

**ON THE LEAST DEGREE OF POLYNOMIALS BOUNDING
ABOVE THE DIFFERENCES BETWEEN LENGTHS AND
MULTIPLICITIES OF CERTAIN SYSTEMS OF
PARAMETERS IN LOCAL RINGS**

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§ 1. Introduction

Let A be a commutative local Noetherian ring with the maximal ideal \mathfrak{m} and M a finitely generated A -module, $d = \dim M$. It is well-known that the difference between the length and the multiplicity of a parameter ideal \mathfrak{q} of M

$$I_M(\mathfrak{q}) = l(M/\mathfrak{q}M) - e(\mathfrak{q}; M)$$

gives a lot of informations on the structure of the module M . For instance, M is a *Cohen-Macaulay* (CM for short) module if and only if $I_M(\mathfrak{q}) = 0$ for some parameter ideal \mathfrak{q} or M is *Buchsbaum* module (see [S-V]) if and only if $I_M(\mathfrak{q})$ is a constant for all parameter ideals \mathfrak{q} of M . In this note we shall investigate this difference, but in a more general situation as follows: Let $x = \{x_1, \dots, x_d\}$ be a system of parameters (s.o.p. for short) on M and $n = (n_1, \dots, n_d)$ a d -tuple of positive integers. We consider the difference

$$I_M(n; x) = l(M/(x_1^{n_1}, \dots, x_d^{n_d})M) - n_1 \cdots n_d e(x; M)$$

as a function in n . In general, this function is not a polynomial in n , even in the case $n_1 = n_2 = \dots = n_d = t$ (see [G-K]). The necessary and sufficient conditions in term of x , for this function to be a polynomial, have been examined in [C₁]. Here we shall show that the least degree of all polynomials in n bounding above $I_M(n; x)$ is independent of the choice of x (Theorem 2.3). This numerical invariant of M will be called the *polynomial type* of M . The aim of this note is to study the polynomial type of a module over a local ring. In Section 2 we define the polynomial

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type of a module and give some properties of this invariant. Using the local cohomology modules of M and the notion of reducing sequence [A-B], we give in Section 3 upper and under bounds for the polynomial type of M . In Section 4 we improve these bounds in the case that A is a factor ring of a CM ring. In particular, if M is equidimensional we can show that the polynomial type of M is just the dimension of the non-Cohen-Macaulay locus of M in $\text{Supp } M$. In the last section we examine the behaviour of the polynomial types by a flat extension.

§ 2. The polynomial type of a module

Throughout this note we denote by A a commutative local Noetherian ring with the maximal ideal \mathfrak{m} and by M a finitely generated A -module with $\dim M = d$.

We begin with the following lemma.

LEMMA 2.1. *Let $x = \{x_1, \dots, x_d\}$ be an s.o.p. on M and $n = (n_1, \dots, n_d)$ a d -tuple of positive integers. Then*

$$l(M/(x_1^{n_1}, \dots, x_d^{n_d})M) \leq n_1 \cdots n_d l(M/(x_1, \dots, x_d)M).$$

Proof. The lemma is proved in [G] for the case $d = 1$. Put

$$E = M/(x_1^{n_1})M \quad \text{and} \quad F = M/(x_2, \dots, x_d)M.$$

Then by induction on d we get

$$\begin{aligned} l(M/(x_1^{n_1}, \dots, x_d^{n_d})M) &= l(E/(x_2^{n_2}, \dots, x_d^{n_d})E) \\ &\leq n_2 \cdots n_d l(E/(x_2, \dots, x_d)E) = n_2 \cdots n_d l(F/(x_1^{n_1})F) \\ &\leq n_1 \cdots n_d l(M/(x_1, \dots, x_d)M) \quad \text{as required.} \end{aligned}$$

From now on, for an s.o.p. $x = \{x_1, \dots, x_d\}$ of M we set

$$I_M(n; x) = l(M/(x_1^{n_1}, \dots, x_d^{n_d})M) - n_1 \cdots n_d e(x; M).$$

In particular, we also set $I_M(x) = I_M(n; x)$ if $n_1 = \dots = n_d = 1$.

COROLLARY 2.2. $I_M(n; x) \leq n_1 \cdots n_d I_M(x)$.

The corollary 2.2 shows that if we consider $I_M(n; x)$ as a function in n then this function is bounded above by the polynomial $n_1 \cdots n_d I_M(x)$. In general we can show the following theorem.

