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CENTRAL EXTENSIONS AND SCHUR'S MULTIPLICATORS OF GALOIS GROUPS

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Introduction

When he developed the theory of central extensions of absolute abelian fields in [1], Fröhlich clearly pointed out a role of Schur's multiplicators of the Galois groups in algebraic number theory. Another role of them was to be well known when the gaps between the everywhere local norms and the global norms of finite Galois extensions were cohomologically described by Tate [10]. The relation of two roles was investigated by Furuta [2], Shirai [9], Heider [3] and others.

Let K/k be a finite Galois extension of algebraic number fields with $\mathfrak{g} = \mathrm{Gal}(K/k)$. Then Schur's multiplicator of $\mathfrak{g}$ is $\mathrm{H}^2(\mathfrak{g}, \boldsymbol{C}^\times)$ under the trivial action of $\mathfrak{g}$ on $\boldsymbol{C}^\times$. We may replace $\boldsymbol{C}^\times$ by the subgroup consisting of all the roots of 1 in $\boldsymbol{C}^\times$, which is naturally isomorphic to the additive group $\boldsymbol{Q}/\boldsymbol{Z}$. Then the derived exact sequence from $0 \to \boldsymbol{Z} \to \boldsymbol{Q} \to \boldsymbol{Q}/\boldsymbol{Z} \to 0$ gives an isomorphism of $\mathrm{H}^2(\mathfrak{g}, \boldsymbol{C}^\times) \simeq \mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})$ onto $\mathrm{H}^3(\mathfrak{g}, \boldsymbol{Z})$, which is the dual of $\mathrm{H}^{-3}(\mathfrak{g}, \boldsymbol{Z})$. The works of Furuta, Shirai and Heider were based on the fact that $\mathrm{H}^{-3}(\mathfrak{g}, \boldsymbol{Z})$ is isomorphic to $\mathrm{H}^{-1}(\mathfrak{g}, K_A^\times/K^\times)$ where $K_A^\times/K^\times$ is the idele class group of K.

Let $\bar{k}$ be the algebraic closure of k, and k_{ab} the maximal abelian extension of k in $\bar{k}$. Heider [3] showed that there always exists a finite central extension L of K/k such that $\mathrm{Gal}(L/L \cap (k_{\mathrm{ab}} \cdot K))$ is isomorphic to the dual of $\mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})$, and, by this fact, gave a new proof Tate's theorem which claims that $\mathrm{H}^2(\mathrm{Gal}(\bar{k}/k), \boldsymbol{Q}/\boldsymbol{Z}) = 0$. (One can see a proof of this theorem in Serre [8].)

In this paper, we start with Tate's theorem to clarify the above mentioned roles of Schur's multiplicators of Galois groups using Hochschild-Serre exact sequences of a simple case, and, as a consequence, give a simple proof of Heider's theorem.

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§ 1. Preliminaries

Let k be either a local field with a finite quotient field or a global field, $\bar{k}$ the algebraic closure of k, and k_{ab} the maximal abelian extension of k in $\bar{k}$. Let K be a finite Galois extension of k and put $\mathfrak{g} = \mathrm{Gal}\,(K/k) = \mathfrak{G}/\mathfrak{G}_K$, $\mathfrak{G} = \mathfrak{G}_k = \mathrm{Gal}\,(\bar{k}/k)$ and $\mathfrak{G}_K = \mathrm{Gal}\,(\bar{k}/K)$.

The maximal genus extension of K/k is $k_{\mathrm{ab}}\cdot K$, whose Galois group over K is $\mathfrak{G}_K/\mathfrak{G}_K \cap \mathfrak{G}'$ where $\mathfrak{G}' = [\mathfrak{G}, \mathfrak{G}]$ is the communtator group of $\mathfrak{G}$. A central extension L of K/k is an extension of K which is normal over k and whose Galois group over K is contained in the center of $\mathrm{Gal}\,(L/k)$. Therefore L is an abelian extension of K. The maximal central extension $\mathrm{MC}_{K/k}$ of K/k contains $k_{\mathrm{ab}}\cdot K$, and its Galois group over K is $\mathfrak{G}_K/[\mathfrak{G}_K, \mathfrak{G}]$ where $[\mathfrak{G}_K, \mathfrak{G}]$ is the group (topologically) generated by the commutators $[\sigma, \tau]$, $\sigma \in \mathfrak{G}_K$, $\tau \in \mathfrak{G}$.

