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COFINITENESS AND FINITENESS OF GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract

Let I be an ideal of a commutative Noetherian local ring R, and M and N two finitely generated modules. Let t be a positive integer. We mainly prove that (i) if $H_I^i(M, N)$ is Artinian for all $i < t$, then $H_I^i(M, N)$ is I-cofinite for all $i < t$ and $\operatorname{Hom}(R/I, H_I^t(M, N))$ is finitely generated; (ii) if $d = \operatorname{pd}(M) < \infty$ and $\dim N = n < \infty$, then $H_I^{d+n}(M, N)$ is I-cofinite. We also prove that if M is a nonzero cyclic R-module, then $H_I^i(N)$ is finitely generated for all $i < t$ if and only if $H_I^i(M, N)$ is finitely generated for all $i < t$.

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1. Introduction

Let R be a commutative Noetherian ring and I a proper ideal of R. In 1969, Grothendieck proposed the following conjecture.

CONJECTURE 1.1. *Let N be a finitely generated R-module and let I be an ideal of R. Then* $\operatorname{Hom}(R/I, H_I^i(N))$ *is finitely generated for all $i \geq 0$.*

Hartshorne provided a counter-example to this conjecture in [9]. He defined an R-module L to be I-cofinite if $\operatorname{Supp}_R\{L\} \subseteq V(I)$ and $\operatorname{Ext}_R^i(R/I, L)$ is a finitely generated R-module for any $i \geq 0$, where $V(I)$ denotes the set of prime ideals of R containing I, and he asked the following question.

QUESTION 1.2. Let N be a finitely generated R-module and let I be an ideal of R. Then is $H_I^i(N)$ I-cofinite for all i?

In general, the answer is also no, even if R is a regular local ring. See [6] for a counter-example to this question.

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The generalized local cohomology module

$$H_I^i(M, N) = \varinjlim_n \operatorname{Ext}_R^i(M/I^n M, N)$$

for all R-modules M and N was introduced by Herzog in [10]. Clearly, it is a generalization of the usual local cohomology module. The study of generalized local cohomology modules was continued by many authors (see, for example, [2, 16]). In [17], Yassemi asked whether Question 1.2 holds for generalized local cohomology. And the cofiniteness of generalized local cohomology modules is studied by Divaani-Aazar and Sazeedeh [8] and Khashyarmanesh and Yassi [11].

At the same time, Conjecture 1.1 inspires us to think about the following question.

QUESTION 1.3. Let M and N be two finitely generated R-modules. When is $\operatorname{Hom}(R/I, H_I^i(M, N))$ finitely generated?

Asadollahi, Khashyarmanesh and Salarian [1] proved that if $H_I^i(M, N)$ is finitely generated for all $i < t$, then $\operatorname{Hom}(R/I, H_I^t(M, N))$ is finitely generated. As an analogue of this result, we show that if $H_I^i(M, N)$ is Artinian for all $i < t$, then $H_I^i(M, N)$ is I-cofinite for all $i < t$ and $\operatorname{Hom}(R/I, H_I^t(M, N))$ is finitely generated. We also prove that if $d = \operatorname{pd}(M) < \infty$ and $\dim N = n < \infty$, then $H_I^{d+n}(M, N)$ is I-cofinite, which is a generalization of [6, Theorem 3].

Throughout this paper, $(R, \mathfrak{m})$ is a commutative Noetherian local ring (with nonzero identity), M and N are finitely generated R-modules and I is a proper ideal of R. We refer the reader to [3] or [4] for any unexplained terminology.

2. Results

We begin this section with some lemmas.