THEOREM 2.3. *Let $x = \{x_1, \dots, x_d\}$ be an s.o.p. on M . Then the least*

degree of all polynomials in n bounding above $I_M(n; x)$ is independent of the choice of x .

Proof (see [C₂], Proposition 2.5). Let $\underline{t} = (t, \dots, t)$ be a d -tuple of the same integers t . Then by [G], Theorem 6, the least degree of all polynomials in t bounding above $I_M(\underline{t}; x)$ is independent of the choice of x . Denote this invariant by $p'(M)$ and by $p(x)$ the least degree of all polynomials in n bounding above $I_M(n; x)$. It is clear that $p'(M) \leq p(x)$. On the other hand, we can easily verify that $I_M(\underline{t}; x) \geq I_M(n; x)$ if $t \geq n_i, i = 1, \dots, d$. It follows that $p'(M) \geq p(x)$. Thus $p(x) = p'(M)$ is independent of the choice of x .

DEFINITION 2.4. The numerical invariant of M given in Theorem 2.3 is called the *polynomial type* of M and we denote it by $p(M)$.

Remark 2.5. (i) If we stipulate that the degree of the zero-polynomial is equal -1 . Then M is a CM module if and only if $p(M) = -1$.

(ii) If $I_M(n; x)$ is a constant for some s.o.p. x on M and for n_1, \dots, n_d sufficiently large then M is called a generalized CM module (see [C-S-T]). Therefore M is a generalized CM module if and only if $p(M) \leq 0$.

(iii) From the limit formula of Lech

$$\lim_{\min(n_i) \rightarrow \infty} (n_1 \cdots n_d)^{-1} l(M/(x_1^{n_1}, \dots, x_d^{n_d})M) = e(x_1, \dots, x_d; M),$$

we easily deduce that $p(M) \leq \dim M - 1$.

LEMMA 2.6. Let \hat{M} be the \mathfrak{m} -adic completion of M then $p(M) = p(\hat{M})$.

Proof. If $x = \{x_1, \dots, x_d\}$ is an s.o.p. on M then x is also an s.o.p. on \hat{M} . The lemma follows from the fact that

$$l(M/(x_1, \dots, x_d)M) = l(\hat{M}/(x_1, \dots, x_d)\hat{M}) \quad \text{and} \quad e(x; M) = e(x; \hat{M}).$$

§ 3. Bounds of $p(M)$

First of all we need some notations as follows.

We denote by α_i the annihilator of the i -th local cohomology module $H_{\mathfrak{m}}^i(M)$ of M with respect to the maximal ideal \mathfrak{m} , and we set

$$\alpha(M) = \alpha_0(M) \cdots \alpha_{d-1}(M).$$

A subset of an s.o.p. x_1, \dots, x_j of M is called a *reducing sequence* if the following condition holds: $x_i \notin P$ for all $P \in \text{Ass}(M/(x_1, \dots, x_{i-1})M)$ with $\dim(A/P) \geq d - i, i = 1, \dots, j$. Note that if $x = \{x_1, \dots, x_d\}$ is an s.o.p.

on M and x_1, \dots, x_{d-1} form a reducing sequence, then x is just a reducing s.o.p. which has been introduced in [A-B]. We set

$$r(M) = \inf\{k/\text{every subset of an s.o.p. of } M \text{ having } (d - k - 1)\text{-elements is a reducing sequence of } M\}.$$

Finally, we denote by $NC(M)$ the *non-Cohen-Macaulay locus* of M , i.e. $NC(M) = \{P \in \text{Supp } M/ M_P \text{ is not a CM module}\}$. The following theorem, which will be often used in this paper, is one of the main results of [C₂].

THEOREM 3.1. *Suppose that A has a dualizing complex. Then*

- (i) $p(M) = r(M) = \dim(A/\alpha(M))$;
- (ii) *if M is equidimensional then $p(M) = \dim(NC(M))$.*

Example 3.2. Ferrand and Raynaud [F-R] have constructed a two-dimensional local integral domain (R, \mathfrak{m}) such that the \mathfrak{m} -adic completion \hat{R} has a one-dimensional associated prime ideal. Thus R is not a generalized CM module; it follows that $p(R) = 1$. But, it is easy to see that $NC(R) = \{\mathfrak{m}\}$, $r(R) = 0$ and $\dim(R/\alpha(R)) = 2$. So we get inequalities

$$\dim(R/\alpha(R)) > p(R) > \dim(NC(R)) = r(R).$$

Thus, in the general case, Theorem 3.1 does not hold without the assumption that A has a dualizing complex. However, we have the following theorem.