When k is a global field, let $k_A^\times$ be the idele group of k and $k^\#$ the kernel of the Artin map $\alpha_k\colon k_A^\times \to \mathrm{Gal}\,(k_{\mathrm{ab}}/k)$ of class field theory. For the finite Galois extension K of k, let $K_A^{\Delta\mathfrak{g}}$ be the closed subgroup of $K_A^\times$ generated by $x^{1-\sigma}$, $x \in K_A^\times$, $\sigma \in \mathfrak{g} = \mathrm{Gal}\,(K/k)$. Then $K_A^{\Delta\mathfrak{g}}\cdot K^\#$ is a closed subgroup of $K_A^\times$, and is contained in $\mathrm{N}_{K/k}^{-1}(k^\#)$ where $\mathrm{N}_{K/k}\colon K_A^\times \to k_A^\times$ is the norm map. By class field theory, we have the canonical isomorphisms

$$\begin{aligned}
&K_A^\times/K^\# \simeq \mathrm{Gal}\,(K_{\mathrm{ab}}/K) \simeq \mathfrak{G}_K/\mathfrak{G}_K'\,,\\
&\mathrm{N}_{K/k}^{-1}(k^\#)/K^\# \simeq \mathrm{Gal}\,(K_{\mathrm{ab}}/k_{\mathrm{ab}}\cdot K) \simeq \mathfrak{G}_K \cap \mathfrak{G}'/\mathfrak{G}_K'\,,\\
&K_A^\times/K_A^{\Delta\mathfrak{g}}\cdot K^\# \simeq \mathrm{Gal}\,(\mathrm{MC}_{K/k}/K) \simeq \mathfrak{G}_K/[\mathfrak{G}_K, \mathfrak{G}]\,,\\
&\mathrm{N}_{K/k}^{-1}(k^\#)/K_A^{\Delta\mathfrak{g}}\cdot K^\# \simeq \mathrm{Gal}\,(\mathrm{MC}_{K/k}/k_{\mathrm{ab}}\cdot K) \simeq \mathfrak{G}_K \cap \mathfrak{G}'/[\mathfrak{G}_K, \mathfrak{G}]\,.
\end{aligned}$$

Let $\mathfrak{P}$ be a prime divisor of K, $\mathfrak{p} = \mathfrak{P} \cap k$, and $\mathrm{N}_{\mathfrak{P}}\colon K_{\mathfrak{P}}^\times \to k_{\mathfrak{p}}^\times$ the norm map where $K_{\mathfrak{P}}$ and $k_{\mathfrak{p}}$ are the completions of K and k, respectively. Let $\mathrm{LG}_{K/k}$ be the subfield of K_{ab} such that

$$\mathrm{Gal}\,(K_{\mathrm{ab}}/\mathrm{LG}_{K/k}) \simeq K^\#\cdot \prod_{\mathfrak{P}} \mathrm{N}_{\mathfrak{P}}^{-1}(1)/K^\#$$

by class field theory. An abelian extension L of K is contained in $\mathrm{LG}_{K/k}$ if and only if $L\cdot K_{\mathfrak{P}}$ is contained in $k_{\mathfrak{p},\mathrm{ab}}\cdot K_{\mathfrak{P}}$ for every $\mathfrak{P}$. Such an extension L of K is just an EL-genus extension of K/k. (See Furuta [2].) It is easy to see

$$\mathrm{N}_{K/k}^{-1}(1) = \mathrm{Ker}\,(\mathrm{N}_{K/k}\colon K_A^\times \to k_A^\times) = K_A^{\Delta\mathfrak{g}}\cdot \prod_{\mathfrak{P}} \mathrm{N}_{\mathfrak{P}}^{-1}(1)\,.$$

If $\mathfrak{P}$ is unramified over k, then $\mathrm{N}_{\mathfrak{P}}^{-1}(1)$ is contained in $K_A^{\Delta\mathfrak{g}}$. Therefore