LEMMA 2.1. *Let M be a finitely generated R-module. If $H_I^i(M)$ is Artinian for all $i < t$, then $H_I^i(M)$ is I-cofinite for all $i < t$ and $\operatorname{Hom}(R/I, H_I^t(M))$ is finitely generated.*

PROOF. This can be deduced from [7, Theorem 2.1] and [13, Proposition 4.3]. □

LEMMA 2.2. *Let M be a finitely generated R-module. If L is Artinian and I-cofinite, then $\operatorname{Ext}_R^i(M, L)$ is I-cofinite for all i.*

PROOF. Since L is Artinian, $\operatorname{Ext}_R^i(M, L)$ is Artinian for all i. By [13, Proposition 4.3], it suffices to prove that $\operatorname{Hom}_R(R/I, \operatorname{Ext}_R^i(M, L))$ is finitely generated. In the following, we show that $\operatorname{Hom}_R(R/I, \operatorname{Ext}_R^i(M, L))$ is of finite length. Since

$$\begin{aligned}\operatorname{Hom}_R(R/I, \operatorname{Ext}_R^i(M, L)) &\cong \operatorname{Hom}_R(R/I, \operatorname{Ext}_R^i(M, L)) \otimes \widehat{R} \\ &\cong \operatorname{Hom}_{\widehat{R}}(\widehat{R}/I\widehat{R}, \operatorname{Ext}_{\widehat{R}}^i(\widehat{M}, L)),\end{aligned}$$

we may assume that R is $\mathfrak{m}$-adic complete.

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Set $E = E(R/\mathfrak{m})$, an injective envelope of $R/\mathfrak{m}$. By [15, Theorem 11.57],

$$\mathrm{Hom}_R(\mathrm{Hom}_R(R/I, \mathrm{Ext}^i_R(M, L)), E) \cong R/I \otimes \mathrm{Tor}^R_i(M, \mathrm{Hom}_R(L, E)).$$

By Matlis duality, $R/I \otimes \mathrm{Tor}^R_i(M, \mathrm{Hom}_R(L, E))$ is finitely generated, so it is enough to show that it is Artinian. Since L is I-cofinite and Artinian, $\mathrm{Hom}_R(R/I, L)$ is of finite length, and then $\mathrm{Hom}_R(\mathrm{Hom}_R(R/I, L), E) \cong R/I \otimes \mathrm{Hom}_R(L, E)$ is of finite length. In particular,

$$\mathrm{Supp}_R\{R/I \otimes \mathrm{Hom}_R(L, E)\} = V(I) \cap \mathrm{Supp}_R\{\mathrm{Hom}_R(L, E)\} = \{\mathfrak{m}\}.$$

Therefore

$$\mathrm{Supp}_R\{R/I \otimes \mathrm{Tor}^R_i(M, \mathrm{Hom}_R(L, E))\} \subseteq V(I) \cap \mathrm{Supp}_R\{\mathrm{Hom}_R(L, E)\} = \{\mathfrak{m}\}.$$

This completes the proof. □

The following lemma plays a key role in the proof of our first main result.

LEMMA 2.3. *Let M be a finitely generated R-module and s be a nonnegative integer. Let L be an R-module such that $H^i_I(L)$ is Artinian and I-cofinite for all $i < s$. If $H^i_I(M, L)$ is Artinian for all $i < s$, then $H^i_I(M, L)$ is I-cofinite for all $i < s$.*

PROOF. The proof is by induction on s. When $s = 1$, by the hypothesis, $H^0_I(L)$ is Artinian and I-cofinite. Then $\mathrm{Hom}(R/I, H^0_I(M, L)) \cong \mathrm{Hom}(M, \mathrm{Hom}(R/I, H^0_I(L)))$, which is of finite length. By [13, Proposition 4.3], the result holds.