THEOREM 3.3. *With the previous notations it holds*

- (i) $\dim(A/\alpha(M)) \geq p(M) \geq r(M)$;
- (ii) *If $NC(M)$ is closed then $p(M) \geq r(M) \geq \dim(NC(M))$.*

Proof. (i) We denote by \hat{A} and \hat{M} the \mathfrak{m} -adic completion of A and M , respectively. Then it is obvious that $\alpha(M)\hat{A} \subseteq \alpha(\hat{M})$. Therefore by Theorem 3.1 and Lemma 2.6.

$$\dim(A/\alpha(M)) = \dim(\hat{A}/\alpha(M)\hat{A}) \geq \dim(\hat{A}/\alpha(\hat{M})) = p(\hat{M}) = p(M).$$

Let $p(M) = k$ and $x = \{x_1, \dots, x_d\}$ be an arbitrary s.o.p. on M . By [A-B], Corollary 4.3, we have

$$I_M(n; x) = l((x_1^{n_1}, \dots, x_{d-1}^{n_{d-1}})M : x_d^{n_d}/(x_1^{n_1}, \dots, x_{d-1}^{n_{d-1}})M) + \sum_{i=1}^{d-1} e(x_i^{n_i+1}, \dots, x_d^{n_d}; (x_1^{n_1}, \dots, x_{i-1}^{n_{i-1}})M : x_i^{n_i}/(x_1^{n_1}, \dots, x_{i-1}^{n_{i-1}})M).$$

Since $I_M(n; x)$ is bounded above by a polynomial in n of degree k , therefore

$$e(x_{i+1}^{n_i+1}, \dots, x_d^{n_d}; (x_1^{n_1}, \dots, x_{i-1}^{n_{i-1}})M : x_i^{n_i} / (x_1^{n_1}, \dots, x_{i-1}^{n_{i-1}})M) = 0$$

for $i = 1, \dots, d - k - 1$. Thus by virtue of [A-B], Proposition 4.7 x_1, \dots, x_{d-k-1} is a reducing sequence; so $p(M) \geq r(M)$. For the proof of (ii) we need two auxiliary lemmas as follows. Note that these lemmas have been proved in $[C_2]$ with the assumption that A has a dualizing complex.

LEMMA 3.4. *Let x_1, \dots, x_j ($j \geq 1$) be an unconditioned reducing sequence of M (i.e. from any permutation of the sequence x_1, \dots, x_j we still get a reducing sequence). Then for every prime ideal P contained in $\text{Ass}(M/(x_1, \dots, x_j)M)$ with $\dim(A/P) = d - j$ we have*

$$\dim(M_P) + \dim(A/P) = d.$$

Proof. Suppose that $\dim(M_P) = k < j$. We can choose the order of the sequence x_1, \dots, x_j such that x_1, \dots, x_k form an s.o.p. on M_P . Since $(x_1, \dots, x_k)M_P \subseteq (x_1, \dots, x_{j-1})M_P \subseteq PM_P$, it follows that $(x_1, \dots, x_{j-1})M_P$ is PA_P -primary. Hence by [M], 7.C, $P \in \text{Ass}(M/(x_1, \dots, x_{j-1})M)$. This contradicts the assumption that x_1, \dots, x_j is a reducing sequence of M . Thus $\dim(M_P) = j$ as required.

LEMMA 3.5. *Let $P \in \text{Supp } M$ with $\dim(A/P) > r(M)$. Then M_P is a CM module and $\dim(M_P) + \dim(A/P) = d$.*

Proof. Let $P \in \text{Supp } M$ with $\dim(A/P) > r(M) = k$. Choose a subset of an s.o.p. x_1, \dots, x_j of M contained in P so that j is maximal. It is easy to verify that $\dim(A/P) = d - j$. Therefore $d - k - 1 \geq j$; so x_1, \dots, x_j is an unconditional reducing sequence of M . Hence $\dim(M_P) + \dim(A/P) = d$ by Lemma 3.4. On the other hand, x_1, \dots, x_j is also a reducing sequence on M_P . Thus by [A-B], Corollary 4.8,

$$\begin{aligned} l(M_P/(x_1, \dots, x_j)M_P) &= e((x_1, \dots, x_j)A_P; M_P) \\ &= l((x_1, \dots, x_{j-1})M_P : x_j / (x_1, \dots, x_{j-1})M_P). \end{aligned}$$

Since x_1, \dots, x_j is a reducing sequence of M we get $PA_P \notin \text{Ass}(M_P/(x_1, \dots, x_{j-1})M_P)$. Therefore $(x_1, \dots, x_{j-1})M_P : x_j = (x_1, \dots, x_{j-1})M_P$. It follows that $l(M_P/(x_1, \dots, x_j)M_P) = e((x_1, \dots, x_j)A_P; M_P)$, in other words, M_P is a CM module. The lemma is proved.

