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RELATIVE KLOOSTERMAN INTEGRALS FOR GL(3): III

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ABSTRACT Let E be a quadratic extension of a number field F with Galois conjugation σ, G' the quasi-split unitary group in three variables, G the group GL(3, E) We let S be the space of the matrices s in G such that $\sigma(s)s = e$ One conjectures a comparison identity between the relative Kuznietsov trace formula for the symmetric space S and the ordinary Kuznietsov trace formula for the group G' (See [10]) We prove the corresponding "fundamental lemmas"

1. **Introduction.** Let F be a non-Archimedean field of odd residue characteristic q. We denote by R_F the ring of integers of F. Let E be an unramified quadratic extension of F, with Galois conjugation $\sigma: z \longmapsto \bar{z}$. We let ψ_F be an additive character of F with conductor R_F and set $\psi_E = \psi_F(z + \bar{z})$. We also write ψ for ψ_E, and $|x|$ for $|x|_E$.

Let G be the group GL(3, E), regarded as an algebraic group over F. Then σ defines an automorphism on GL(3, E). Let S be the variety:

$$\{s \in \mathrm{GL}(3, E) \mid s\bar{s} = e\}.$$

The group G acts transitively on S by $s \longmapsto \bar{g}^{-1}sg$, (see [2]). The group GL(3, F) is the fixator of e under this action. We also denote it by H.

We denote by B the group of upper triangular matrices in GL(3, E), by A the group of diagonal matrices, and by N the group of upper triangular matrices with unit diagonal. Let θ be the character of N defined by:

$$\theta\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \psi(x - y).$$

The group N acts on S, thus S is a disjoint union of N-orbits. A set of representatives for the orbits of S consists of the matrices $w\mathbf{a}$, where w is a permutation matrix with $w^2 = e$ and $\mathbf{a} \in A$ with $w\mathbf{a}w\bar{\mathbf{a}} = e$. (See [12]). We also note that the scalar matrices ue with $u\bar{u} = 1$ acts on S by multiplication. An N-orbit of an element $s \in S$ is called *relevant* if θ is trivial on the fixator N_s of s in N. We set:

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad d_a = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix}.$$

The relevant N-orbits are those with a representative of the form uwd_a (regular orbits) and those with a representative of the form ud_{-1} (singular orbits).

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Let Φ be a smooth function of compact support on S We define the orbital integrals

$$I(a,\Phi) = \int_N \Phi(\bar{n}^{-1} w d_a n)\theta(n)\, dn$$
$$I_s(\Phi) = \int_{N/N_{d_{-1}}} \Phi(\bar{n}^{-1} d_{-1} n)\theta(n)\, dn$$

Let G' be the group

$$\{g \in G \mid \bar{g}^t w g = w\}$$

Thus G' is a quasi-split unitary group We denote by B', N' and A' the intersection of B, N and A with G' The group N' consists of the matrices of the form

$$\begin{pmatrix} 1 & x & t - \frac{x\bar{x}}{2} \\ 0 & 1 & -\bar{x} \\ 0 & 0 & 1 \end{pmatrix}, \quad t + \bar{t} = 0$$

We define a character θ' of N' by $\theta'(n') = \psi(x)$ Let f' be a smooth function of compact support on G', we consider the following integrals

$$J(a,f') = \int_{N' \times N'} f'(n_1^{-1} w d_a n_2)\theta'(n_1 n_2)\, dn_1\, dn_2,$$
$$J_s(f') = \int_{N'} f'(n^{-1} d_{-1})\theta'(n)\, dn$$

With the above notations, we say Φ and f' have matching orbital integrals if

$$I(a,\Phi) = J(a,f'), \quad I_s(\Phi) = J_s(f')$$

For a smooth function of compact support f on G, we define a function Φ_f on S by

$$\Phi_f(\bar{g}^{-1} g) = \int_H f(hg)\, dh$$

Then Φ_f is a smooth function of compact support on S If f and f' are Hecke functions on G and G' respectively, which satisfy the stable base change correspondence (described in Section 2 7), then we say Φ_f and f' are associated Our main result is

THEOREM 1 *If Φ_f and f' are associated, then they have matching orbital integrals*

If f and f' are the unit elements of the respective Hecke algebras, this is proved in [7]

To motivate our result, we temporarily go to a global setting Let E be a quadratic extension of a number field F, Φ a smooth function of compact support on $S(F_{\mathbf{A}})$ which is a product of local functions Then the function K_Φ on $G(F_{\mathbf{A}})$ defined by

$$K_\Phi(g) = \sum_{\xi \in S(F)} \Phi(\bar{g}^{-1}\xi g)$$

is invariant on the left under $G(F)$ Functions of this form can be constructed for any symmetric space S (viewed as the fixator of an antiautomorphism of a reductive group G), and have been studied by Sarnak in connection with Diophantine problems on the

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variety S. Since the space spanned by the functions K_Φ is right invariant, it is natural to perform a spectral analysis of the functions K_Φ, writing them as the sum of their projections on the cuspidal spectrum and an Eisenstein part, (more precisely, as a sum of a discrete part and a continous part). In our case, since the action of G on S is transitive, the space V of an automorphic cuspidal representation π of G has a nonzero projection on the space spanned by those K_Φ if and only if there is a ϕ in V such that:

$$\int_{H(F)\backslash H(F_{\mathbf{A}})} \phi(h)\, dh \neq 0.$$

Conjecturally such *distinguished* representations are characterized as stable base change of the automorphic representations of G'. (See [2] for more details). To prove the conjecture, one considers a smooth function of compact support f' on $G'(F_{\mathbf{A}})$ which is a product of local functions. One associates to f' the kernel function:

$$K_{f'}(g_1, g_2) = \sum_{\xi \in G'(F)} f'(g_1^{-1}\xi g_2).$$

Suppose the local components of f' have matching integrals with the local components of Φ. At a split place, this relation is simply a convolution relation; at an inert place, this relation is of the above type. Then:

$$\int K_\Phi(n)\theta(n)\, dn = \iint K_{f'}(n_1', n_2')\theta(n_1' n_2')\, dn_1'\, dn_2'.$$

From this identity, one wants to derive an analoguous identity for the discrete parts of the kernel functions K_Φ and $K_{f'}$, and then obtain the conjecture. (See [8], [10] for global identities of this type; for the case at hand, see [2], [3] for details).

Our result is surely sufficient to prove the conjecture under the following *very restrictive* assumptions: if a place v of F is inert in E it is non-Archimedean of odd residue characteristic and unramified in E; furthermore, we restrict ourselves to representations π of G which are supercuspidal at a finite place of E above a split place of F, and unramified at each place of E above a place of F which is inert. At any rate, our theorem is an essential step in the proof of the above global identity (the "fundamental lemma").

Instead of considering a variant of the Kuznietsov trace formula as we did above, one could consider "relative Poincaré series", that is, functions of the form K_Φ where the infinite components of Φ are suitable non-compactly supported functions. Our result would then be useful to compare the Fourier coefficient of these Poincaré series with the Fourier coefficients of the usual Poincaré series for G'. In this context, we may regard the integrals $J(a, f')$ as Kloosterman integrals on G', that is, local analogues of the Kloosterman sums, (see [4], [5] and [13]). The integrals $I(a, \Phi)$ are then a new kind of Kloosterman integrals and we show they are identical with the Kloosterman integrals for G'.