$K_A^{\Delta\mathfrak{g}}$ is a subgroup of $N_{K/k}^{-1}(1)$ of finite index. By [7, Proposition 4], we have Masuda's result [6]:

$$\begin{aligned}\mathrm{Gal}\,(\mathrm{MC}_{K/k} \cap \mathrm{LG}_{K/k}/k_{\mathrm{ab}}\cdot K) &\simeq N_{K/k}^{-1}(k^{\#})/N_{K/k}^{-1}(1)\cdot K^{\#}\\ &\simeq k^{\times} \cap N_{K/k}(K_A^{\times})/N_{K/k}(K^{\times})\,.\end{aligned}$$

§ 2. The structure of $\mathrm{Gal}\,(\mathrm{MC}_{K/k}/k_{\mathrm{ab}}\cdot K)$

Let K/k be a fiinte Galois extension of either local fields or global fields.

The additive group $\boldsymbol{Q}/\boldsymbol{Z}$ with the discrete topology is considered a $\mathfrak{G}$-module with the trivial action. As a special case of Hochschild-Serre exact sequences [5], we have an exact sequence

$$\mathrm{Hom}\,(\mathfrak{G}, \boldsymbol{Q}/\boldsymbol{Z}) \xrightarrow[r_1]{} \mathrm{Hom}\,(\mathfrak{G}_K, \boldsymbol{Q}/\boldsymbol{Z})^{\mathfrak{G}} \xrightarrow[t_2]{} \mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z}) \xrightarrow[l_2]{} \mathrm{H}^2(\mathfrak{G}, \boldsymbol{Q}/\boldsymbol{Z})\,.$$

The homomorphism r_1, is the restriction, and l_2 is the inflation (or the lift). Let $\xi: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{G}_K$ be a factor system of the group extension $\mathfrak{G}_K \to \mathfrak{G} \to \mathfrak{g}$. For $\varphi \in \mathrm{Hom}\,(\mathfrak{G}_K, \boldsymbol{Q}/\boldsymbol{Z})^{\mathfrak{G}}$, $\varphi \circ \xi$ is then a cocycle in $\mathrm{Z}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})$. The homomorphism t_2 assigns the class of $-\varphi \circ \xi$ to φ. It is obvious that

$$\begin{aligned}\mathrm{Hom}\,(\mathfrak{G}_K, \boldsymbol{Q}/\boldsymbol{Z})^{\mathfrak{G}} &= \{\psi \in \mathrm{Hom}\,(\mathfrak{G}_K, \boldsymbol{Q}/\boldsymbol{Z}) \mid \psi|_{[\mathfrak{G}_K, \mathfrak{G}]} = 0\}\,,\\ \mathrm{Im}\,(r_1) &= \{\varphi \in \mathrm{Hom}\,(\mathfrak{G}_K, \boldsymbol{Q}/\boldsymbol{Z}) \mid \varphi|_{\mathfrak{G}_K \cap \mathfrak{G}'} = 0\}\,.\end{aligned}$$

Therefore the cokernel of r_1 is naturally identified with

$$(\mathfrak{G}_K \cap \mathfrak{G}'/[\mathfrak{G}_K, \mathfrak{G}])^* = \mathrm{Hom}\,(\mathfrak{G}_K \cap \mathfrak{G}'/[\mathfrak{G}_K, \mathfrak{G}], \boldsymbol{Q}/\boldsymbol{Z})$$

which is the dual group of the abelian group $\mathfrak{G}_K \cap \mathfrak{G}'/[\mathfrak{G}_K, \mathfrak{G}]$.

The following theorem is the basis of this paper:

TATE'S THEOREM.

$$\mathrm{H}^2(\mathfrak{G}_k, \boldsymbol{Q}/\boldsymbol{Z}) = \{0\}\,.$$

(See Serre [8].) Let us denote the dual of the finite abelian group $\mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})$ by $\mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})^*$. By Tate's Theorem, we have, by t_2,

THEOREM 1 (Local case). *There exists canonical isomorphisms*

$$\mathrm{Gal}\,(\mathrm{MC}_{K/k}/k_{\mathrm{ab}}\cdot K) \simeq N_{K/k}^{-1}(1)/K^{\Delta\mathfrak{g}} \simeq \mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})^*$$

where $K^{\Delta\mathfrak{g}}$ *is the subgroup of* $K^{\times}$ *generated by* $a^{1-\sigma}$, $a \in K^{\times}$, $\sigma \in \mathfrak{g}$.

