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Dynamically stable models for galaxies

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Abstract. A popular approach to model galaxies is Schwarzschild's method. For this method, a grid of sample orbits of stars in an external potential is calculated, and a model for the stellar system is obtained through attributing specific weights to the orbits in a superposition of them. The models created with Schwarzschild's method can fit many observed properties of the modeled stellar system with high precision. However, systems that are stationary as Schwarzschild models may therefore exhibit a strong time evolution if they are translated into more realistic self-gravitating models. The issue is highlighted with the Galactic center as an example.

Keywords. galaxies: kinematics and dynamics, methods: numerical

1. Schwarzschild's Method

In 1979, a new method was proposed by Schwarzschild (1979) to model the kinematics and dynamics of galaxies. Schwarzschild's method relies on the following steps: 1.) A potential is created based on the luminosity profile of the galaxy. 2.) The trajectories of test particles in the potetial are calculated for a variety of initial conditions. The orbits are generally not closed, so that sufficiently many orbits have to be calculated in order to understand the whereabouts of the particle. Note that as test particles, the addition of the particles does not alter the potential, in contrast to reality. 3.) For each test particle, the positions and velocities are recorded in an orbital library for constant time steps. If the time span is long enough (see point 2.)), the recorded positions for each particle can be translated into a density profile and velocity profiles. Up to which order the velocity profiles are considered is a matter of choice. However, generally the solution gets more precise with the number of orders to be considered, but the probability not to find a solution at all also increases. 4.) Linear combinations of the density profiles for the test particles are created and compared with sets of parameters covering a range; e.g. the mass-to-light ratio of the galaxy, the mass of a super-massive black hole in its center, the degree of its triaxiality, and so on. For each set of these parameters, a χ^2-fit is performed, resulting in the best linear combination for the given set of parameters. The set of parameters with the least χ^2 is then chosen for the according parameters of the galaxy. Naturally, the fits are not exact, so that it becomes a matter of the limit one sets whether the agreement of the parameters with the fit is deemed aceptable or not.

Schwarzschild's method is succesful in finding a kinematic and dynamic basis for many observational parameters of galaxies, and therefore it is still widely used (e.g. in van den Bosch *et al.* 2008, Seth *et al.* 2014, Feldmeier-Krause *et al.* 2017 and Kowalczyk *et al.* 2019). However, none of the above papers (and many more papers dealing with Schwarzschild's method) request that the object they model is in virial equilibrium, while they do assume the density profile of object to be time-independent, or stationary. This can lead to situations where the object is stationary, as it should be as a superposition of orbits in an external time-independent potential, but has a virial ratio $\neq 0.5$. The virial ratio, $Q = T/|W|$ where T is the kinetic energy and W is the potential energy, has however to be 0.5 because of the virial theorem in gravitational force fields. Thus, a physically viable solution (i.e. $Q = 0.5$) is achieved only by accident with Schwarzschild's method, unless $Q = 0.5$ is made an additional condition that the fits have to fulfill to some extent.

2. The Virial Theorem

If the forces are conservative, i.e. they only depend on the location (and not on the velocities, for example), a bound system fullfills the virial theorem as

$$-\left\langle \frac{1}{2} \int_V \rho(\mathbf{x})\, \mathbf{x} \cdot (\nabla\Phi)\, d^3\mathbf{x} \right\rangle + 2\,\langle T \rangle = 0. \tag{2.1}$$

Here, $\rho(\mathbf{x})$ is the matter density at the location $\mathbf{x}$ and $\nabla\Phi$ is the gradient of the potential Φ, T is the total kinetic energy and the angled brackets signify a time average. If the potential is additionally homogeneous, the gradient of the potential can be replaced by the potential itself times a factor k, so that the virial theorem becomes

$$-\,k \left\langle \frac{1}{2} \int_V \rho(\mathbf{x})\Phi(\mathbf{x})\, d^3\mathbf{x} \right\rangle + 2\,\langle T \rangle = 0. \tag{2.2}$$

The expression in the angled brackets is just the potential energy and the gravitational potential happens to be a homogeneous potential with the factor $k = -1$. Thus, the virial theorem for the gravitational potential becomes

$$\langle W \rangle + 2\,\langle T \rangle = 0. \tag{2.3}$$

If the system is also stationary, the time average is not necessary, so that the virial theorem reads

$$W + 2T = 0, \quad \text{and} \quad Q = \frac{T}{|W|} = 0.5, \tag{2.4}$$

respectively.