We now go back to the local setting and introduce a few more notations. Set $K = \mathrm{GL}(3, R_E)$, $K' = K \cap G'$. If f is a Hecke function on G, then Φ_f is K-invariant. Let $\mathcal{U}$

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be the space of K-invariant functions of compact support on S. Similarly for f' a Hecke function on G', we define:

$$\Phi'_{f'}(g) = \int_{N'} f'(n^{-1}g)\theta'(n)\, dn.$$

Then $\Phi'_{f'}$ is left N'-equivariant and right K'-invariant and of compact support modulo N'. Let $\mathcal{V}$ be the space of such functions on G'. If Φ' is in $\mathcal{V}$, we set:

$$J(a, \Phi') = \int_{N'} \Phi'(wd_a n)\theta'(n)\, dn, \quad J_s(\Phi') = \Phi'(d_{-1}).$$

Thus $J(a, \Phi'_{f'}) = J(a, f')$, $J_s(\Phi'_{f'}) = J_s(f')$.

The material is arranged as follows. In Section 2, we reduce the problem to the comparison of the integrals $I(a, \Phi)$ and $J(a, \Phi')$. We compute the formal Mellin transform of $J(a, \Phi')$ for Φ' in $\mathcal{V}$ in Section 3, and in Section 4, the formal Mellin transform of $I(a, \Phi)$ for Φ in $\mathcal{U}$. The computation in Section 4 does not include the case treated in [7]. The Mellin transforms here turn out to be the square of Gauss sum times an elementary factor. The main difficulty is to show that the same factor occurs for both integrals, (Lemma 2). Comparing the Mellin transforms, we obtain our result.

Professor Jacquet suggested this problem to me. From the beginning of this work to the final version, I benefitted from his aid. I would like to express my gratitude here.

2. Correspondence of the Hecke algebras.

2.1. Let π be an irreducible unramified representation of G with a K-fixed vector v_0. If f is a Hecke function on G, that is, a K-biinvariant function of compact support on G, we define $\hat{f}(\pi)$ by:

$$\pi(f)v_0 = \hat{f}(\pi)v_0.$$

Suppose that π is *distinguished*, that is, there is a linear form $\lambda \neq 0$ on the space V of π which is H-invariant. Then the central character of π must be trivial and π is self-contragradient ([2]). It follows that π is a component of $\mathrm{Ind}(G, B; \chi, 1, \chi^{-1})$ where $\chi(x) = |x|^t$, t a complex number. Let Ω be the function on S defined by:

$$\Omega(\bar{g}^{-1}g) = \lambda\big(\pi(g)v_0\big).$$

Then Ω is invariant under K and an eigenfunction of the Hecke algebra in the sense that:

$$\int_G f(g)\Omega(\bar{g}^{-1}sg)\, dg = \hat{f}(\pi)\Omega(s). \tag{1}$$

Conversely, we will show that given any $t \in \mathbf{C}^\times$ and the corresponding representation π of the above form, there is, up to a scalar factor, a unique nonzero function Ω on S which is K-invariant and satisfies (1); (even though π may not be distinguished).

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2.2. Fix a uniformizer ϖ in F. For an integer $r \geq 0$ we set:

$$d_r = d_a \text{ where } a = \varpi^r.$$

It is easily checked that the matrices of the form wd_r form a set of representatives for the orbits of K in S and thus a K-invariant function Ω on S is determined by the values:

$$\Omega_r = \Omega(wd_r).$$

Note that such a function is invariant under the center of G' acting on S and satisfies:

$$\Omega(s) = \Omega({}^t\bar{s}).$$

If follows that, given π as above, (1) is satisfied for all Hecke functions if and only if it is satisfied for the characteristic function T of the set $K\,\mathrm{diag}[\varpi, 1, 1]K$. In turn, (1) reads:

$$\int_G T(g)\Omega(\bar{g}^{-1}wd_rg)\,dg = q^2(q^{-2t} + 1 + q^{2t})\Omega_r.$$

This equation is equivalent to the following difference equation:

$$q^4\Omega_{r+1} + q^2\Omega_r + \Omega_{r-1} = q^2(q^{-2t} + 1 + q^{2t})\Omega_r, \quad r \geq 1, \tag{2}$$

$$(q^2 + q + 1)\Omega_0 + (q^4 - q)\Omega_1 = q^2(q^{-2t} + 1 + q^{2t})\Omega_0. \tag{3}$$

These relations determine Ω_r in terms of Ω_0; thus, our assertion on the existence and uniqueness of Ω is established. In particular, we may assume $\Omega_0 = 1$. Solving the difference equation, we get:

PROPOSITION 1.

$$\Omega_r = \frac{(1 - q^{2t-1})(1 - q^{2-2t})}{(q - q^4)(q^{-2t-2} - q^{2t-2})}q^{-r(2t+2)} + \frac{(1 - q^{-2t-1})(1 - q^{2+2t})}{(q - q^4)(q^{2t-2} - q^{-2t-2})}q^{-r(-2t+2)}. \tag{4}$$

2.3. Let ds denote the invariant measure on S such that $K \cap S$ has volume 1. Then for any Hecke function f, we have (Ω as in (4)):

$$\hat{f}(\pi) = \int_S \Phi_f(s)\Omega(s)\,ds. \tag{5}$$

To apply this equation, we need to compute the volume c_r of the K-orbit of wd_r in S. Let Φ_r be the characteristic function of this orbit; we have:

$$\begin{aligned}\int_G \int_S \Phi_r(\bar{g}^{-1}sg)\,ds T(g)\,dg &= \int_G T(g)dg \int_S \Phi_r(s)\,ds \\ &= (q^4 + q^2 + 1)c_r.\end{aligned}$$

Just as before, the above equation yields a difference equation. Solving the equation with the initial condition $c_0 = 1$, we find:

$$c_r = \begin{cases} q^{4r}(1 - q^{-3}) & \text{if } r \geq 1 \\ 1 & \text{if } r = 0. \end{cases}$$

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2.4. Note the functions Φ_r form a basis of the space $\mathcal{U}$. For Φ_r, the integral in (5) equals:

$$\int_S \Phi_r(s)\Omega(s)\,ds = 1, \quad \text{if } r = 0 \tag{6}$$

$$\int_S \Phi_r(s)\Omega(s)\,ds = \frac{1}{q^4}\left[\frac{q^{r(2-2t)}(q^{2-2t}-1)}{q^{-2t-2}-q^{2t-2}} + \frac{q^{r(2+2t)}(q^{2+2t}-1)}{q^{2t-2}-q^{-2t-2}}\right]$$

$$-\frac{1}{q^3}\left[\frac{q^{(r-1)(2-2t)}(q^{2-2t}-1)}{q^{-2t-2}-q^{2t-2}} + \frac{q^{(r-1)(2+2t)}(q^{2+2t}-1)}{q^{2t-2}-q^{-2t-2}}\right], \quad \text{if } r \geq 1. \tag{7}$$

2.5. Let $t \in \mathbb{C}^x$ and χ as above. Denote by π' the unramified component of the representation of G' induced by the character of B':

$$ud_an \longmapsto \chi(a).$$

Let W be the corresponding Whittaker function:

$$W(ngk) = \bar{\theta}'(n)W(g), \quad W(e) = 1$$

and:

$$\hat{f}'(\pi')W(g) = \int_{G'} f'(x)W(gx)\,dx.$$

In particular:

$$\hat{f}'(\pi') = \int_{G'} f'(x)W(x)\,dx.$$

2.6. In the introduction, we defined $\mathcal{V}$ a space of functions on G'. If Φ' is in $\mathcal{V}$, it is determined by the function on $E^\times$ also noted Φ' and defined by:

$$\Phi'(g) = \theta'(\bar{n})\Phi'(a)$$

if

$$g = nd_ak.$$

Note that $\Phi'(a) = 0$ for $|a| > 1$. With this notation (and a similar notation for W), we get:

$$\hat{f}'(\pi') = \int_{E^\times} \Phi'_{f'}(a)W(a)|a|^{-2}\,d^\times a.$$

For an integer $r \geq 0$, we denote by Φ'_r the function defined by:

$$\Phi'_r(a) = \begin{cases} 1 & \text{if } |a| = q^{-2r} \\ 0 & \text{otherwise.} \end{cases}$$