LEMMA. *Suppose that K/k is a finite Galois extension of global fields. Then we have*

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(i) $N^{-1}_{K/K}(k^{\#}) = N^{-1}_{K/k}(k^{\times})\cdot k^{\#}$;

(ii) $N^{-1}_{K/k}(k^{\times}) \cap K^{\#} = K^{\times}\cdot(K^{\#} \cap K_A^{\Delta\mathfrak{g}})$;

(iii) $N^{-1}_{K/k}(k^{\#})/K_A^{\Delta\mathfrak{g}}\cdot K^{\#} \simeq N^{-1}_{K/k}(k^{\times})/K_A^{\Delta\mathfrak{g}}\cdot K^{\times}$.

Proof. If k is a function field, then $k^{\#} = k^{\times}$ and $K^{\#} = K^{\times}$. Therefore we consider the case of algebraic number fields. In the proof of Proposition 4 of [7], we showed (i). Let us see (ii). Put $d = [K:k]$. Let z be an element of $N^{-1}_{K/k}(k^{\times}) \cap K^{\#}$. By [7, Proposition 1], we have $a \in K^{\times}$ and $y \in K^{\#}$ such that $z = a\cdot y^d$. Put $x = N_{K/k}(y) \in k^{\#}$. Then $y^d = x\cdot\prod_{\sigma\in\mathfrak{g}} y^{1-\sigma}$. Obviously $x \in N^{-1}_{K/k}(k^{\times}) \cap K^{\#}$. Therefore $x^d = N_{K/k}(x) \in k^{\times}$. By [7, Proposition 2, (1)], we have $b \in k^{\times}$ such that $x^d = b^d$. Put $w = x\cdot b^{-1}$. Then $w \in k^{\#}$ and $w^d = 1$. Therefore, by [7, Proposition 2, (2)], we have $c \in k^{\times}$ and $v \in k^{\times}_{\infty+}$ such that $w = c\cdot v$. Here $k^{\times}_{\infty+}$ is the connected component of the unity of the Archimedean part of $k^{\times}_A$. Therefore $w^d = 1$ implies $v^d = 1$. Let $\mathfrak{p}$ be an Archimedean prime divisor. If $\mathfrak{p}$ is real, then, at $\mathfrak{p}$, v should be equal to 1 which is the only root of 1 in $\boldsymbol{R}^{\times}_{+}$. If $\mathfrak{p}$ is complex, then $\mathfrak{p}$ splits completely in K. Therefore $v \in K_A^{\Delta\mathfrak{g}}$ since $v^d = 1$. Thus we have $z = (abc)\cdot v\cdot\prod_{\sigma\in\mathfrak{g}} y^{1-\sigma}$ with $abc \in K^{\times}$ and $v\cdot\prod y^{1-\sigma} \in K_A^{\Delta\mathfrak{g}} \cap K^{\#}$. This shows that the left hand side of (ii) is contained in the right. The converse in clear. One can see (iii) at once by (i) and (ii).

THEOREM 2 (Global case). *There exist canonical isomorphisms*

$$\mathrm{Gal}\,(\mathrm{MC}_{K/k}/k_{\mathrm{ab}}\cdot K) \simeq N^{-1}_{K/k}(k^{\times})/K_A^{\Delta\mathfrak{g}}K^{\times} \simeq \mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})^* .$$

Let us see the last isomorphism explicitly. Let $\zeta_{K/k}$ be a cocycle of the canonical class of $\mathrm{H}^2(\mathfrak{g}, K_A^{\times}/K^{\times})$. (See Hochschild-Nakayama [4]). Composing the Artin map $\bar{\alpha}_k\colon K_A^{\times}/K^{\times} \to \mathfrak{G}_K/\mathfrak{G}'_K$, we have a cocycle $\bar{\alpha}_K \circ \zeta_{K/k}$ of $\mathrm{Z}^2(\mathfrak{g}, \mathfrak{G}_K/\mathfrak{G}'_K)$, which is cohomologuous to the cocycle obtained by the factor system ξ modulo $\mathfrak{G}'_K$. Therefore, the isomorphism

