QUOTIENTS OF F-SPACES

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(Received 6 October, 1976)

Let X be a non-locally convex F-space (complete metric linear space) whose dual X' separates the points of X. Then it is known that X possesses a closed subspace N which fails to be weakly closed (see [**3**]), or, equivalently, such that the quotient space X/N does not have a point separating dual. However the question has also been raised by Duren, Romberg and Shields [**2**] of whether X possesses a proper closed weakly dense (PCWD) subspace N, or, equivalently a closed subspace N such that X/N has trivial dual. In [**2**], the space H_p $(0<p<1)$ was shown to have a PCWD subspace; later in [**9**], Shapiro showed that ℓ_p $(0<p<1)$ and certain spaces of analytic function have PCWD subspaces. Our first result in this note is that every separable non-locally convex F-space with separating dual has a PCWD subspace.

It was for some time unknown whether an F-space with trivial dual could have non-zero compact endomorphisms. This problem was equivalent to the existence of a non-zero compact operator $T: X \to Y$, where X has trivial dual; for if such an operator exists, then we may suppose T has dense range and then the space $X \oplus Y$ has trivial dual and admits the compact endomorphism $(x, y) \to (0, Tx)$. Let us say that an F-space X *admits compact operators* if there is a non-zero compact operator with domain X. The most commonly arising spaces with trivial dual L_p $(0<p<1)$, do not admit compact operators ([**4**]). However in [**7**] it was shown that the spaces H_p $(0<p<1)$ possess quotients with trivial dual but admitting compact operators; equivalently there is a compact operator T with domain H_p whose kernel $T^{-1}(0)$ is a PCWD subspace of H_p. The construction depended on certain special properties of H_p. However we show here that every separable non-locally convex locally bounded F-space with a base of weakly closed neighbourhoods of zero admits a compact operator whose kernel is a PCWD subspace, and thus has a quotient with trivial dual but admitting compact operators. This result applies to H_p and to any locally bounded space with a basis; in a sense there are many examples of compact endomorphisms in spaces with trivial dual.

Let X be an F-space; we denote an F-norm (in the sense of [**3**]) defining the topology on X by $\|\cdot\|$. The following lemma is proved in [**3**] and [**8**].

LEMMA 1. *Let $|\cdot|$ be an F-norm on X which defines a topology weaker than the original topology. Suppose (x_n) is a sequence in X such that $|x_n| \to 0$ but $\|x_n\| \geq \varepsilon > 0 (n \in \mathbb{N})$. Then there is a subsequence (u_n) of (x_n) which is a strongly regular M-basic sequence; i.e. there exist continuous linear functionals $(\varphi_n : n \in \mathbb{N})$ on the closed linear span E of $(u_n : n \in \mathbb{N})$ such that*

(a) $\varphi_i(u_j) = \delta_{ij} \quad (i, j \in \mathbb{N})$,

(b) $\lim_{n\to\infty} \varphi_n(x) = 0 \quad (x \in E)$,

(c) *if $x \in E$, and $\varphi_n(x) = 0$ for all $n \in \mathbb{N}$, then $x = 0$.*

Glasgow Math. J. **19** (1978) 103–108

https://doi.org/10.1017/S0017089500003463 Published online by Cambridge University Press

It is well-known that most familiar properties of compact operators do not require local convexity (see e.g. [**12**]). Thus the following lemma is almost certainly known, although we include a proof for completeness. We note our later use of Lemma 2 is essentially equivalent to using a stability theorem for M-basic sequences due to Drewnowski [**1**].

LEMMA 2. *Let X and Y be F-spaces and let $A:X\to Y$ be a linear embedding (i.e. a homeomorphism onto its range). Let $K:X\to Y$ be compact; then $A+K$ has closed range.*

Proof. Let U be a balanced neighbourhood of 0 in X such that $\overline{K(U)}$ is compact in Y. Let $N=(A+K)^{-1}(0)$ and consider the quotient map $q:X\to X/N$ and the induced map $S:X/N\to Y$ so that $Sq=A+K$. We show that S is an embedding. Suppose not; then there is a sequence (x_n) in X/N such that $Sx_n\to 0$ but for some neighbourhood V of 0 in X with $V\subset\frac{1}{2}U$ we have $x_n\notin q(V)(n\in\mathbb{N})$. Select a sequence a_n with $0<a_n\leq 1$ such that $a_nx_n\notin q(V)$ but $a_nx_n\in q(U)$. Then $a_nx_n=q(u_n)$, where $u_n\in U$, and there is a subsequence $(u_{p(n)})$ of (u_n) such that $Ku_{p(n)}\to y$ in Y. Hence $Au_{p(n)}\to -y$ and $u_{p(n)}\to v$, where $Av=-y$. Thus $q(v)=0$ and $q(u_{p(n)})=a_{p(n)}x_{p(n)}\to 0$, a contradiction.