Using the explicit formula for the Whittaker function, we get:

$$\int_{E^\times} \Phi'_r(a)W(a)|a|^{-2}\,d^\times a = (1-q^{4t})^{-1}q^{-r(2t-2)} + (1-q^{-4t})^{-1}q^{-r(-2t-2)}. \tag{8}$$

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2.7. The homomorphism of the stable base change $f \longmapsto f'$ is defined by the condition $\hat{f}(\pi) = \hat{f}'(\pi')$. With the above notations, this condition is equivalent to:

$$\int_S \Phi_f(s)\Omega(s)\,ds = \int_{E^\times} \Phi'_{f'}(a)W(a)|a|^{-2}\,d^\times a.$$

If Φ is in $\mathcal{U}$ and Φ' is in $\mathcal{V}$, we write $\Phi \leftrightarrow \Phi'$ if:

$$\int_S \Phi(s)\Omega(s)\,ds = \int_{E^\times} \Phi'(a)W(a)|a|^{-2}\,d^\times a.$$

Thus $\Phi_f \leftrightarrow \Phi'_{f'}$ by definition. Let $\Phi^r = \sum_{i=0}^r \Phi_i$, $r \geq 0$. We set $\Phi^r = 0$ and $\Phi'_r = 0$ if $r < 0$.

PROPOSITION 2.

$$\Phi^r \leftrightarrow \Phi'_r - q\Phi'_{r-1}.$$

In particular, the correspondence $\Phi \leftrightarrow \Phi'$ is a bijection between $\mathcal{U}$ and $\mathcal{V}$.

PROOF. The first assertion follows easily from (6), (7) and (8). The second follows from the fact the functions in question form a basis of $\mathcal{U}$ and $\mathcal{V}$ respectively. ■

We see that to prove Theorem 1, it suffices to show that for $\Phi \in \mathcal{U}$, $\Phi' \in \mathcal{V}$ and $\Phi \leftrightarrow \Phi'$:

$$I(a,\Phi) = J(a,\Phi'), \quad I_s(\Phi) = J_s(\Phi').$$

By the previous proposition, all we need to prove is the following:

PROPOSITION 3.

$$I(a,\Phi^r) = J(a,\Phi'_r - q\Phi'_{r-1}) \tag{9}$$

$$I_s(b,\Phi^r) = J_s(b,\Phi'_r - q\Phi'_{r-1}). \tag{10}$$

The proof of this proposition will be the task of the remaining of this paper. The easy verification of the equality between I_s and J_s is left to the reader.

3. **Mellin transform of $J(a,\Phi'_r)$.** Let Φ'_r be the function defined in Section 2.6. Recall we defined:

$$J(a,\Phi'_r) = \int_{N'} \Phi'_r(wd_an)\theta'(n)\,dn.$$

We will compute the *formal* Mellin transforms of J and I as functions on $E^\times$. For a justification of our computation, we refer to [7]. Let χ be any character of $E^\times$, we write $\chi(z) = \chi_0(z)|z|^t$, where χ_0 has module one and is trivial if χ is unramified. We also set $X = q^{2t}$. The Mellin transform of J is then a formal Laurent series in X:

$$\hat{J}(\chi) = \int_{E^\times} J(a)\chi(a)\,d^\times a. \tag{1}$$

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We use $\hat{J}_r(\chi)$ to denote the Mellin transform of $J(a, \Phi'_r)$ with respect to χ. An easy but lengthy computation using the right K'-invariance and left N'-equivariance of the function Φ'_r shows that:

$$\hat{J}_r(\chi) = \int \Phi'_r(\bar{a}^{-1})\chi(a)\, d^\times a \tag{2}$$

$$+ \sum_{s=1}^{\infty} \int q^s(1-q^{-1})\Phi'_r(\bar{a}^{-1}\varpi^s)\chi(a)\, d^\times a \tag{3}$$

$$+ \int_{|y|>1,\, t+\bar{t}=0} \Phi'_r\left\{\left[a\left(t - \frac{y\bar{y}}{2}\right)\right]^{-1}\right\}$$

$$\int \psi\left\{\left[a\left(t - \frac{y\bar{y}}{2}\right)\right]^{-1} \bar{y} + y\right\}\chi(a)\, d^\times a\, dy\, dt. \tag{4}$$

If χ is ramified, clearly (2) and (3) give 0. If χ is unramified, (2) gives X^r, and (3) gives:

$$X^r \frac{(q-1)X^{-1}}{1-qX^{-1}}.$$

In (4), we make the change of variable $a \longmapsto (t - \frac{y\bar{y}}{2})^{-1}$; the integral becomes:

$$\int \Phi'_r(\bar{a}^{-1})\psi(y + \bar{y}a^{-1})\chi\left(a\left(t - \frac{y\bar{y}}{2}\right)^{-1}\right) dy\, dt\, d^\times a$$

with

$$|y| > 1, \quad t + \bar{t} = 0.$$

By changing t to $t(y\bar{y}/2)$ and a to $a^{-1}\bar{y}$, we get:

$$\int |y\bar{y}|_F \psi(y)\chi^{-1}(y)\, dy \int \psi(a)\chi^{-1}(a)\, d^\times a \int \chi\left(\frac{2}{t-1}\right) dt \tag{5}$$

with

$$|y| > 1, t + \bar{t} = 0, \quad |ay^{-1}| = q^{-2r}.$$

We first suppose that χ is ramified with conductor m. By the well known vanishing property of Gauss sums (integrals), we may impose a restriction $|a| = |y| = q^{2m}$ on the domain of (5). Thus, if $r \neq 0$, the expression (5) is 0. If $r = 0$, (5) is:

$$(1 - q^{-2})^{-1}\left[\int_{|y|=q^{2m}} \psi(y)\chi^{-1}(y)\, dy\right]^2 \int_{t+\bar{t}=0} \chi\left(\frac{2}{t-1}\right) dt. \tag{6}$$

Now we consider the unramified case. We have:

$$\int_{|y|=q^{2m}} \psi(y)\chi^{-1}(y)\, dy \neq 0$$

only if $m \leq 1$. Therefore (5) does not vanish only if $|y| = q^2$, while:

$$\int_{|y|=q^2} |y\bar{y}|_F \psi(y)\chi^{-1}(y)\, dy = -q^2X^{-1}.$$

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If $r \geq 1$, since $|ay^{-1}| = q^{-2r}$ and $|y| = q^2$, we get $|a| \leq 1$ and:

$$\int_{|a|=q^{2-2r}} \psi(a)\chi^{-1}(a)\, d^\times a = X^{r-1}.$$

If $r = 0$, then $|a| = q^2$, we get:

$$\int_{|a|=q^2} \psi(a)\chi^{-1}(a)\, d^\times a = -(1-q^{-2})^{-1}q^{-2}X^{-1}.$$

To get the explicit result, we apply the following lemma:

LEMMA 1. *If χ is unramified, we have:*

$$\int_{t+\bar{t}=0} \chi\Big(\frac{2}{t-1}\Big)\, dt = \frac{1-X^{-1}}{1-qX^{-1}}. \tag{7}$$

Thus when χ is unramified, (5) gives:

$$-q^2X^{r-2}\frac{1-X^{-1}}{1-qX^{-1}} \quad \text{if } r > 0 \tag{8}$$

$$(1-q^{-2})^{-1}X^{-2}\frac{1-X^{-1}}{1-qX^{-1}} \quad \text{if } r = 0. \tag{9}$$

We sum up the above results into the following:

If χ is ramified with conductor m, then:

$$\hat{J}_r(\chi) = 0, \quad \text{if } r \geq 1 \tag{10}$$

$$\hat{J}_0(\chi) = (1-q^{-2})^{-1}\Big[\int_{|y|=q^{2m}} \psi(y)\chi^{-1}(y)\, dy\Big]^2 \int_{t+\bar{t}=0} \chi\Big(\frac{2}{t-1}\Big)\, dt. \tag{11}$$

If χ is unramified, then:

$$\hat{J}_r(\chi) = (1+qX^{-1})(1-X^{-1})X^r, \quad \text{if } r \geq 1 \tag{12}$$

$$\hat{J}_0(\chi) = (1-qX^{-1})^{-1}(1-X^{-1})[1+(1-q^{-2})^{-1}X^{-2}]. \tag{13}$$

We notice the difference between the results here and those in [7]. In [7], the term $|a| = 1$ is excluded.