Given the above result, on first sight it looks like a mistake to find virial ratios that are constantly $\neq 0.5$ when applying Schwarzschild's method, since Schwarzschild's method depicts a gravitational potential. However, in Schwarzschild's method the potential is external and the particles are mass-less, whereas in reality the particles create themselves the potential in which they orbit. To better understand the consequences, consider an external Plummer (1911) profile. The Plummer profile is characterized by the gravitational potential

$$\Phi_{\mathrm{P,ext}}(r) = -\frac{GM}{(r^2 + b^2)^{1/2}}, \tag{2.5}$$

where M is the mass, G is the gravitational constant and b is a constant distance. Now imagine a bunch of test particles, which can feel the potential generated by the Plummer profile, but do not alter it significantly because their mass is too small. Thus, the test particles orbit in the Plummer potential, but they should fulfill the following additional two conditions: 1.) The test particles are distributed according to a Plummer profile with the same parameter b like the external Plummer profile, $\Phi_{\mathrm{P,ext}}$, and 2.) the distribution of the test particles is stationary.

There are different ways how to achieve the fulfillment of the additional two conditions. One way is to distribute the test aparticles so that they fulfill the Jeans equation for systems that are stationary, spherically symmetric, and isotropic in the second velocity moments. It can be shown that in this case, equation (2.4) is fulfilled (e.g. in Kroupa 2008), so that the system of test particles is stationary and in virial equilibrium. Thus, it would maintain its structure also without an external potential, save by the influence of two-body interactions. This fact is used for modeling star clusters which are in virial equilibrium at the beginning of a N-Body simulation.

A different way to build up a stationary Plummer profile of test particles is to put all test particles on circular orbits. Since a circular orbit does not change its radius, the density profile is stationary. However, it will *not* fulfill equation (2.4). To see this, first consider the orbits with $r >> b$. In this case, b becomes negligible and the potential becomes the gravitational potential for a point mass. A circular orbit around a gravitational point mass fulfills $Q = T/|W| = 0.5$, as can be verified by setting the gravitational force, $F_G = (GMm)/r^2$, equal to the centripetal force, $F_c = (mv^2)/r$. In these equations, G is the gravitational constant, M is the mass of the central body, m is the mass of the orbiting particle and r is the distance between the central mass and the orbiting particle. However, for $r << b$, the mass M within r goes to 0 and thus also the kinetic energy goes to 0, while the gravitational potential is the deepest. In consequence, Q starts from 0 for $r = 0$ and approaches 0.5 for $r \to \infty$, but never reaches it.

Thus, gravitational subsystems can be stationary and yet have $Q \neq 0.5$, provided they are embedded in an external potential. The external potential may even be gravitational, too. However, the subsystems and the external system must together fulfill $Q = 0.5$ at all times if they are stationary, because of the virial theorem. Schwarzschild's models are examples of systems where the particles are treated as mass-less with respect to the external potential. It is then the external potential that carries the mass in Schwarzschild's method. In other words, Schwarzschild's models fulfill the virial theorem only as in equation (2.1), but not as in equation (2.2), in contrast to a self-gravitating model.

3. Application to the Galactic center

To further test the above statements, we use the best-fitting Schwarzschild model for the Galactic center by Feldmeier-Krause *et al.* (2017) and transform it into a self-gravitating model. In more detail, the method is as follows: 1.) Discretize the orbits into N particles and let them randomly fill the space according to the weights put to the orbits by Schwarzschild's method. We took $N = 10^3$ and $N = 10^4$ particles. 2.) Attribute the mass of the external potential in Schwarzschild's method equally to the N particles. The external potential is then not needed any longer and the system becomes self-gravitating. 3.) Follow the time-evolution of the self-gravitating system in some N-body integrator. We chose NBODY6 (Aarseth 1999) for this last step and followed the evolution for $\approx$110 Myr.