Proof of (ii). By part (i) of Theorem we only need to show that

$r(M) \geq \dim(NC(M))$. This inequality immediately follows from Lemma 3.5. The proof of Theorem 3.3 is now complete.

COROLLARY 3.6. *Let $P \in \text{Supp } M$ with $\dim(A/P) > p(M)$. Then M_P is a CM module and $\dim(M_P) + \dim(A/P) = d$.*

§ 4. The case A is the homomorphic image of a CM ring

In this section we shall improve our previous inequalities in the case that A is the homomorphic image of a CM ring. We begin with the following theorem.

THEOREM 4.1. *Let A be the homomorphic image of a CM ring and k a positive integer. Then the following conditions are equivalent:*

- (i) $p(M) \leq k$.
- (ii) Any subset of an s.o.p. of M having $(d - k - 1)$ -elements is a reducing sequence.
- (iii) For any $P \in \text{Supp } M$ with $\dim(A/P) > k$, M_P is a CM module and $\dim(M_P) + \dim(A/P) = d$.
- (iv) For any $P \in \text{Supp } M$ with $\dim(A/P) = k + 1$, M_P is a CM module and $\dim(M_P) + \dim(A/P) = d$.

Proof. By the proof of Theorem 3.3 we already get the following implications (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv). Since the CM property is stable under generalization, (iii) is equivalent to (iv). Thus we have only to show that (iii) \Rightarrow (i). As above, we denote by \hat{A} and \hat{M} the \mathfrak{m} -adic completion of A and M , respectively. Put $\dim(\hat{A}/\mathfrak{a}\hat{M}) = k'$ and suppose that $k' > k$. Let $P \in \text{Ass}(\hat{A}/\mathfrak{a}\hat{M})$ so that $\dim(\hat{A}/P\hat{A}) = k'$ and $P \cap A = \mathfrak{p}$. Then there exists a $Q \in \text{Ass}(\hat{A}/\mathfrak{p}\hat{A})$ such that $Q \subseteq P$. Therefore $Q \cap A = \mathfrak{p}$. Since $\hat{A}/\mathfrak{p}\hat{A}$ is unmixed by [N], 34.9, we get

$$\dim(A/\mathfrak{p}) = \dim(\hat{A}/\mathfrak{p}\hat{A}) = \dim(\hat{A}/Q) \leq \dim(\hat{A}/P\hat{A}) = k'.$$

Note that \hat{A} is catenary, so we deduce from 3.5 and the assumption that

$$\begin{aligned} d &= \dim(M_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) = \dim(M_{\mathfrak{p}}) + \dim(\hat{A}/Q) \\ &= \dim(M_{\mathfrak{p}}) + \dim(\hat{A}/P) + \dim(\hat{A}_P/Q\hat{A}_P) \\ &= \dim(M_{\mathfrak{p}}) + \dim(\hat{A}/P) + \dim(\hat{A}_P/\mathfrak{p}\hat{A}_P) \\ &= \dim(\hat{M}_P) + \dim(\hat{A}/P). \end{aligned}$$

On the other hand, since $\dim(\hat{A}/\mathfrak{p}) \geq k' > k$, $M_{\mathfrak{p}}$ is a CM module. So \hat{M}_P is CM because any fiber of the canonical homomorphism $A \rightarrow \hat{A}$ is CM.

Therefore

$$d = \dim(\hat{M}_P) + \dim(\hat{A}/P) = \text{depth}(\hat{M}_P) + \dim(\hat{A}/P).$$

Applying [S], Satz 2.4.6 it follows that $\alpha(\hat{M}) \not\subseteq P$ which is a contradiction. Thus $k \geq k'$. By virtue of Lemma 2.6 and Theorem 3.1 we deduce that $p(M) = p(\hat{M}) = k' \leq k$ as required.