Suppose that $s > 1$, and the result holds for the case $s - 1$. The short exact sequence

$$0 \longrightarrow H^0_I(L) \longrightarrow L \longrightarrow L/H^0_I(L) \longrightarrow 0$$

yields the long exact sequence

$$\cdots \longrightarrow H^i_I(M, H^0_I(L)) \longrightarrow H^i_I(M, L) \longrightarrow H^i_I(M, L/H^0_I(L)) \longrightarrow \cdots .$$

Since $H^0_I(L)$ is I-torsion, $H^i_I(M, H^0_I(L)) \cong \mathrm{Ext}^i_R(M, H^0_I(L))$. Then, by Lemma 2.2, $H^i_I(M, H^0_I(L))$ is I-cofinite and Artinian for all i. By [13, Corollary 4.4], it is enough for us to prove that $H^i_I(M, L/H^0_I(L))$ is I-cofinite for all $i < s$. So we can assume that $\Gamma_I(L) = 0$. Taking an injective hull E of L, then we have the short exact sequence $0 \longrightarrow L \longrightarrow E \longrightarrow E/L \longrightarrow 0$. Consequently, from the long exact sequences of the above short exact sequence, $H^{i+1}_I(M, L) \cong H^i_I(M, E/L)$ and $H^{i+1}_I(L) \cong H^i_I(E/L)$ for all i. Thus, $H^i_I(E/L)$ is Artinian and I-cofinite, and $H^i_I(M, E/L)$ is Artinian for all $i < s - 1$. Now by the induction hypothesis, the result is proved. □

The following lemma has already been proved. However, we cannot find the original proof, so we give our own.

LEMMA 2.4. *Assume that $0 \longrightarrow L_1 \longrightarrow L_2 \longrightarrow L_3 \longrightarrow 0$ is an exact sequence of finitely generated R-modules. Then we have the long exact sequence*

$$\cdots \longrightarrow H^i_I(L_3, N) \longrightarrow H^i_I(L_2, N) \longrightarrow H^i_I(L_1, N) \longrightarrow H^{i+1}_I(L_3, N) \longrightarrow \cdots .$$

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PROOF. Let $0 \longrightarrow N \longrightarrow E^\bullet$ be a minimal injective resolution of N. Note that $\Gamma_I(E)$ is injective if E is injective. Then $\operatorname{Hom}(-, \Gamma_I(E))$ is an exact functor. So $0 \longrightarrow \operatorname{Hom}(L_3, \Gamma_I(E^\bullet)) \longrightarrow \operatorname{Hom}(L_2, \Gamma_I(E^\bullet)) \longrightarrow \operatorname{Hom}(L_1, \Gamma_I(E^\bullet)) \longrightarrow 0$ is an exact sequence of R-complexes. By a well-known theorem of homology theory, we have a long exact sequence

$$\cdots \longrightarrow H^i(\operatorname{Hom}(L_3, \Gamma_I(E^\bullet))) \longrightarrow H^i(\operatorname{Hom}(L_2, \Gamma_I(E^\bullet))) \longrightarrow H^i(\operatorname{Hom}(L_1, \Gamma_I(E^\bullet))) \longrightarrow H^{i+1}(\operatorname{Hom}(L_3, \Gamma_I(E^\bullet))) \longrightarrow \cdots .$$

Suppose that M is a finitely generated R-module. Then

$$\Gamma_I(\operatorname{Hom}(M, E^\bullet)) \cong \operatorname{Hom}(M, \Gamma_I(E^\bullet)),$$

and so

$$H_I^i(M, N) = \varinjlim_n \operatorname{Ext}_R^i(M/I^n M, N) \cong H^i(\Gamma_I(\operatorname{Hom}(M, E^\bullet))) \cong H^i(\operatorname{Hom}(M, \Gamma_I(E^\bullet))).$$