$$t^*\colon \mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})^* \longrightarrow N^{-1}_{K/k}(k^{\times})/K_A^{\Delta\mathfrak{g}}\cdot K^{\times}$$

of the theorem is the dual of the isomorphism

$$t\colon (N^{-1}_{K/k}(k^{\times})/K_A^{\Delta\mathfrak{g}}K^{\times})^* \longrightarrow \mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})$$

which is described as follows: For $\varphi \in (N^{-1}_{K/k}(k^{\times})/K_A^{\Delta\mathfrak{g}}K^{\times})^*$, take $\tilde{\varphi} \in \mathrm{Hom}\,(K_A^{\times}/K^{\times}, \boldsymbol{Q}/\boldsymbol{Z})$ such that

$$\tilde{\varphi}|_{N^{-1}_{K/k}(k^{\times})/K^{\times}} = \varphi \circ \mathrm{proj}\colon N^{-1}_{K/k}(k^{\times})/K^{\times} \longrightarrow N^{-1}_{K/k}(k^{\times})/K_A^{\Delta\mathfrak{g}}\cdot K^{\times} .$$

Then the class of the cocycle $-\tilde{\varphi}\circ\zeta_{K/k}$ in $\mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})$ is well-determined by φ, and is the image $t(\varphi)$ of φ.

§ 3. The structure of $\mathrm{Gal}(\mathrm{MC}_{K/k} \cap \mathrm{LG}_{K/k}/k_{\mathrm{ab}} \cdot K)$

From now on, we restrict ourselves to the case of global fields.

Since $\bar{k}_{\mathfrak{p}} = \bar{k} \cdot k_{\mathfrak{p}}$ and $K_{\mathfrak{P},\mathrm{ab}} = K_{\mathrm{ab}} \cdot K_{\mathfrak{P}}$, the imbedding $\tilde{\iota}_{\mathfrak{P}}\colon \mathrm{Gal}(\bar{k}_{\mathfrak{p}}/k_{\mathfrak{p}}) \to \mathfrak{G} = \mathrm{Gal}(\bar{k}/k)$ defined by $\tilde{\iota}_{\mathfrak{P}}(\sigma) = \sigma|_{\bar{k}}$ for $\sigma \in \mathrm{Gal}(\bar{k}_{\mathfrak{p}}/k_{\mathfrak{p}})$ induces an imbedding $\mathrm{Gal}(K_{\mathfrak{P},\mathrm{ab}}/K_{\mathfrak{P}}) \to \mathrm{Gal}(K_{\mathrm{ab}}/K)$ which corresponds to the imbedding $\iota_{\mathfrak{P}}\colon K_{\mathfrak{P}}^{\times} \to K_A^{\times}$ by the Artin maps. Furthermore, $\tilde{\iota}_{\mathfrak{P}}$ induces the natural imbedding map

$$\tilde{\iota}_{\mathfrak{P}}\colon \mathrm{Gal}(K_{\mathfrak{P}}/k_{\mathfrak{p}}) \longrightarrow \mathfrak{g} = \mathrm{Gal}(K/k)\ ,$$

whose image $\mathfrak{g}_{\mathfrak{P}}$ is the decomposition group of $\mathfrak{P}$. If we choose the representation map $S\colon \mathfrak{g} \to \mathfrak{G}$ so that $S(\mathfrak{g}_{\mathfrak{P}})$ is contained in $\tilde{\iota}_{\mathfrak{P}}(\mathrm{Gal}(\bar{k}_{\mathfrak{p}}/k_{\mathfrak{p}}))$, then the factor system $\xi_{\mathfrak{P}}$ of the group extension $\tilde{\iota}_{\mathfrak{P}}(\mathrm{Gal}(\bar{k}_{\mathfrak{p}}/K_{\mathfrak{P}})) \to \tilde{\iota}_{\mathfrak{P}}(\mathrm{Gal}(\bar{k}_{\mathfrak{p}}/k_{\mathfrak{p}})) \to \mathfrak{g}_{\mathfrak{P}}$ is just the restriction of the factor system ξ of $\mathfrak{G}_K \to \mathfrak{G} \to \mathfrak{g}$. Therefore, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{N}_{\mathfrak{P}}^{-1}(1)/K_{\mathfrak{P}}^{\Delta \mathfrak{g}_{\mathfrak{P}}} & \xrightarrow[\iota_{\mathfrak{P}}]{} & \mathrm{N}_{K/k}^{-1}(k^{\times})/K_A^{\Delta \mathfrak{g}} \cdot K^{\times} \\
\downarrow \wr & & \downarrow \wr \\
\mathrm{H}^2(\mathfrak{g}_{\mathfrak{P}}, \boldsymbol{Q}/\boldsymbol{Z})^* & \xrightarrow[\tilde{\iota}_{\mathfrak{P}}^*]{} & \mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})^*\ ,
\end{array}$$