4. **Mellin transform of $I(a, \Phi^r)$.** We now evaluate the Mellin transform of the integral I:

$$I(a, \Phi^r) = \int_N \Phi^r(\bar{n}^{-1}wan)\theta(n)\, dn.$$

The case $r = 0$ is treated in [7]. Thus here we only consider the case when $r > 0$. We note that our method does not apply to the case $r = 0$.

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4.1. We keep the notations of Section 3. We set:

(1) $$\hat{I}_r(\chi) = \int_{N\times E^\times} \Phi^r(\bar{n}^{-1}wd_a n)\theta(n)\,dn\chi(a)\,d^\times a.$$

Let

$$n = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

then:

$$\bar{n}^{-1}wd_a n = \begin{pmatrix} a\bar{z'} & a\bar{z'}x - \bar{x} & a\bar{z'}z - \bar{x}y + \bar{a}^{-1} \\ -a\bar{y} & 1 - ax\bar{y} & -a\bar{y}z + y \\ a & ax & az \end{pmatrix}.$$

Here we have set $z+z' = xy$. Recall Φ^r is the characteristic function of the set of elements in S with norm bounded above by q^{2r}. Thus our integral is:

(2) $$\int \psi(x-y)\chi(a)\,dx\,dy\,dz\,d^\times a$$

with domain of integration:

$$|a| \le q^{2r} \quad |ax| \le q^{2r} \quad |ay| \le q^{2r}$$
$$|axy| \le q^{2r} \quad |az| \le q^{2r} \quad |az'| \le q^{2r}$$
$$|a\bar{z'}x - \bar{x}| \le q^{2r} \quad |a\bar{y}z - y| \le q^{2r} \quad z + z' = xy$$
$$|a\bar{z'}z - \bar{x}y + \bar{a}^{-1}| \le q^{2r}.$$

We make the following changes of variables: $z \longmapsto xyz$, $z' \longmapsto xyz'$, $a \longmapsto ax^{-1}\bar{y}^{-1}$, and $y \longmapsto -y$; the integral becomes:

(3) $$\hat{I}_r(\chi) = \chi(-1)\int \psi(x+y)|xy|\chi(ax^{-1}\bar{y}^{-1})\,dx\,dy\,dz\,d^\times a$$

with domain of integration:

(4) $$|ax^{-1}y^{-1}| \le q^{2r} \quad |ay^{-1}| \le q^{2r} \quad |ax^{-1}| \le q^{2r}$$
(5) $$|a| \le q^{2r} \quad |az| \le q^{2r} \quad |az'| \le q^{2r}$$
(6) $$|x|\,|a\bar{z'} - 1| \le q^{2r} \quad |y|\,|1 - az| \le q^{2r}$$
(7) $$|xy|\,|a\bar{z'}z - 1 + \bar{a}^{-1}| \le q^{2r} \quad z + z' = 1.$$

4.2. We first consider the case when χ is ramified. Suppose its conductor is m. Then (3) equals:

$$\chi(-1)q^{4m}\Big[\int_{|y|=q^{2m}} \psi(y)\chi^{-1}(y)\,dy\Big]^2 \int \chi(a)\,dz\,d^\times a.$$

Here the last integral is over the set:

(8) $$|az| \le q^{2r} \quad |a| \le q^{2r}$$
(9) $$|1 - az| \le q^{2r-2m}$$
(10) $$|1 - a + a\bar{z}| \le q^{2r-2m} \quad |az - az\bar{z} - 1 + \bar{a}^{-1}| \le q^{2r-4m}.$$

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If $|a| = q^{2r}$, then by (9), $|z| < 1$. But then we have $|1 - a + a\bar{z}| = |a|$, which contradicts our restriction. Thus we have $|a| \leq q^{2r-2}$. We change $z \mapsto a^{-1}z$; then what we need to compute is the following:

$$L_r = \int |a|^{-1}\chi(a)\, dz\, d^{\times}a \tag{11}$$

with the conditions:

$$|a| \leq q^{2(r-1)} \quad |1 - z| \leq q^{2(r-m)} \tag{12}$$

$$|1 - a + az/\bar{a}| \leq q^{2r-2m} \quad |a|^{-1}|\bar{a}z - z\bar{z} - \bar{a} + 1| \leq q^{2r-4m}. \tag{13}$$

By the second condition in (12), the condition $|1 - a + az/\bar{a}| \leq q^{2(r-m)}$ is equivalent to $|1 - a + a/\bar{a}| \leq q^{2(r-m)}$.

We first look at the integral (11) with the extra condition $|1 - z| < q^{2r-2m}$. It is clear there exists a γ in E such that $|\gamma| = q^{-2(m-1)}$, $|1 + \gamma| = 1$, $\chi(1 + \gamma) \neq 1$. By making the change of variable $a \mapsto (1 + \gamma)a$, we easily see this part of the integral gives zero.

Thus we may impose $|1 - z| = q^{2r-2m}$ on the domain of (11). We change z to $z + 1$, then the domain of the integral L_r becomes:

$$|a| \leq q^{2r-2} \quad |1 - a + a/\bar{a}| \leq q^{2r-2m} \quad |z| = q^{2r-2m} \tag{14}$$

$$|a|^{-1}|\bar{a}z - z\bar{z} - z - \bar{z}| \leq q^{2r-4m}. \tag{15}$$

By (15), we may write:

$$\frac{\bar{a}z}{z\bar{z} + z + \bar{z}} = 1 + u\varpi^m, \quad u \in R_E.$$

After substituting this expression for a into the conditions (14), we find that miraculously, the condition $|1 - a + a/\bar{a}| \leq q^{2r-2m}$ becomes vacuous. (We note this does not happen in the case $r = 0$.) Thus we arrive at:

$$L_r = (1 - q^{-2})^{-1}q^{-2m}\int_{|z|=q^{2r-2m}} \left|\frac{z + \bar{z} + z\bar{z}}{\bar{z}}\right|^{-1} \chi\left(\frac{z + \bar{z} + z\bar{z}}{\bar{z}}\right) dz. \tag{16}$$

The following lemma which we prove in (4.3) is the crux of the matter.