THEOREM 1. *Let X be a non-locally convex separable F-space with separating dual. Then X has a PCWD subspace, and hence a non-trivial quotient with trivial dual.*

Proof. Denote by μ, the Mackey topology on X (see [**10**]), i.e. the topology induced by all convex neighbourhoods of 0. Then μ is metrizable and may be given by an F-norm $|\cdot|$. Since X is non-locally convex there is a sequence (x_n) in X such that $\|x_n\|\geq\varepsilon>0$, but $|x_n|\to 0$. We apply Lemma 1 to deduce the existence of a strongly regular M-basic subsequence (u_n) of (x_n); as in Lemma 1, let E be the closed linear span of $\{u_n:n\in\mathbb{N}\}$ and $\{\varphi_n:n\in\mathbb{N}\}$ be the biorthogonal functionals on E. Let E_0 be the closed linear span of $\{u_{2n};\ n\in\mathbb{N}\}$.

Now let $\{v_n:n\in\mathbb{N}\}$ be a dense countable subset of X and choose ε_n such that $0<\varepsilon_n<1$ and $\|\varepsilon_nv_n\|\leq 2^{-n}$. Since $|u_n|\to 0$, we may find an increasing sequence $\ell(n)$ such that $|\varepsilon_n^{-1}u_{2\ell(n)}|\to 0$. Now define $K:E_0\to X$ by

$$Kx=\sum_{n=1}^{\infty}\varepsilon_n\varphi_{2\ell(n)}(x)v_n.$$

By the Banach–Steinhaus theorem, the functionals $(\varphi_{2\ell(n)};\ n\in\mathbb{N})$ are equicontinuous on E_0 and hence K is compact (it maps the zero-neighbourhood $U=\{x:|\varphi_{2\ell(n)}(x)|\leq 1,\ n\in\mathbb{N}\}$ into a relatively compact set). Now let $J:E_0\to X$ be the inclusion map and $N=(J+K)(E_0)$.

Then N is closed by Lemma 2. If $N=X$, then $J+K$ is open and $(J+K)(U)$ is a neighbourhood of 0 in X. Let $q:X\to X/E_0$ be the quotient map; then $q(J+K)(U)\subset qK(U)$ is relatively compact and hence $\dim X/E_0<\infty$. However $\dim X/E_0\geq\dim E/E_0=\infty$ and thus N is a proper closed subspace. If $x\in X$, then there is a sequence $v_{n_k}\to x$; then $\varepsilon_{n_k}^{-1}(u_{2\ell(n_k)}+\varepsilon_{n_k}v_{n_k})\in N$ and converges to x in the Mackey topology; hence N is weakly dense.

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REMARK. We do not know whether every non-locally convex F-space has a non-trivial quotient with trivial dual.

THEOREM 2. *Let B be a locally bounded F-space with a basis and let X be a non-locally convex subspace of B. Then there is a compact operator T on X whose kernel is a PCWD subspace of X.*

Thus X has a quotient space with trivial dual but admitting compact operators.

Proof. The proof is a modification of the preceding theorem. Let (e_n) be a basis of B and let (e_n') be the linear functionals biorthogonal to (e_n). Let $\|\cdot\|$ be a p-homogeneous norm on B defining the topology, where $p<1$. Without loss of generality we assume that the basis (e_n) is monotone, i.e.

$$\sup_{1\le k<\infty}\left\|\sum_{n=1}^{k} a_n e_n\right\| = \left\|\sum_{n=1}^{\infty} a_n e_n\right\|.$$

Let σ be the topology on B induced by the functionals $x\to e_n'(x)$ and let $\bar{B}$ be the σ-completion of B. Let $B^\gamma\subset\bar{B}$ be the set of $x\in\bar{B}$ such that

$$\|x\| = \sup_{1\le k<\infty}\left\|\sum_{n=1}^{k} e_n'(x)e_n\right\|<\infty.$$

Thus B^γ is a p-normed space with unit ball $U=\{x\in B^\gamma:\|x\|\le 1\}$. Then U is σ-compact and we may define on B^γ the topology β which is the largest topology agreeing with σ on each set $kU(k\in\mathbb{N})$. Then β is a Hausdorff vector topology ([**11**]) and is locally p-convex (see e.g. [**6**] for the explicit form of the neighbourhoods of 0). We shall show that it is possible to choose a PCWD subspace N of X such that N is closed in the β-topology relative to X. Once this is achieved X/N admits a locally p-convex topology $\tilde{\beta}$ (the quotient β-topology) in which the unit ball is precompact. As in [**7**] there is a non-zero compact operator $S:X/N\to Y$, where Y is a p-Banach space. Then if $q:X\to X/N$ is the quotient map, $Sq:X\to Y$ is compact and its kernel is PCWD.