Hence, we obtain the long exact sequence that we wished to prove. □

For any submodule K of a finitely generated R-module L, we use $K :_L \langle \mathfrak{m} \rangle$ to denote the submodule $\{x \in L \mid \mathfrak{m}^n x \subseteq K$ for some $n > 0\}$. A sequence $x_1, \ldots, x_n$ of elements in $\mathfrak{m}$ is said to be an $\mathfrak{m}$-filter regular sequence on a module N if

$$(x_1, \ldots, x_{i-1})N :_N x_i \subseteq (x_1, \ldots, x_{i-1})N :_N \langle \mathfrak{m} \rangle$$

for all $i = 1, \ldots, n$. The f-depth of an ideal I on a module N is defined as the length of any maximal $\mathfrak{m}$-filter regular sequence on N in I; we denote it by $\text{f-depth}(I, N)$. As the analogue of a result in the local cohomology modules case, the authors of [5] proved that $\text{f-depth}(I + \operatorname{Ann} M, N)$ is equal to $\operatorname{Min}\{r \mid H_I^r(M, N)$ is not Artinian$\}$. On the other hand, from the definition of generalized local cohomology modules, it is well known that, for all i,

$$H^i_{I+\operatorname{Ann} M+\operatorname{Ann} N}(M, N) \cong H^i_{I+\operatorname{Ann} M}(M, N) \cong H^i_I(M, N).$$

Now we are in the position to present our first main result.

THEOREM 2.5. *Let M and N be two finitely generated R-modules. If, for some nonnegative integer t, $H_I^i(M, N)$ is Artinian for all $i < t$, then $H_I^i(M, N)$ is I-cofinite for all $i < t$ and* $\operatorname{Hom}(R/I, H_I^t(M, N))$ *is finitely generated.*

PROOF. Since $H_I^i(M, N)$ is Artinian for all $i < t$, $H^i_{I+\operatorname{Ann} M}(N)$ is Artinian for all $i < t$ by [5, Theorem 2.2]. Then by Lemma 2.1, $H^i_{I+\operatorname{Ann} M}(N)$ is $(I + \operatorname{Ann} M)$-cofinite for all $i < t$. Therefore, for any $i < t$, $H^i_{I+\operatorname{Ann} M}(M, N)$ is $(I + \operatorname{Ann} M)$-cofinite by Lemma 2.3. In particular, $\operatorname{Hom}(R/(I + \operatorname{Ann} M), H^i_{I+\operatorname{Ann} M}(M, N))$ is

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finitely generated for all $i < t$. Note that $\mathrm{Hom}(R/I, H_I^i(M, N)) \cong \mathrm{Hom}(R/(I + \mathrm{Ann}\, M), H_{I+\mathrm{Ann}\, M}^i(M, N))$, so that $\mathrm{Hom}(R/I, H_I^i(M, N))$ is finitely generated for all $i < t$. Thus, by [13, Proposition 4.3], it follows that $H_I^i(M, N)$ is I-cofinite for all $i < t$ from the hypothesis that $H_I^i(M, N)$ is Artinian for all $i < t$.

In the following, we prove that $\mathrm{Hom}(R/I, H_I^t(M, N))$ is finitely generated. Since, for any i,

$$H_{I+\mathrm{Ann}\, M}^i(M, N) \cong H_I^i(M, N)$$

and

$$\mathrm{Hom}(R/I, H_I^i(M, N)) \cong \mathrm{Hom}(R/(I + \mathrm{Ann}\, M), H_{I+\mathrm{Ann}\, M}^i(M, N)),$$

we can assume that $\mathrm{Ann}\, M \subseteq I$.

Let $0 \longrightarrow K \longrightarrow R^n \longrightarrow M \longrightarrow 0$ be an exact sequence of finitely generated R-modules. By Lemma 2.4, we have the exact sequence

$$\cdots \longrightarrow H_I^{t-1}(K, N) \xrightarrow{\alpha} H_I^t(M, N) \longrightarrow H_I^t(R^n, N) \longrightarrow \cdots .$$