LEMMA 2. *Let χ be a ramified character with conductor m, then:*

$$\int_{|z|=q^{2r-2m}} \left|\frac{z + \bar{z} + z\bar{z}}{\bar{z}}\right|^{-1} \chi\left(\frac{z + \bar{z} + z\bar{z}}{\bar{z}}\right) dz = \begin{cases} 0 & \text{if } r > 1 \\ -q^{1-2m}\int_{t+\bar{t}=0} \chi\left(\frac{2}{1-t}\right) dt & \text{if } r = 1. \end{cases} \tag{17}$$

From the lemma we get that when the character χ is ramified, the Mellin transform $\hat{I}_r(\chi)$ for $r \geq 1$ is:

$$\hat{I}_r(\chi) = 0, \quad \text{if } r \geq 2 \tag{18}$$

$$\hat{I}_1(\chi) = -q(1 - q^{-2})^{-1}\int_{t+\bar{t}=0} \chi\left(\frac{2}{t - 1}\right) dt \times \left[\int_{|y|=q^{2m}} \psi(y)\chi^{-1}(y)\, dy\right]^2. \tag{19}$$

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4.3. In this section we prove Lemma 2. Let us set $E = F(\tau)$ with $|\tau| = 1$. Let $z = \alpha+\beta\sqrt{\tau}$ with α and β in F. Denote the character $\chi(z)|z|^{-1}$ by $\chi'(z)$, then the left hand side of (17) is:

$$\int \chi'\Big[(\alpha+\beta\sqrt{\tau})(1+\frac{2\alpha}{\alpha^2-\beta^2\tau})\Big]\,d\alpha\,d\beta \tag{20}$$

with $\max\{|\alpha|_F, |\beta|_F\} = q^{r-m}$.

We separate (20) into two parts, P_r^1 and P_r^2, where P_r^1 denotes the contribution of the set of the pairs (α,β) with $|\beta|_F \leq |\alpha|_F = q^{r-m}$, and P_r^2 that of the set of the pairs (α,β) with $q^{r-m} = |\beta|_F > |\alpha|_F$. For P_r^1, we change β to $\alpha\beta$, then:

$$P_r^1 = \int |\alpha|_F\chi'\Big[(1+\beta\sqrt{\tau})(\alpha+\frac{2}{1-\beta^2\tau})\Big]\,d\alpha\,d\beta \tag{21}$$

with $|\alpha|_F = q^{r-m}, |\beta| \leq 1, \alpha,\beta \in F$. Since $|1-\beta^2\tau| = 1$, another change of variable gives:

$$P_r^1 = q^{r-m}\int \chi'\Big[(\frac{2}{1-\beta\sqrt{\tau}})(1+\alpha)\Big]\,d\alpha\,d\beta \tag{22}$$

with $|\alpha|_F = q^{r-m}$, $|\beta| \leq 1$, $\alpha,\beta \in F$. Similarly, we may simplify the expression for P_r^2 and get:

$$P_r^2 = q^{r-m}\int |\alpha|_F\chi'\Big[\Big(\frac{2}{1-\alpha^{-1}\sqrt{\tau}}\Big)(1+\beta)\Big]\,d\alpha\,d\beta \tag{23}$$

with $|\alpha\beta|_F = q^{r-m}$, $|\alpha| < 1$, $\alpha,\beta \in F$.

Denote by m' the conductor of the character $\chi|_{F^\times}$; set $m' = 0$ when this restriction gives an unramified character of $F^\times$. To continue, we first state some lemmas.

LEMMA 3. *If χ is a character of $F^\times$ ramified with conductor m', then:*

$$\int_{|\beta|_F=q^s,\beta\in F} \chi(1+\beta)\,d\beta = \begin{cases} (1-q^{-1})q^s & \text{if } s \leq -m' \\ -q^{-m'} & \text{if } s = -m'+1 \\ 0 & \text{if } s > -m'+1. \end{cases} \tag{24}$$

LEMMA 4. *With the above notations, if $0 < m' \leq m$, then for any integer $s > m-m'$:*

$$\int_{|\beta|_F=q^s} \chi(1+\beta\sqrt{\tau})\,d\beta = 0. \tag{25}$$

PROOF. We take a $\gamma' \in F$ such that $\chi(1+\gamma') \neq 1$, $|\gamma'|_F = q^{1-m'}$ and $|1+\gamma'|_F = 1$. Then for $s > m-m'$, we have:

$$\int_{|\beta|=q^s} \chi(1+\gamma')\chi(1+\beta\sqrt{\tau})\,d\beta = \int_{|\beta|=q^s} \chi\big(1+(1+\gamma')\beta\sqrt{\tau}\big)\,d\beta.$$

Using the change of variable $\beta \longmapsto (1+\gamma')^{-1}\beta$, we see that the right hand side equals:

$$\int_{|\beta|=q^s} \chi(1+\beta\sqrt{\tau})\,d\beta.$$

From the fact that $\chi(1+\gamma') \neq 1$, we derive our assertion. ■

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LEMMA 5. *With the above notations,*

(1) If $0 < m' < m$, *then for any integer* s *satisfying* $-m < s < m - m'$:

$$\int_{|\beta|_F \le q^s} \chi(1+\beta\sqrt{\tau})\, d\beta = 0. \tag{26}$$

(2) If $m' = 0$, $m > 1$, *then for any integer* s *satisfying* $-m < s < m - 1$, *the same identity holds.*

PROOF. We observe when $m' < m$ and $m > 1$, there exists a $\gamma \in E$ such that $\gamma + \bar{\gamma} = 0$, $|\gamma| = q^{2-2m}$ and $\chi(1+\gamma) \neq 1$. Let γ be as such, then:

$$\begin{aligned}\int_{|\beta|_F \le q^s} \chi[(1+\gamma)(1+\beta\sqrt{\tau})]\, d\beta &= \int_{|\beta|_F \le q^s} \chi(1+\gamma\beta\sqrt{\tau}+\gamma+\beta\sqrt{\tau})\, d\beta \\ &= \int_{|\beta|_F \le q^s} \chi\Big(1+\frac{\gamma+\beta\sqrt{\tau}}{1+\gamma\beta\sqrt{\tau}}\Big)\, d\beta.\end{aligned}$$

After the change of variable

$$\beta \longmapsto \frac{\beta(1-\gamma^2)}{1-\gamma\beta\sqrt{\tau}} - \frac{\gamma}{\sqrt{\tau}}$$

the right hand side equals:

$$\int_{|\beta|_F \le q^s} \chi(1+\beta\sqrt{\tau})\, d\beta.$$

Then the fact that $\chi(1+\gamma) \neq 0$ implies our assertions. ■

From the above lemma, one easily derives the following lemma:

LEMMA 6. *With the above notations,*

(1) If $0 < m' < m$, *then for any integer* s *satisfying* $2 - m \le s < m - m'$:

$$\int_{|\beta|_F = q^s} \chi(1+\beta\sqrt{\tau})\, d\beta = 0. \tag{27}$$

(2) If $m' = 0$, $m > 1$, *then for any integer* s *satisfying* $2 - m \le s < m - 1$, *the same identity holds.*

(3) In both cases, we have:

$$\int_{|\beta|_F = q^{1-m}} \chi(1+\beta\sqrt{\tau})\, d\beta = -q^{-m}. \tag{28}$$

We will also make use of the following identity true for any $s < 0$:

$$\int_{|\alpha|_F = q^s} \chi'\Big(\frac{2}{1-\alpha^{-1}\sqrt{\tau}}\Big)\, d\alpha = \int_{|\alpha|_F = q^{-s}} \chi\Big(\frac{2}{1-\alpha\sqrt{\tau}}\Big)\, d\alpha. \tag{29}$$

Now we proceed to prove Lemma 2 case by case.

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First, let us assume $m = m'$. From Lemma 3, it follows easily that $P_r^2 = 0$, and P_r^1 is not zero only if $r = 1$; in which case we have:

$$P_1^1 = -q^{1-2m} \int_{|\beta|_F \le 1} \chi'\Big(\frac{2}{1-\beta\sqrt{\tau}}\Big) d\beta$$

$$= -q^{1-2m} \int_{t+\bar{t}=0, |t|\le 1} \chi\Big(\frac{2}{1-t}\Big)\, dt. \tag{30}$$

Applying Lemma 4 to the case at hand ($m = m'$), we get:

$$\int_{t+\bar{t}=0,|t|>1} \chi\Big(\frac{2}{1-t}\Big)\, dt = 0.$$

This identity together with (30) prove Lemma 2 in the case $m = m'$.