It remains to determine N. Let $|\cdot|$ be a norm defining the Mackey topology on X. By assumption $|\cdot|$ and $\|\cdot\|$ are nonequivalent on X, and hence also on any closed subspace of finite codimension. Thus it is possible to construct a sequence (x_n) in X such that $\|x_n\|=1$ $(n\in\mathbb{N})$, $|x_n|\to 0$ and $\lim_{n\to\infty} e_i'(x_n)=0$ for $1\le i<\infty$. By standard arguments we may determine a subsequence (y_n) of (x_n) and a block basic sequence (u_n) of (e_n) such that

$$u_n=\sum_{i=p_{n-1}+1}^{p_n} a_i e_i,$$

where $p_0=0<p_1<p_2\dots$, and $\|y_n-u_n\|\le\frac{1}{16}2^{-n}$.

Let E be the closed linear span in B of $\{u_n:n\in\mathbb{N}\}$ and E_0 the closed linear span of $\{u_{2n}:n\in\mathbb{N}\}$. Denote by E^γ and E_0^γ their respective closures in (B^γ,β). Let (u_n') denote the biorthogonal functionals on E^γ; then each u_n' is β-continuous in E^γ (it is a finite linear

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combination of the e'_n) and

$$\|u'_n(x)u_n\| \le 2\|x\| \quad (x \in E^\gamma),$$

so that as $\|u_n\| \ge \frac{1}{2}$, $|u'_n(x)|^p \le 4\|x\|$.

Let (v_n) be a countable dense subset of $X\backslash\{0\}$ and choose $\varepsilon_n > 0$ so that

$$\varepsilon_n^p = \tfrac{1}{16}2^{-n}\|v_n\|^{-1} \quad (n \in \mathbb{N}).$$

Then choose an increasing sequence $\ell(n)$ such that

$$|\varepsilon_n^{-1} y_{2\ell(n)}| \to 0.$$

Next define $K: E^\gamma \to B^\gamma$ by

$$Kx = \sum_{n=1}^{\infty} u'_n(x)(y_n - u_n) + \sum_{n=1}^{\infty} \varepsilon_n u'_{2\ell(n)}(x) v_n.$$

Thus K is compact for the p-norm topology of E^γ and $K(E^\gamma) \subset B$. If $\|x\| \le 1$

$$\begin{aligned}\|Kx\| &\le \sum_{n=1}^{\infty} |u'_n(x)|^p \|y_n - u_n\| + \sum_{n=1}^{\infty} \varepsilon_n^p |u'_{2\ell(n)}(x)|^p \|v_n\| \\ &\le \tfrac{1}{2}\|x\|.\end{aligned}$$

Let $J: E^\gamma \to B^\gamma$ be the inclusion map, and let $N = (J+K)(E_0)$. By Lemma 2, N is closed in B. In fact it is easy to see that $N \subset X$ and an argument similar to the proof of Theorem 1 shows that N is weakly dense in X (note that $y_{2\ell(n)} + \varepsilon_n v_n \in N$). Suppose $N = X$; then as $(J+K)(E_0) \subset (J+K)(E) \subset X$ we have $(J+K)(E) = (J+K)(E_0)$ and there exists $x \in E$, with $x \ne 0$, such that $(J+K)x = 0$. However $\|Kx\| \le \frac{1}{2}\|x\|$ and so this is impossible. Hence N is a PCWD subspace of X.

It remains to show that N is relatively β-closed. Consider first $N^\gamma = (J+K)(E^\gamma)$. To show that N^γ is β-closed in B^γ it is only necessary to show that $N^\gamma \cap U$ is σ-closed. Suppose $z_n \in E^\gamma$, $\|(J+K)z_n\| \le 1$ and $(J+K)(z_n) \to u(\beta)$. Then as $\|(J+K)z_n\| \ge \frac{1}{2}\|z_n\|$ we have $\|z_n\| \le 2$; by passing to a subsequence we may suppose $z_n \to z(\sigma)$ and $z \in E^\gamma$. Since each (u'_n) is σ-continuous on E^γ, K is continuous on bounded sets for the σ-topology. Hence $(J+K)z_n \to (J+K)z(\sigma)$ and $(J+K)z = u$. Thus N^γ is β-closed in B^γ. Now consider $N^\gamma \cap B \supset N$; if $x \in N^\gamma \cap B$, then $x = (J+K)u$, $u \in E^\gamma$ and $u = x - Ku \in B \cap E^\gamma = E$. Thus $x \in N$ and so $N = N^\gamma \cap B$ is β-closed in B and hence in X, as required.