It is well-known that if A is the homomorphic image of a CM ring then $NC(M)$ is closed in $\text{Spec } A$. Then we get the following corollary.

COROLLARY 4.2. *Suppose that A is the homomorphic image of a CM ring and M is equidimensional. Then it holds*

$$p(M) = r(M) = \dim(NC(M)).$$

Proof. Note that if A is catenary and M is equidimensional then for all $P \in \text{Supp } M$ we have $\dim(M_P) + \dim(A/P) = d$. Therefore by Theorem 4.1 it follows that $p(M) = \dim(NC(M))$. Hence $p(M) = r(M) = \dim(NC(M))$ by Theorem 3.3.

The following corollary is one of the main results in [C-S-T].

COROLLARY 4.3. *Suppose that A is the homomorphic image of a CM ring then following conditions are equivalent:*

- (i) M is a generalized CM module.
- (ii) $p(M) \leq 0$.
- (iii) Every s.o.p. on M is reducing.
- (iv) $\dim(NC(M)) = 0$ and $\dim(M_P) + \dim(A/P) = d$ for all $P \in \text{Supp } M$.

Proof. The corollary immediately follows from Theorem 4.1 and the definition of a generalized CM module.

§ 5. Flat extensions

Let $(A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ be a flat, local homomorphism. We denote by F the fiber $A/\mathfrak{m} \otimes_A B$. In this section we examine the relationship between polynomial types $p(A)$, $p(B)$ and $\dim F$ by such a flat homomorphism.

THEOREM 5.1. *Let $f : (A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ be a local, flat extension. Then:*

- (i) *If $p(B) \geq \dim F$ then $p(B) \geq p(A) + \dim F$. Moreover, the equality holds if $\alpha(A)B \subseteq \alpha(B)$.*
- (ii) *If $p(B) < \dim F$ then A is a CM ring.*

Proof. Denote \hat{A} and \hat{B} the \mathfrak{m} -adic (the \mathfrak{n} -adic) completion of $A(B)$,

respectively. Let $\hat{f}: \hat{A} \rightarrow \hat{B}$ be the induced flat homomorphism. Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ \hat{A} & \xrightarrow{\hat{f}} & \hat{B}. \end{array}$$

Then by Lemma 2.6 we have $p(A) = p(\hat{A})$, $p(B) = p(\hat{B})$. It is also easily to see that $\hat{B} \otimes_{\hat{A}} \hat{A}/\hat{\mathfrak{m}} \cong \hat{F}$, \hat{F} is the \mathfrak{n} -adic completion of F . Hence, without loss of the generality we can suppose that A and B are complete. Therefore they are homomorphic images of regular rings.

(i) Let $\mathfrak{p} \in \text{Spec } A$ such that $\dim(A/\mathfrak{p}) > p(B) - \dim F$. Since the going down Theorem holds between A and B we can choose a $P \in \text{Spec } B$ such that $\dim(B/P) = \dim(B/\mathfrak{p}B)$ and $P \cap A = \mathfrak{p}$. Therefore we get

$$\dim(B/P) = \dim(B/\mathfrak{p}B) = \dim(A/\mathfrak{p}) + \dim F > p(B).$$

By virtue of Theorem 4.1, B_P is CM and $\dim(B_P) + \dim(B/P) = \dim B$; it follows that

$$\begin{aligned} \dim B &= \dim(B_P) + \dim(B/P) = \dim(A_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) + \dim F \\ &= \dim A + \dim F. \end{aligned}$$

Therefore $\dim(A_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) = \dim A$. Hence $p(A) \leq p(B) - \dim F$ by Theorem 4.1. Suppose now that $\alpha(A)B \subseteq \alpha(B)$. Let $P \in \text{Spec } B$ such that $\alpha(B) \subseteq P$ and $\dim(B/P) = p(B)$. Note that A and B are catenary; we have for $\mathfrak{p} = P \cap A$

$$\begin{aligned} \dim(B/P) &\leq \dim(B/\mathfrak{p}B) - \dim(B_P/\mathfrak{p}B_P) \\ &= \dim(A/\mathfrak{p}) + \dim F - \dim(B_P/\mathfrak{p}B_P). \end{aligned}$$

Therefore

$$\begin{aligned} \dim(A/\mathfrak{p}) &\geq \dim(B/P) - \dim F + \dim(B_P/\mathfrak{p}B_P) \\ &= p(B) - \dim F + \dim(B_P/\mathfrak{p}B_P) \geq p(B) - \dim F. \end{aligned}$$

Since $\alpha(A)B \subseteq \alpha(B)$ it follows that $\alpha(A) \subseteq P \cap A = \mathfrak{p}$. Thus

$$p(A) = \dim(A/\alpha(A)) \geq \dim(A/\mathfrak{p}) \geq p(B) - \dim F$$

by Theorem 3.1. So we get $p(A) = p(B) - \dim F$ as required.

