbf{G}^\circ$ and normalised by σ we may assume $x \in \mathbf{U}\sigma$. So $x = x_0\sigma$ where x_0 is a product of root elements for $\mathbf{G}^\circ$. If root elements for representatives from all σ-orbits of simple roots occur in x_0, then x is regular and hence contained in a unique Borel subgroup of $\mathbf{G}$ by [36, Prop. II.10.2], so $\mathrm{Sub}_{\mathbf{G}}(x) = \mathbf{C}_{\mathbf{B}}(\sigma)$. Representatives for the nonregular outer unipotent classes in types A_2, A_3 and A_4 are given in [24, Tab. 4]. It transpires that in each case at least one of the given representatives in the finite group for a class in the algebraic group has the property that its image under any reflection in $\bar{S}$ is still in $\mathbf{U}\sigma$, and thus x lies in a unipotent normal subgroup of any $\mathbf{P}^\sigma$, where $\mathbf{P}$ runs over the normalisers in $\mathbf{G}$ of the minimal σ-stable standard parabolic subgroups of $\mathbf{G}^\circ$. Since these generate $\mathbf{G}$, we have $\mathrm{Sub}_{\mathbf{G}}(x) = \mathbf{G}$. □

Theorem 6.7. *Let $\mathbf{G}$ be such that $\mathbf{G}^\circ$ is simple and of index p in $\mathbf{G}$, and $x \in \mathbf{G}^\circ\sigma$ unipotent. Then $\mathrm{Sub}_{\mathbf{G}}(x) = \mathbf{G}$ unless x is regular, when $\mathrm{Sub}_{\mathbf{G}}(x) = \mathbf{C}_{\mathbf{B}}(\sigma)$.*

Proof. First, by [36, Prop. II.10.2] the regular unipotent elements in $\mathbf{G}^\circ\sigma$ are picky, arguing as in the proof of Proposition 3.1. Now assume $x \in \mathbf{G}^\circ\sigma$ is unipotent nonregular. Again we may assume $x \in \mathbf{U}\sigma$. If $\mathbf{C}_{\mathbf{G}^\circ}(\sigma)$ has rank 1 or 2, we are done by Proposition 6.6, or $\mathbf{G}^\circ$ is of type D_4, $p = 3$ and σ induces triality. In the latter case, inspection of the representatives given in [24, Tab. 8] shows that x can be chosen to lie in the maximal normal unipotent subgroups of $\mathbf{C}_{\mathbf{P}}(\sigma)$, where $\mathbf{P}$ runs over the normalisers in $\mathbf{G}$ of the minimal σ-stable standard parabolic subgroups of $\mathbf{G}^\circ$ and thus $\mathrm{Sub}_{\mathbf{G}}(x) = \mathbf{G}$.

So we may now assume that $p = 2$, $\mathbf{C}^\circ_{\mathbf{G}}(\sigma)$ has rank at least 3, and that the claim has been shown for all groups of smaller rank. As x is not regular, by [36, Prop. II.10.2] it is a product of root elements not involving representatives from at least one orbit of simple roots $\alpha \in \bar{\Delta}$. Consider the Dynkin diagram of $\mathbf{C}^\circ_{\mathbf{G}}(\sigma)$, with nodes labelled by $\bar{\Delta}$. If α is not an end node, then we can conclude verbatim by the arguments in the proof of Proposition 3.6. If α is an end node, then the image of x in the standard Levi subgroup corresponding to α and the adjacent node $\beta \in \bar{\Delta}$ is nonregular, so has a conjugate not involving a root element for β (by [24, Tab. 4]) and hence we are reduced to the previous case since $\mathbf{C}^\circ_{\mathbf{G}}(\sigma)$ has rank at least 3. Note that this also holds trivially if this Levi is of type A_2^2. □

6.3. Subnormalisers of semisimple elements

We return to the setting of connected reductive groups $\mathbf{G}$ in arbitrary characteristic and now consider semisimple elements.

What do subnormalisers of semisimple elements look like in connected reductive groups? They are related to the *p-closed subsystems of* Φ (see [27, Def. 13.2]); as before it is easy and convenient to reduce to the case of simple groups.