We now turn to the case $m' < m$. In the case when $0 < m' < m$, applying Lemma 5(1) with $s = 0$ we see $P_r^1 = 0$; Lemma 6(1) and Lemma 4 allow us to impose the restriction $|\alpha|_F = q^{m'-m}$ on the domain of integration for P_r^2; thus:

$$P_r^2 = q^{r-2m+m'} \int_{|\beta|_F = q^{r-m'}, |\alpha|_F = q^{m'-m}} \chi'\Big[\Big(\frac{2}{1-\alpha^{-1}\sqrt{\tau}}\Big)(1+\beta)\Big]\, d\alpha\, d\beta.$$

By Lemma 3 the integral is zero unless $r = 1$; in which case it is:

$$P_1^2 = -q^{1-2m} \int_{|\alpha|_F = q^{m'-m}} \chi'\Big(\frac{2}{1-\alpha^{-1}\sqrt{\tau}}\Big)\, d\alpha$$

$$= -q^{1-2m} \int_{|\alpha|_F = q^{m-m'}} \chi\Big(\frac{2}{1-\alpha\sqrt{\tau}}\Big)\, d\alpha.$$

Applying Lemma 4 and Lemma 6(1) again, we see the restriction on the domain can be removed. If we set $t = \alpha\sqrt{\tau}$, the above expression becomes the right hand side of (17).

In the case when $m' = 0$, $m \ge 2$, applying Lemma 5(2) with $s = 0$, we get $P_r^1 = 0$. We first consider the case when $r \ge 2$, then according to Lemma 6(2), P_r^2 has the form:

$$q^{r-m} \int_{|\alpha\beta|_F = q^{r-m}, |\alpha|_F \le q^{1-m}} |\alpha|_F \chi'\Big[\Big(\frac{2}{1-\alpha^{-1}\sqrt{\tau}}\Big)(1+\beta)\Big]\, d\alpha\, d\beta$$

$$= q^{r-m} \int_{|\alpha\beta|_F = q^{r-m}, |\alpha|_F \le q^{1-m}} |\alpha|_F \chi'\Big(\frac{2\beta}{1-\alpha^{-1}\sqrt{\tau}}\Big)\, d\alpha\, d\beta.$$

By the change of variable $\beta \longmapsto \beta\alpha^{-1}$, we get:

$$P_r^2 = q^{r-m} \int_{|\beta|_F = q^{r-m}, |\alpha|_F \le q^{1-m}} \chi'\Big(\frac{2\beta\sqrt{\tau}}{\alpha\sqrt{\tau}-\tau}\Big)\, d\alpha\, d\beta.$$

Applying Lemma 5(2) with $s = 1 - m$, we see this integral is 0. For the case when $r = 1$, from Lemma 6(2) we get:

$$P_1^2 = q^{1-m} \int_{|\alpha\beta|_F = q^{1-m}, |\alpha|_F \le q^{1-m}} |\alpha|_F \chi'\Big[\Big(\frac{2}{1-\alpha^{-1}\sqrt{\tau}}\Big)(1+\beta)\Big]\, d\alpha\, d\beta$$

$$= q^{1-m} \int_{|\alpha\beta|_F = q^{1-m}, |\alpha|_F \le q^{-m}} |\alpha|_F \chi'\Big[\Big(\frac{2}{1-\alpha^{-1}\sqrt{\tau}}\Big)(1+\beta)\Big]\, d\alpha\, d\beta \tag{31}$$

$$+ q^{1-m} \int_{|\beta|_F = 1, |\alpha|_F = q^{1-m}} |\alpha|_F \chi'\Big[\Big(\frac{2}{1-\alpha^{-1}\sqrt{\tau}}\Big)(1+\beta)\Big]\, d\alpha\, d\beta. \tag{32}$$

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The expression (32) can be written as

$$(33)\, q^{1\ \ m}\int_{|\beta|_F<1\ |\alpha|_F=q^{1\ \ m}} |\alpha|_F \chi'\Big[\Big(\frac{2}{1-\alpha^{\ \ 1}\sqrt{\tau}}\Big)(1+\beta)\Big]\,d\alpha\,d\beta$$

$$(34)\qquad -q^{1\ \ m}\int_{|\beta|_F<1\ |\alpha|_F=q^{1\ \ m}} |\alpha|_F \chi'\Big[\Big(\frac{2}{1-\alpha^{\ \ 1}\sqrt{\tau}}\Big)(1+\beta)\Big]\,d\alpha\,d\beta$$

For (33), we change $\beta \longmapsto \beta - 1$ and get

$$(35)\quad q^{1\ \ m}\int_{|\beta|_F=1\ |\alpha|_F=q^{1\ \ m}} |\alpha|_F \chi'\Big(\frac{2\beta}{1-\alpha^{-1}\sqrt{\tau}}\Big)\,d\alpha\,d\beta$$

$$(36)\qquad +q^{1\ \ m}\int_{|\beta|_F<1\ |\alpha|_F=q^{1\ \ m}} |\alpha|_F \chi'\Big(\frac{2\beta}{1-\alpha^{-1}\sqrt{\tau}}\Big)\,d\alpha\,d\beta$$

Using the argument in the $r \geq 2$ case, we see the sum of (31) and (35) is 0 As for (36), we change $\beta \longmapsto \beta\alpha^{-1}$ and get

$$q^{1\ \ m}\int_{|\beta|_F\leq q^{\ \ m}\ |\alpha|_F=q^{1\ \ m}} \chi'\Big(\frac{2\beta}{\alpha-\sqrt{\tau}}\Big)\,d\alpha\,d\beta$$

Applying Lemma 6(3), the above expression equals

$$(37)\qquad -q^{1\ \ 2m}\int_{|\beta|_F<q^{\ \ m}} \chi'\Big(\frac{2\beta}{-\sqrt{\tau}}\Big)\,d\beta$$

As for (34), it equals

$$(38)\qquad -q^{1\ \ 2m}\int_{|\alpha|_F=q^{1\ \ m}} \chi'\Big(\frac{2}{1-\alpha^{\ \ 1}\sqrt{\tau}}\Big)\,d\alpha$$

Thus P_1^2 is the sum of (37) and (38) Note when $m'=0, m\geq 2$, we may apply Lemma 5(2) to the case $s = m-2$ and get

$$\begin{aligned}\int \chi(\frac{2}{1-t})\,dt &= \int_{|\alpha|_F>q^{m\ \ 1}} \chi\Big(\frac{2}{1-\alpha\sqrt{\tau}}\Big)\,d\alpha\\ &= \int_{|\alpha|_F-q^{1\ \ m}} \chi'\Big(\frac{2}{1-\alpha^{-1}\sqrt{\tau}}\Big)\,d\alpha + \int_{|\beta|_F<q^{\ \ m}} \chi'\Big(\frac{2}{1-\beta^{\ \ 1}\sqrt{\tau}}\Big)\,d\beta\\ (39)\qquad &= \int_{|\alpha|_F=q^{1\ \ m}} \chi'\Big(\frac{2}{1-\alpha^{\ \ 1}\sqrt{\tau}}\Big)\,d\alpha + \int_{|\beta|_F<q^{\ \ m}} \chi'\Big(\frac{2\beta}{-\sqrt{\tau}}\Big)\,d\beta\end{aligned}$$