(ii) Let $P \in \text{Spec } B$ such that $\dim(B/P) = \dim F$ and $P \cap A = \mathfrak{m}$. As $\dim(B/P) > p(B)$, B_P is CM. Therefore A is CM by [M], 21.C. The proof of Theorem 5.1 is complete.

COROLLARY 5.2. *With the same A, B, F as in Theorem 5.1, if $\dim F = 0$ then $p(A) = p(B)$.*

Proof. Since $\dim F = 0$ we can easily show that $\alpha(A)B \subseteq \alpha(B)$. Therefore $p(A) = p(B)$ by Theorem 5.1, (i).

COROLLARY 5.3. *Let $\mathfrak{p} \in \text{Spec } A$ with $\dim(A/\mathfrak{p}) \leq p(A)$ then*

$$p(A_{\mathfrak{p}}) \leq p(A) - \dim(A/\mathfrak{p}).$$

Proof. Let \hat{A} be the \mathfrak{m} -adic completion of A . Choose a $P \in \text{Spec } \hat{A}$ such that $\dim(\hat{A}/P) = \dim(\hat{A}/\mathfrak{p}\hat{A}) = \dim(A/\mathfrak{p})$ and consider the local flat homomorphism $A_{\mathfrak{p}} \rightarrow \hat{A}_P$. Since $\dim(\hat{A}_P/\mathfrak{p}\hat{A}_P) = 0$ it follows by Corollary 5.2 that $p(A_{\mathfrak{p}}) = p(\hat{A}_P)$. Hence we can assume that A is a complete ring. Let now $\mathfrak{q} \subseteq \mathfrak{p}$ a prime ideal such that $\dim(A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}) > p(A) - \dim(A/\mathfrak{p})$. Then we have

$$\dim(A/\mathfrak{q}) \geq \dim(A/\mathfrak{p}) + \dim(A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}) > p(A).$$

Thus $A_{\mathfrak{q}}$ is CM by Corollary 3.6. Furthermore, since A is complete A is catenary; therefore $\dim(A_{\mathfrak{p}}) = \dim(A_{\mathfrak{q}}) + \dim(A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}})$. Hence, applying Theorem 4.1 for $A_{\mathfrak{p}}$, we deduce that $p(A_{\mathfrak{p}}) \leq p(A) - \dim(A/\mathfrak{p})$ as required.

Theorem 5.1 has some interesting consequences. First, the following two corollaries say about the relationship between the polynomial types. Their proofs are trivial therefore we omit it.

COROLLARY 5.4. *With the same A, B as in Theorem 5.1 then $p(A) \leq p(B)$.*

COROLLARY 5.5. *With the same A, B, F as in Theorem 5.1, assume that A is not a CM ring then $p(A) = p(B)$ if and only if $\dim F = 0$.*

The following corollary is a generalization of Theorem 1, (b) in [D-E].

COROLLARY 5.6. *With the same notations as above then A is a generalized CM ring if $p(B) \leq \dim F$.*

Proof. If $p(B) < \dim F$ then A is CM by Theorem 5.1, (ii). If $p(B) = \dim F$ then by Theorem 5.1, (i)

$$\dim F = p(B) \geq p(A) + \dim F.$$

It follows that $p(A) \leq 0$; so A is generalized CM by Remark 2.5.

EXAMPLE 5.7. Let (A, \mathfrak{m}) be a local ring of dimension $d > 0$ and let

X a transcendental over A . We define $B = A[X]_{(m, X)A[X]}$ then $A \rightarrow B$ is a local, flat homomorphism. It is easy to see that the fiber $F = B \otimes_A A/\mathfrak{m}$ is regular and of dimension 1. If A is CM, since F is regular, then B is also CM. Therefore the equality $p(B) = p(A) + \dim F$ does not hold in this case. If A is not CM then by Theorem 5.1, (i), $p(B) \geq p(A) + \dim F = p(A) + 1$. Especially, if A is generalized CM, i.e. $p(A) = 0$ then $p(B) \geq 1$; so B is not a generalized CM ring.

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