where $\tilde{\iota}_{\mathfrak{P}}^*$ is the dual of $\mathrm{res}_{\mathfrak{P}}\colon \mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z}) \to \mathrm{H}^2(\mathfrak{g}_{\mathfrak{P}}, \boldsymbol{Q}/\boldsymbol{Z})$. Let

$$\tilde{\iota}^*\colon \textstyle\prod_{\mathfrak{P}} \mathrm{H}^2(\mathfrak{g}_{\mathfrak{P}}, \boldsymbol{Q}/\boldsymbol{Z})^* \longrightarrow \mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})^*$$

be the dual of

$$\tilde{\iota} = \prod \mathrm{res}_{\mathfrak{P}}\colon \mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z}) \longrightarrow \prod_{\mathfrak{P}} \mathrm{H}^2(\mathfrak{g}_{\mathfrak{P}}, \boldsymbol{Q}/\boldsymbol{Z})\ .$$

Then we recover the following well-known result by Theorems 1 and 2 with the last paragraph of § 1:

THEOREM 3.

$$\begin{aligned}
\mathrm{Gal}(\mathrm{MC}_{K/k} \cap \mathrm{LG}_{K/k}/k_{\mathrm{ab}} \cdot K) &\simeq k^{\times} \cap \mathrm{N}_{K/k}(K_A^{\times})/\mathrm{N}_{K/k}(K^{\times}) \\
&\simeq \mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})^*/\tilde{\iota}^*(\prod_{\mathfrak{P}} \mathrm{H}^2(\mathfrak{g}_{\mathfrak{P}}, \boldsymbol{Q}/\boldsymbol{Z})^*)\ .
\end{aligned}$$

§ 4. Finite central extensions

Let L be a finite abelian extension of K which is normal over k. Then L corresponds to an open subgroup U of $K_A^{\times}$ containing $K^{\#}$ such that $U^{\sigma} = U$ for $\forall \sigma \in \mathfrak{g}$ by $U = \mathrm{N}_{L/K}(L_A^{\times}) \cdot K^{\times}$. Let L_0 be the maximal central extension of K/k contained in L, L_0' the maximal EL-genus extension of

K/k in L_0, and L_1 the genus field of L with respect to K/k. Because $L_0 = L \cap \mathrm{MC}_{K/k}$, $L_0' = L \cap \mathrm{MC}_{K/k} \cap \mathrm{LG}_{K/k}$ and $L_1 = L \cap (k_{\mathrm{ab}} \cdot K)$, we have $\mathrm{Gal}(L_0/K) \simeq K_A^\times / U \cdot K_A^{\Delta\mathfrak{g}}$, $\mathrm{Gal}(L_0'/K) \simeq K_A^\times / U \cdot \mathrm{N}_{K/k}^{-1}(1)$, and

$$\mathrm{Gal}(L_1/K) \simeq K_A^\times / U \cdot \mathrm{N}_{K/k}^{-1}(k^\sharp) = K_A^\times / U \cdot \mathrm{N}_{K/k}^{-1}(k^\times) .$$