Corollary. *Every non-locally convex separable locally bounded F-space with a base of weakly closed neighbourhoods of* 0 *has a quotient with trivial dual which admits compact operators.*

Proof. These spaces are precisely those isomorphic to subspaces of a locally bounded F-space with a basis (see [**5**], Theorem 7.4).

Remarks. (1) There exist locally bounded non-locally convex F-spaces with bases such that every non-trivial quotient admits compact operators. Indeed any pseudo-reflexive space as constructed in [**8**] has this property, as it possesses a topology β in

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which the unit ball is compact and which defines the same closed subspaces as the original topology.

(2) There exist separable locally bounded non-locally convex F-spaces X with separating duals such that every compact operator on X has a weakly closed kernel. To see this construct an Orlicz function φ on $[0,\infty)$ so that

(a) $x^{-\frac{1}{2}}\varphi(x)$ is increasing,
(b) $\limsup_{x\to\infty} x^{-1}\varphi(x)=\infty$,
(c) $\liminf_{x\to\infty} x^{-1}\varphi(x)=1$,
(d) $\sup_{x\geq 1} \dfrac{\varphi(2x)}{\varphi(x)}<\infty$.

Then $L_\varphi(0,1)$ is a locally bounded non-locally convex space with separating dual and, by the results of [**4**], every compact operator on L_φ factors through the inclusion map $L_\varphi \subsetneq L_1$. Hence the kernel of any compact operator is weakly closed.

(3) In the proof of the theorem the Mackey topology on X may be replaced by any strictly weaker metrizable topology. In particular if X is locally p-convex ($p<1$) but not locally r-convex for any $r>p$, then we may consider the topology induced by all absolutely r-convex neighbourhoods of 0 ($r>p$). We then obtain by the same arguments the following result.

THEOREM 3. *Let X be a closed subspace of a locally bounded F-space with a basis. Suppose X is locally p-convex ($p<1$) but not locally r-convex for any $r>p$. Then there is a non-zero compact operator $T:X\to Y$ (where Y is a locally bounded F-space) such that $X/\ker T$ admits no operators into any locally r-convex space for $r>p$.*

Let $X=\ell_p$ and U be the closed unit ball of ℓ_p. If we construct T as in the theorem then $\overline{T(U)}$ is a compact p-convex subset of Y, which cannot be linearly embedded into a locally r-convex space. For if S is such an embedding (i.e. S is linear on the linear span of $\overline{T(U)}$ and continuous on $\overline{T(U)}$), then ST is continuous on ℓ_p and $\ker(ST)\subset\ker T$. By the theorem $ST=0$. On the other hand $\overline{T(U)}$ can be linearly embedded in a locally p-convex space (see [**6**]).

Note added in proof. (See the Remark preceding Theorem 2): An example of a non-locally convex F-space with no non-trivial quotient with trivial dual has been constructed by J. Roberts (Springer Lecture Notes No. 604, pp. 76–81). Similar examples were also obtained independently by M. Ribe and the author.

REFERENCES

1. L. Drewnowski, On minimally comparable F-spaces, *J. Functional Analysis* **26** (1977) 315–332.

2. P. L. Duren, B. W. Romberg and A. L. Shields, Linear functionals on H_p spaces with $0<p<1$, *J. Reine Angew. Math.* **38** (1969) 32–60.

3. N. J. Kalton, Basic sequences in F-spaces and their applications, *Proc. Edinburgh Math. Soc.* (2) **19** (1974) 151–167.

https://doi.org/10.1017/S0017089500003463 Published online by Cambridge University Press

4. N. J. Kalton, Compact and strictly singular operator on Orlicz spaces, *Israel J. Math.*, **26** (1977) 126–136.

5. N. J. Kalton, Universal spaces and universal bases in metric linear spaces, *Studia Math.*, to appear.

6. N. J. Kalton, Compact p-convex sets, *Quart. J. Math. Oxford Ser.* 2 **28** (1977) 301–308.

7. N. J. Kalton and J. H. Shapiro, An F-space with trivial dual and non-trivial compact endomorphisms, *Israel J. Math.* **20** (1975) 282–291.

8. N. J. Kalton and J. H. Shapiro, Bases and basic sequences in F-spaces, *Studia Math.*, **56** (1976) 47–61.

9. J. H. Shapiro, Examples of proper closed weakly dense subspace in non-locally convex F-spaces, *Israel J. Math.* **7** (1969) 369–380.

10. J. H. Shapiro, Extension of linear functionals on F-spaces with bases, *Duke Math. J.* **37** (1970) 639–645.

11. L. Waelbroeck, *Topological Vector Spaces and Algebras,* Lecture Notes in Mathematics No. 230 (Springer-Verlag, 1970).

12. J. H. Williamson, Compact linear operators in linear topological spaces, *J. London Math. Soc.* **29** (1954) 149–156.

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