Theorem 6.8. *Let $\mathbf{G}$ be a simple algebraic group and $s \in \mathbf{G}$ semisimple. Then there exists a p-closed subsystem $\Psi \subseteq \Phi$ such that $\mathrm{Sub}_{\mathbf{G}}(s) = \mathbf{N}_{\mathbf{G}}(\mathbf{G}(\Psi))$, the normaliser of the subsystem subgroup $\mathbf{G}(\Psi)$ corresponding to Ψ, where $\Psi = \emptyset$, $\Psi = \Phi$, or Ψ consists of all roots of a fixed length.*

Proof. Let $s \in \mathbf{G}$ be semisimple and $\mathbf{T}$ a maximal torus of $\mathbf{G}$ containing s (which exists by [27, Cor. 6.11]). Then $\mathbf{N}_{\mathbf{G}}(\mathbf{T})$ is contained in $\mathrm{Sub}_{\mathbf{G}}(s)$, so $\mathbf{T}$ normalises $\mathrm{Sub}_{\mathbf{G}}(s)$. Let $\Psi := \{\alpha \in \Phi \mid \mathbf{U}_\alpha \leq \mathrm{Sub}_{\mathbf{G}}(s)\}$. Then, Ψ is a p-closed subsystem of Φ, by definition. Since $\mathrm{Sub}_{\mathbf{G}}(s)$ contains $\mathbf{N}_{\mathbf{G}}(\mathbf{T})$ and hence representatives for all Weyl group elements, Ψ consists of full W-orbits of roots, that is, of all short, all long, or all roots, or $\Psi = \emptyset$ (see [27, Cor. A.18]). Now $\mathrm{Sub}_{\mathbf{G}}(s)^\circ$ is a subsystem subgroup since it is normalised by the maximal torus $\mathbf{T}$. According to [27, Cor. 13.7] this shows that $\mathrm{Sub}_{\mathbf{G}}(s) = \mathbf{N}_{\mathbf{G}}(\mathbf{G}(\Psi))$. □

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Corollary 6.9. *In the situation of Theorem 6.8, either* $\mathrm{Sub}_{\mathbf{G}}(s) = \mathbf{G}$*, or one of the following holds:*

1. *s is regular and* $\mathrm{Sub}_{\mathbf{G}}(s) = \mathbf{N}_{\mathbf{G}}(\mathbf{T})$*;*
2. $\mathbf{G} = B_n$ *and* $\mathrm{Sub}_{\mathbf{G}}(s) = D_n$*, or* $\mathrm{Sub}_{\mathbf{G}}(s) = A_1^n$ *when* $p = 2$*;*
3. $\mathbf{G} = C_n$ *with* $n, p > 2$ *and* $\mathrm{Sub}_{\mathbf{G}}(s) = A_1^n$*;*
4. $\mathbf{G} = G_2$ *and* $\mathrm{Sub}_{\mathbf{G}}(s) = A_2$*; or*
5. $\mathbf{G} = F_4$ *and* $\mathrm{Sub}_{\mathbf{G}}(s) = D_4$.

Conversely, the cases (1)–(5) can only occur when $\mathbf{C}_{\mathbf{G}}(s)$ *lies in a subsystem subgroup of the given type.*

Proof. By Theorem 6.8 we have $\mathrm{Sub}_{\mathbf{G}}(s) = \mathbf{N}_{\mathbf{G}}(\mathbf{G}(\Psi))$ for some p-closed subsystem Ψ of Φ consisting of full orbits of roots under the Weyl group. If $\Psi = \emptyset$ then $\mathbf{G}(\Psi) = \mathbf{T}$ and thus $\mathrm{Sub}_{\mathbf{G}}(s) = \mathbf{N}_{\mathbf{G}}(\mathbf{T})$. For the rest of the proof assume $\Psi \neq \emptyset$. We need to understand the possible Ψ, where, of course, we may assume Φ has two root lengths. The structure of such subsystems is given in [27, Tab. B.2]. By [27, Thm 13.14 and Prop. 13.15] these are p-closed precisely under the conditions listed in the statement. Note that all of these subgroups do indeed contain the full normaliser of a maximal torus of $\mathbf{G}$.

Since $\mathrm{Sub}_{\mathbf{G}}(s)$ contains $\mathbf{C}_{\mathbf{G}}(s)$ the additional claim follows. □

Example 6.10. We have not been able to find sufficient conditions for the occurrence of cases (1)–(5) in Corollary 6.9. Let $\mathbf{G} = \mathrm{GL}_n$ and $s \in \mathbf{G}$ a diagonal element with eigenvalues all the nth roots of unity, where we assume the characteristic of $\mathbf{G}$ does not divide n. Thus s is regular semisimple and $\mathbf{C}_{\mathbf{G}}(s) = \mathbf{T}$, the diagonal maximal torus. But a conjugate of s lies in the group of permutation matrices, the Weyl group of $\mathbf{G}$. Hence $\mathbf{N}_{\mathbf{G}}(\mathbf{T})$ contains two elements from the $\mathbf{G}$-class of s not conjugate in $\mathbf{N}_{\mathbf{G}}(\mathbf{T})$ and thus $\mathrm{Sub}_{\mathbf{G}}(s) = \mathbf{G}$ by Theorem 6.8, even though s is regular.

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