Some results are shown in Fig. (1). The random irregularities to the curves are a result of two-body relaxation, by which particles can leave the Galactic center. This is in principle in contradiction to the virial theorem, which assumes a bound system. However, due to the virial theorem, the remaining bound particles always try to fulfill $\langle Q \rangle = 0.5$. Eventually, two-body relaxation leads to the dissolution of the system, which can especially be seen from the expansion of the uppermost lagrange radius for the system with 10^3 particles. The red solid curves show also more regular, but declining motions around $Q = 0.5$. They represent the main result shown here: If the mass from the external potential in the Schwarzschild model is directly taken to the self-gravitating model, the self-gravitating model oscillates and is therefore not stationary. This is in contrast to the Schwarzschild model. The problem can be solved by adjusting the mass from the

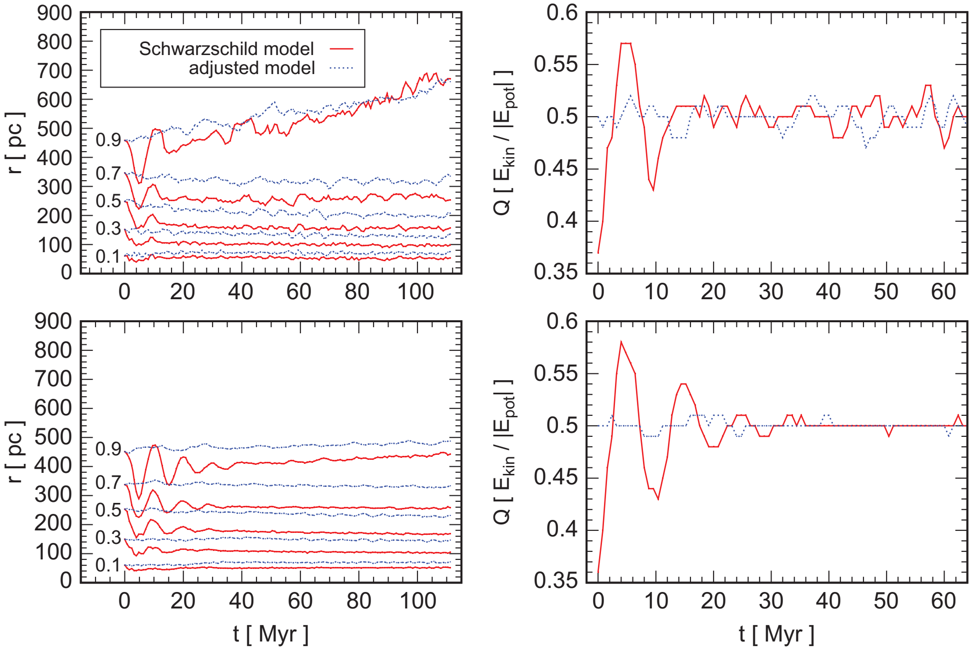

Figure 1. Test of the best-fitting Schwarzschild model for the Galactic center by Feldmeier-Krause *et al.* (2017), using the direct N-body code NBODY6 (see Aarseth 1999). The red solid curves show the Schwarzschild model without modifications and the blue dashed curves show the same Schwarzschild model, but with an adjusted mass so that it fulfills the condition for stationarity from the beginning. The upper panels are for 1000 and the lower panels are for 10000 equal-mass particles. The panels to the left show how different Lagrange-radii change with time and the panels to the right show the corresponding value for Q. The numbers to the left of the lines in the left panels show which fraction of the total mass is enclosed by the Lagrange radii.

external potential properly a posteriori (see the blue dashed curves). However, a better way to find proper Schwarzschild solutions is to make (close) stationarity of the models according to the virial theorem an additional condition, so that it has to be fulfilled along with the other conditions from which the best Schwarzschild model is selected.

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