Thus in the present case, Lemma 2 follows from comparing (37) and (38) with (39)

For the final case, $m = 1, m' = 0$, the following simple lemma plays an important role

LEMMA 7 *With the above notations, if $m = 1, m' = 0$, then*

$$(40)\qquad -\int_{|\beta|_F<1} \chi'\Big(\frac{2}{1-\beta\sqrt{\tau}}\Big)\,d\beta = \int_{|\alpha|_F<1} \chi'\Big(\frac{2}{\alpha-\sqrt{\tau}}\Big)\,d\alpha = q^{\ \ 1}\chi'\Big(-\frac{2}{\sqrt{\tau}}\Big)$$

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PROOF. Using the fact that $\int_{|z|=1} \chi'(2/z)\,dz = 0$, we get:

$$\int_{|\alpha|_F<1,|\beta|_F=1} \chi'\Big(\frac{2}{\alpha-\beta\sqrt{\tau}}\Big)\,d\alpha\,d\beta + \int_{|\alpha|_F=1,|\beta|_F\le 1} \chi'\Big(\frac{2}{\alpha-\beta\sqrt{\tau}}\Big)\,d\alpha\,d\beta = 0.$$

Since $\chi'|_F$ is unramified, we may derive:

$$(1-q^{-1})\int_{|\alpha|_F<1} \chi'\Big(\frac{2}{\alpha-\sqrt{\tau}}\Big)\,d\alpha + (1-q^{-1})\int_{|\beta|_F\le 1} \chi'\Big(\frac{2}{1-\beta\sqrt{\tau}}\Big)\,d\beta = 0.$$

This gives the first identity in the lemma, the second is trivial. ∎

Thus in this case, we have:

$$P_r^1 = -q^{r-2}\chi'\Big(-\frac{2}{\sqrt{\tau}}\Big)\int_{|\alpha|_F=q^{r-1}} \chi'(1+\alpha)\,d\alpha \tag{41}$$

$$P_r^2 = q^{r-1}\int_{|\alpha\beta|_F=q^{r-1},|\alpha|_F<1} |\alpha|_F\chi'\Big(-\frac{2}{\alpha^{-1}\sqrt{\tau}}\Big)\chi'(\beta)\,d\alpha\,d\beta. \tag{42}$$

In (42) we make the change of variable $\beta \longmapsto \beta\alpha^{-1}$ and get:

$$\begin{aligned} P_r^2 &= q^{r-1}\int_{|\beta|_F=q^{r-1},|\alpha|_F<1} \chi'\Big(-\frac{2\beta}{\sqrt{\tau}}\Big)\,d\alpha\,d\beta \\ &= q^{r-2}\chi'\Big(-\frac{2}{\sqrt{\tau}}\Big)\int_{|\beta|_F=q^{r-1}} \chi'(\beta)\,d\beta. \end{aligned} \tag{43}$$

In the case when $r>1$, clearly $P_r^1+P_r^2=0$. For the case $r=1$, the sum gives:

$$\begin{aligned} &q^{-1}\chi'\Big(-\frac{2}{\sqrt{\tau}}\Big)\Big[\int_{|\alpha|_F=1} \chi'(\alpha)\,d\alpha - \int_{|\alpha|_F=1} \chi'(1+\alpha)\,d\alpha\Big] \\ &\quad = q^{-1}\chi'\Big(-\frac{2}{\sqrt{\tau}}\Big)\Big[-\int_{|\alpha|_F<1} \chi'(\alpha)\,d\alpha + \int_{|\alpha|_F<1} \chi'(1+\alpha)\,d\alpha\Big] \\ &\quad = q^{-1}\chi'\Big(-\frac{2}{\sqrt{\tau}}\Big)\Big[q^{-1} - \int_{|\alpha|_F<1} \chi'(\alpha)\,d\alpha\Big]. \end{aligned} \tag{44}$$

Meanwhile, it follows from Lemma 7 that when $m=1$ and $m'=0$, we have:

$$\begin{aligned} \int \chi\Big(\frac{2}{1-t}\Big)\,dt &= \int_{|\alpha|_F\le 1} \chi\Big(\frac{2}{1-\alpha\sqrt{\tau}}\Big)\,d\alpha + \int_{|\alpha|_F>1} \chi\Big(\frac{2}{1-\alpha\sqrt{\tau}}\Big)\,d\alpha \\ &= -q^{-1}\chi\Big(-\frac{2}{\sqrt{\tau}}\Big) + \chi\Big(-\frac{2}{\sqrt{\tau}}\Big)\int_{|\alpha|_F>1} \chi^{-1}(\alpha)\,d\alpha. \end{aligned}$$

Changing $\alpha \longmapsto \alpha^{-1}$, we get:

$$\int \chi\Big(\frac{2}{1-t}\Big)\,dt = -q^{-1}\chi'\Big(-\frac{2}{\sqrt{\tau}}\Big) + \chi'\Big(-\frac{2}{\sqrt{\tau}}\Big)\int_{|\alpha|_F<1} \chi'(\alpha)\,d\alpha. \tag{45}$$

Lemma 2 in this case now follows from comparing (44) with (45). ∎

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4.4. If χ is unramified, then we may impose $|x| \leq q^2$ and $|y| \leq q^2$. We decompose the domain into four parts. We write:

$$\hat{I}_r(\chi) = M_r^1 + M_r^2 + M_r^3 + M_r^4$$

where M_r^i is the same integral as $\hat{I}_r$, but with an extra restriction on its domain of integration:

$$\text{if } i = 1 \quad |x| = q^2 \quad |y| = q^2 \tag{46}$$

$$\text{if } i = 2 \quad |x| = q^2 \quad |y| \leq 1 \tag{47}$$

$$\text{if } i = 3 \quad |x| \leq 1 \quad |y| = q^2 \tag{48}$$

$$\text{if } i = 4 \quad |x| \leq 1 \quad |y| \leq 1. \tag{49}$$

We certainly have $M_r^3 = M_r^2$. We first consider M_r^3:

$$M_r^3 = -X^{-1} \sum_{s=0}^{\infty} X^s(1 - q^{-2})q^{2-4s} \int \chi(a)\, dz\, d^\times a. \tag{50}$$

After throwing out the vacuous conditions, we find the restrictions for the above integral are:

$$|a| \leq q^{2r-2s}, \quad |az| \leq q^{2r-2}, \quad |az - az\bar{z} - 1 + \bar{a}^{-1}| \leq q^{2r-2+2s}.$$

Changing z to $a^{-1}z$, we get:

$$M_r^3 = -X^{-1} \sum_{s=0}^{\infty} X^s(1 - q^{-2})q^{2-4s} \int |a|^{-1} \chi(a)\, dz\, d^\times a$$

with domain of integration:

$$|a| \leq q^{2r-2s}, \quad |z| \leq q^{2r-2}, \quad |a|^{-1}|\bar{a}(z-1) - z\bar{z} + 1| \leq q^{2r+2s-2}.$$

Note the last condition is equivalent to $|a|^{-1}|1 - z\bar{z}| \leq q^{2r+2s-2}$.