Therefore, we have

$$\begin{aligned} \mathrm{Gal}(L_0/L_1) &\simeq U \cdot \mathrm{N}_{K/k}^{-1}(k^\times) / U \cdot K_A^{\Delta\mathfrak{g}} \\ &\simeq \mathrm{N}_{K/k}^{-1}(k^\times) / (U \cap \mathrm{N}_{K/k}^{-1}(k^\times)) \cdot K_A^{\Delta\mathfrak{g}} , \\ \mathrm{Gal}(L_0'/L_1) &\simeq U \cdot \mathrm{N}_{K/k}^{-1}(k^\times) / U \cdot \mathrm{N}_{K/k}^{-1}(1) \\ &\simeq \mathrm{N}_{K/k}^{-1}(k^\times) / (U \cap \mathrm{N}_{K/k}^{-1}(k^\times)) \cdot \mathrm{N}_{K/k}^{-1}(1) \\ &\simeq k^\times \cap \mathrm{N}_{K/k}(K_A^\times) / (k^\times \cap \mathrm{N}_{K/k}(U)) \\ &= k^\times \cap \mathrm{N}_{K/k}(K_A^\times) / (k^\times \cap \mathrm{N}_{L/k}(L_A^\times)) \cdot \mathrm{N}_{K/k}(K^\times) . \end{aligned}$$

By Theorems 2 and 3, we have

THEOREM 4. *There exist canonical exact sequences*

$$(1) \quad \begin{aligned} 1 \longrightarrow K^\times \mathrm{N}_{L/K}(L_A^\times) \cap \mathrm{N}_{K/k}^{-1}(k^\times) / K^\times \mathrm{N}_{L/K}(L_A^\times) \cap K_A^{\Delta\mathfrak{g}} \cdot K^\times \\ \longrightarrow \mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})^* \longrightarrow \mathrm{Gal}(L_0/L_1) \longrightarrow 1 ; \end{aligned}$$

$$(2) \quad \begin{aligned} 1 \longrightarrow \mathrm{N}_{L/k}(L_A^\times) \cap k^\times / \mathrm{N}_{L/k}(L_A^\times) \cap \mathrm{N}_{K/k}(K^\times) \\ \longrightarrow \mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})^* / \hat{\imath}^*(\prod_{\mathfrak{P}} \mathrm{H}^2(\mathfrak{g}_{\mathfrak{P}}, \boldsymbol{Q}/\boldsymbol{Z})^*) \longrightarrow \mathrm{Gal}(L_0'/L_1) \longrightarrow 1 . \end{aligned}$$

Remark 1. Furuta [2] proceeded investigations on the structure of $\mathrm{Gal}(L_0'/L_1)$ to obtain its totally cohomological expression.

Remark 2. Shirai [9] investigated the exact sequence (1) more precisely in the case that L is a certain ray class field of K.

If L is a finite central extension of K/k, then $U = K^\times \mathrm{N}_{L/K}(L_A^\times)$ contains $K_A^{\Delta\mathfrak{g}} K^\times$. Therefore we have

COROLLARY. *For a finite central extension L of K/k, there is a canonical exact sequence*

$$\begin{aligned} 1 \longrightarrow K^\times \mathrm{N}_{L/K}(L_A^\times) \cap \mathrm{N}_{K/k}^{-1}(k^\times) / K_A^{\Delta\mathfrak{g}} \cdot K^\times \\ \longrightarrow \mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})^* \longrightarrow \mathrm{Gal}(L/L \cap (k_{\mathrm{ab}} \cdot K)) \longrightarrow 1 . \end{aligned}$$

Especially, $[L : L \cap (k_{\mathrm{ab}} \cdot K)] \leqq |\mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})|$.

Let us say that a finite central extension L of K/k is abundant if $[L : L \cap (k_{\mathrm{ab}} \cdot K)] = |\mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})|$.

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THEOREM 5. *There always exist abundant central extensions of K over k. A finite central extension L of K/k is abundant if and only if $L\cdot k_{\mathrm{ab}} = \mathrm{MC}_{K/k}$. If this is the case, then* $\mathrm{Gal}(L/L \cap (k_{\mathrm{ab}}\cdot K))$ *is isomorphic to the dual of Schur's multiplicator* $\mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})$ *of the Galois group* $\mathfrak{g} = \mathrm{Gal}(K/k)$.