It is clear that $H_I^i(R^n, N) \cong \bigoplus_{i=1}^n H_I^i(N)$ for any i. Then $H_I^i(R^n, N)$ and $H_I^i(K, N)$ are Artinian for all $i < t$ by [5, Theorem 2.2]. By virtue of the former part of the result, we know that $H_I^i(K, N)$ is I-cofinite for all $i < t$. Let L denote the image of α in the above long exact sequence. By [13, Corollary 4.4], L is I-cofinite. From the exact sequence

$$0 \longrightarrow L \xrightarrow{\alpha} H_I^t(M, N) \longrightarrow H_I^t(R^n, N) \longrightarrow \cdots ,$$

we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}(R/I, L) \longrightarrow \mathrm{Hom}(R/I, H_I^t(M, N)) \longrightarrow \mathrm{Hom}(R/I, H_I^t(R^n, N)).$$

By Lemma 2.1, the right term of the above exact sequence is finitely generated. Then the result follows from the above exact sequence. □

The following lemma is a generalization of [14, Lemma 3.4].

Lemma 2.6 ([12, Theorem 3.2]). *Let M be a finitely generated R-module such that $d = \mathrm{pd}(M) < \infty$. Let N be a finitely generated R-module and assume that n is an integer, and $x_1, x_2, \ldots, x_n$ is an I-filter regular sequence on N. Then*

$$H_I^{i+n}(M, N) \cong H_I^i(M, H_{(x_1, x_2, \ldots, x_n)}^n(N))$$

for all $i \geq d$. □

Proposition 2.7. *Let I be an ideal of R, and let M, N be two finitely generated R-modules such that $d = \mathrm{pd}(M) < \infty$ and $\dim N = n < \infty$. Then $H_I^{d+n}(M, N) \cong \mathrm{Ext}_R^d(M, H_I^n(N))$. In particular, $H_I^{d+n}(M, N)$ is Artinian.*

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PROOF. For this integer n, it is well known that there exists a sequence $x_1, x_2, \ldots, x_n$ in I such that it is an I-filter regular sequence on N. Note that $H^n_{(x_1,x_2,\ldots,x_n)}(N)$ is Artinian when $n = \dim N$. By virtue of [14, Lemma 3.4],

$$H^n_{(x_1,x_2,\ldots,x_n)}(N) \cong H^0_I(H^n_{(x_1,x_2,\ldots,x_n)}(N)) \cong H^n_I(N).$$

Therefore, by Lemma 2.6,

$$H^{d+n}_I(M, N) \cong H^d_I(M, H^n_{(x_1,x_2,\ldots,x_n)}(N)) \cong H^d_I(M, H^n_I(N)) \cong \mathrm{Ext}^d_R(M, H^n_I(N)).$$

This completes the proof. □

The following theorem is our second main result, which generalizes [6, Theorem 3].

THEOREM 2.8. *Let I an ideal of R, and let M, N be two finitely generated R-modules such that $d = \mathrm{pd}(M) < \infty$ and $\dim N = n < \infty$. Then $H^{d+n}_I(M, N)$ is I-cofinite.*

PROOF. By [6, Theorem 3], we know that $H^n_I(N)$ is I-cofinite. Then by Lemma 2.2 and Proposition 2.7, the result follows. □

In the last part of this note, we discuss the finiteness of $H^i_I(M, N)$.

LEMMA 2.9. *Let N be a finitely generated R-module and M a nonzero cyclic R-module. Let t be a positive integer. If $H^i_I(N)$ is finitely generated for all $i < t$, then $H^t_I(N)$ is finitely generated if and only if $\mathrm{Hom}(M, H^t_I(N))$ is finitely generated.*

PROOF. The 'only if' part is clear. Now suppose that $\mathrm{Hom}(M, H^t_I(N))$ is finitely generated. Note that $\mathrm{Hom}(M, H^t_I(N))$ is I-torsion, then there exists an integer n such that $I^n \mathrm{Hom}(M, H^t_I(N)) = 0$. Assume that M is generated by an element m. For any $x \in H^t_I(N)$, we can find an element $f \in \mathrm{Hom}(M, H^t_I(N))$ such that $f(m) = x$. Since $I^n f = 0$, $I^n x = 0$, and so $I^n H^t_I(N) = 0$. Since $H^i_I(N)$ is finitely generated for all $i < t$, by [3, Proposition 9.1.2], there exists an integer r, $I^r H^i_I(N) = 0$ for all $i < t$. Thus, $I^r H^i_I(N) = 0$ for all $i < t + 1$. Again by [3, Proposition 9.1.2], $H^i_I(N)$ is finitely generated for all $i < t + 1$. In particular, $H^t_I(N)$ is finitely generated. □