We separate the domain into two parts according to whether $|a| \geq q^{2-2r-2s}$ or not. The part with $|a| \geq q^{2-2r-2s}$ contributes:

$$\sum_{l=1-s-r}^{r-s} X^l q^{-2l} \int_{|z| \leq q^{2r-2},\, |z\bar{z}| \leq q^{2(r+s+l-1)}} dz = \sum_{l=0}^{2r-1} X^{l+1-r-s} q^{-2(l+1-r-s)} q^{2[l/2]}$$

here $[x]$ denote the integral part of a real number x. The part $|a| < q^{2-2r-2s}$ contributes:

$$\sum_{l=-s-r}^{-\infty} X^l q^{-2l} \int_{|1-z\bar{z}| \leq q^{2(r+s+l-1)}} dz = \sum_{l=-1}^{-\infty} X^{l+1-r-s} q^{-2(l+1-r-s)} q^l (1 + q^{-1}).$$

By adding the above two terms and summing over s, we get the following expression for M_r^3:

$$M_r^2 = M_r^3 = -X^{-r} q^{2r} \left[\sum_{l=0}^{2r-1} q^{2[-l/2]} X^l + \sum_{l=-1}^{-\infty} (1 + q^{-1}) q^{-l} X^l \right]. \tag{51}$$

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Now we turn to the computation of M_r^4. After changing z to $a^{-1}z$, we find the contribution of the subset with $|x| = q^{-2s}$ and $|y| = q^{-2t}$ is:

$$X^s(1-q^{-2})q^{-4s}X^t(1-q^{-2})q^{-4t}\int |a|^{-1}\chi(a)\,dz\,d^\times a \tag{52}$$

where the integral is over the set:

$$|a| \le q^{2r-2s-2t}, \quad |z| \le q^{2r}, \quad |\bar{a}(1-z)+1-z\bar{z}| \le |a|q^{2r+2s+2t}.$$

The last inequality may be replaced by $|1-z\bar{z}| \le q^{2r+2s+2t}|a|$. Again we separate the domain into two parts according to whether $q^{2r+2s+2t}|a| \ge 1$ or not. A similar argument shows that:

$$M_r^4 = X^{-r}q^{2r}\Big[\sum_{l=0}^{2r} q^{2[-l/2]}X^l + \sum_{l=-1}^{-\infty}(1+q^{-1})q^{-l}X^l\Big]. \tag{53}$$

Finally we deal with M_r^1. After changing z to $a^{-1}z$, we have:

$$M_r^1 = X^{-2}q^4\int |a|^{-1}\chi(a)\,dz\,d^\times a \tag{54}$$

with

$$|z| \le q^{2r-2}, \quad |a| \le q^{2r-2}, \quad |\bar{a}(z-1)+1-z\bar{z}| \le |a|q^{2r-4}. \tag{55}$$

We have noticed in the ramified case that the result for $r = 1$ is different from the result for $r > 1$. Here we even have to compute M_1^1 separately. We first treat the easier case $r \ge 2$.

We again separate the domain into two parts. For the subset with $|z| \le q^{2r-4}$, the third condition in (55) is equivalent to $|1-z\bar{z}| \le |a|q^{2r-4}$. We may proceed just as before; the contribution of this set is:

$$X^{-r}q^{2r}\Big[\sum_{l=0}^{2r-3} q^{2[-l/2]}X^l + \sum_{l=-1}^{-\infty}(1+q^{-1})q^{-l}X^l\Big]. \tag{56}$$

On the subset with $|z| = q^{2r-2}$, we have $|a| = |z|$ and this set contributes:

$$X^{-2}q^4\int |a|^{-1}\chi(a)\,dz\,d^\times a$$

with

$$|a| = |z| = q^{2r-2}, \quad |a-z| \le q^{2r-4}.$$

A simple computation shows it is:

$$q^2X^{r-3}. \tag{57}$$

We still have M_1^1 left. We first consider the contribution from the subset with $|1-z| \le q^{-2}$. With this restriction, the inequality $|\bar{a}(1-z)+1-z\bar{z}| \le |a|q^{-2}$ in (55) simplifies to $|1-z\bar{z}| \le |a|q^{-2}$; thus this subset contributes:

$$\begin{aligned} X^{-2}q^4\int |a|^{-1}\chi(a)\,d^\times a\,dz &= X^{-2}q^4\sum_{s=0}^{-\infty} q^{s-2}q^{-2s}X^s \\ &= X^{-2}q^2(1-qX^{-1})^{-1}. \end{aligned} \tag{58}$$

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To find the contribution of the subset with $|z-1| = 1$, we change z to $z+1$; the integral over this subset becomes:

$$\int_{z \in R_E^\times} |a|^{-1} \chi(a)\, dz\, d^\times a$$

with

$$|a| \leq 1, \quad |\bar{a}z - z\bar{z} - z - \bar{z}| \leq |a| q^{-2}.$$

The second inequality gives:

$$a = \frac{z + \bar{z} + z\bar{z}}{\bar{z}}(1 + u\varpi)$$

with $u \in R_E$. Thus the above integral equals:

$$q^{-2}(1-q^{-2})^{-1} \int_{z \in R_E^\times} \chi'\Big(\frac{z+\bar{z}+z\bar{z}}{\bar{z}}\Big)\, dz.$$

Let $z = \alpha + \beta\sqrt{\tau}$. We again separate the above integral into two parts according to whether $|\alpha| < 1$ or not. Over the set with $|\alpha| < 1$, we have $|z + \bar{z} + z\bar{z}| = |z|$; therefore this part gives:

$$q^{-2}(1-q^{-2})^{-1} \int_{|\alpha|<1, |\beta|=1} 1\, d\alpha\, d\beta = q^{-2}(1-q^{-2})^{-1} q^{-1}(1-q^{-1}). \tag{59}$$

For the part $|\alpha| = 1$ and $|\beta| \leq 1$, we change β to $\beta\alpha$ and get the contribution from this subset:

$$\begin{aligned} &q^{-2}(1-q^{-2})^{-1} \int_{|\alpha|=1, |\beta| \leq 1} \chi'\Big(\frac{2}{\alpha} + 1 - \beta^2\tau\Big)\, d\alpha\, d\beta \\ &\qquad = q^{-2}(1-q^{-2})^{-1}\Big[(1-q^{-1})\frac{qX^{-1}}{1-qX^{-1}} + 1 - 2q^{-1}\Big]. \end{aligned} \tag{60}$$

Combining the results in (58), (59) and (60), we get our last term:

$$M_1^1 = X^{-2} q^2 (1-qX^{-1})^{-1}(1-q^{-2})^{-1}(2 - q^{-1} - 2q^{-2} + q^{-1}X^{-1}). \tag{61}$$

Finally, we write down the expression for $\hat{I}_r(\chi)$ when χ is unramified.

If $r \geq 2$, by (51), (53), (56) and (57), we have:

$$\hat{I}_r(\chi) = q^2 X^{r-3} + X^r - X^{r-1} - q^2 X^{r-2}. \tag{62}$$

If $r = 1$, then the equations (51), (53) and (61) give us:

$$\hat{I}_1(\chi) = (1 - X^{-1})(X - X^{-1}q^2) \tag{63}$$

$$\qquad - (1-qX^{-1})^{-1}(1-q^{-2})^{-1}(1-X^{-1}) q^3 X^{-2}. \tag{64}$$

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5. **Conclusion.** In Sections 3 and 4, we computed the Mellin transforms of $J(a, \Phi'_r)$ and $I(a, \Phi^r)$. To prove that $J(a, \Phi'_r - q\Phi'_{r-1}) = I(a, \Phi^r)$, all we need to show ıs for arbitrary χ, and for all nonnegative ıntegers r:

$$\hat{J}_r(\chi) - q\hat{J}_{r-1}(\chi) = \hat{I}_r(\chi). \tag{1}$$

The case $r = 0$ ıs treated ın [7]. For $r > 0$, if the character χ is ramıfied, (1) follows from the equations (3.10), (3.11), (4.18) and (4.19). If χ is unramıfied, when $r \geq 2$, by (3.12) and (4.62) the equality (1) holds. If χ is unramified and $r = 1$ our equality follows from the equations (3.12), (3.13), (4.63) and (4.64).

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