PROPOSITION 2.10. *Let N be a finitely generated R-module and let t be a positive integer. If M is a nonzero cyclic R-module, then $H^i_I(N)$ is finitely generated for all $i < t$ if and only if $H^i_I(M, N)$ is finitely generated for all $i < t$.*

PROOF. The 'only if' part has been proved in [11, Theorem 1.1(iv)]. Now we suppose that $H^i_I(M, N)$ is finitely generated for all $i < t$. By induction on t, we can assume that $H^i_I(N)$ is finitely generated for all $i < t - 1$. Then by [11, Theorem 1.1(iii)], it follows that $\mathrm{Hom}(M, H^{t-1}_I(N))$ is finitely generated from the fact that $H^{t-1}_I(M, N)$ is finitely generated. Then $H^{t-1}_I(N)$ is finitely generated by Lemma 2.9. □

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References

[1] J. Asadollahi, K. Khashyarmanesh and Sh. Salarian, 'On the finiteness properties of the generalized local cohomology modules', *Comm. Algebra* **30** (2002), 859–867.

[2] M. H. Bijan-Zadeh, 'A common generalization of local cohomology theories', *Glasgow Math. J.* **21** (1980), 173–181.

[3] M. P. Brodmann and R. Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications* (Cambridge University Press, Cambridge, 1998).

[4] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Revised edn (Cambridge University Press, Cambridge, 1996).

[5] L. Z. Chu and Z. M. Tang, 'On the Artinianness of generalized local cohomology', *Comm. Algebra* **35** (2007), 3821–3827.

[6] D. Delfino and T. Marley, 'Cofinite modules and local cohomology', *J. Pure Appl. Algebra* **121** (1997), 45–52.

[7] M. T. Dibaei and S. Yassemi, 'Associated primes and cofiniteness of local cohomology modules', *Manuscripta Math.* **117** (2005), 199–205.

[8] K. Divaani-Aazar and R. Sazeedeh, 'Cofiniteness of generalized local cohomology modules', *Colloq. Math.* **99** (2004), 283–290.

[9] R. Hartshorne, 'Affine duality and cofiniteness', *Invent. Math* **9** (1970), 145–164.

[10] J. Herzog, Komplexe Auflösungen und Dualität in der lokalen Algebra, Universität Regensburg, Preprint, 1974.

[11] K. Khashyarmanesh and M. Yassi, 'On the finiteness property of generalized local cohomology modules', *Algebra Colloq.* **12** (2005), 293–300.

[12] K. Khashyarmanesh, M. Yassi and A. Abbasi, 'Filter regular sequence and generalized local cohomology modules', *Comm. Algebra* **32** (2004), 253–259.

[13] L. Melkersson, 'Modules cofinite with respect to an ideal', *J. Algebra* **285** (2005), 649–668.

[14] U. Nagel and P. Schenzel, 'Cohomological annihilators and Castelnuovo Mumford regularity', in: *Commutative Algebra: Syzygies, Multiplicities, and Birational Algebra (South Hadley, MA, 1992)* (American Mathematical Society, Providence, RI, 1994), pp. 307–328.

[15] J. Rotman, *Introduction to Homological Algebra* (Academic Press, New York, 1979).

[16] N. Suzuki, 'On the generalized local cohomology and its duality', *J. Math. Kyoto. Univ.* **18** (1978), 71–85.

[17] S. Yassemi, 'Cofinite modules', *Comm. Algebra* **29** (2001), 2333–2340.

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