Proof. Let $\pi\colon K_A^\times \to K_A^\times/K_A^{\Delta\mathfrak{g}}\cdot K^\#$ be the projection. Since

$$[\mathrm{N}_{K/k}^{-1}(k^\#)\colon K_A^{\Delta\mathfrak{g}}\cdot K^\#] = [\mathrm{N}_{K/k}^{-1}(k^\times)\colon K_A^{\Delta\mathfrak{g}}\cdot K^\times] = |\mathrm{H}^2(\mathfrak{g}, \boldsymbol{Q}/\boldsymbol{Z})|$$

is finite, $\pi(\mathrm{N}_{K/k}^{-1}(k^\#))$ is a finite subgroup of $\pi(K_A^\times)$. Therefore, there exists an open subgroup $\overline{U}$ of $\pi(K_A^\times)$ such that $\overline{U} \cap \pi(\mathrm{N}_{K/k}^{-1}(k^\#)) = \{1\}$. Put $U = \pi^{-1}(\overline{U})$. Then U is an open subgroup of $K_A^\times$, and contains $K_A^{\Delta\mathfrak{g}}\cdot K^\#$. Therefore, $[K_A^\times\colon U]$ is finite. Let L be the abelian extension of K corresponding to U. Then L is a finite central extension of K/k. By the choice of U, we have $U \cap \mathrm{N}_{K/k}^{-1}(k^\#) = K_A^{\Delta\mathfrak{g}}\cdot K^\#$. Since $U = K^\times \mathrm{N}_{L/K}(L_A^\times)$ we easily see that $K^\times\mathrm{N}_{L/K}(L_A^\times) \cap \mathrm{N}_{K/k}^{-1}(k^\times) = K_A^{\Delta\mathfrak{g}}\cdot K^\times$, by the lemma of Section 2. Therefore, it follows from the above corollary that L is abundant.

If L is a finite central extension of K/k, then $L\cdot k_{\mathrm{ab}} = \mathrm{MC}_{K/k}$ if and only if $\mathrm{Gal}(K_{\mathrm{ab}}/L) \cap \mathrm{Gal}(K_{\mathrm{ab}}/k_{\mathrm{ab}}\cdot K) = \mathrm{Gal}(K_{\mathrm{ab}}/\mathrm{MC}_{K/k})$, which is equivalent to the condition,

$$K^\times\mathrm{N}_{L/K}(L_A^\times) \cap \mathrm{N}_{K/k}^{-1}(k^\#) = K_A^{\Delta\mathfrak{g}}\cdot K^\#\,,$$

by class field theory. Therefore, one can easily see the theorem by the lemma of Section 2 and the above corollary to Theorem 4.

Remark 3. By Theorem 5, it is fundamental for the theory of central extensions of K/k to obtain an abundant central extension of the minimum degree. It might be of any worth to note that there exist infinitely many abundant central extensions of K/k whose Galois groups over K are isomorphic. In fact: Let L be an abundant central extension, and M an intermediate field of L/K. Take an abelian extension k_1 of k such that $k_1 \cap L = k$ and $\mathrm{Gal}(k_1/k) \simeq \mathrm{Gal}(L/M)$. Then $\mathrm{Gal}(L\cdot k_1/M) \simeq \mathrm{Gal}(L/M) \times \mathrm{Gal}(k_1\cdot M/M)$ where the last group is isomorphic to $\mathrm{Gal}(k_1/k)$. Fix an isomorphism $\varphi\colon \mathrm{Gal}(L/M) \to \mathrm{Gal}(k_1\cdot M/M)$. Then the subfield M_1 of $L\cdot k_1$ fixed by the subgroup $\{(\sigma, \varphi(\sigma)) \mid \sigma \in \mathrm{Gal}(L/M)\}$ of $\mathrm{Gal}(L\cdot k_1/M)$ is an abundant central extension of K/k such that $M_1 \cap L = M$ and $\mathrm{Gal}(M_1/K) \simeq \mathrm{Gal}(L/K)$.

PROBLEM. Is there an abundant central extension L of K/k such that $L \cap (k_{\mathrm{ab}}\cdot K) = K$? If not, then what determines the structure of

Gal $(L \cap (k_{ab} \cdot K)/K)$ for an abundant central extension of the minimum degree ?

Let k be the rational number field $\boldsymbol{Q}$, and K the m-th cyclotomic field with $(m, 16) \neq 8$. Shirai [9, Theorem 32] gave a natural abundant central extension $\hat{K}_{m p_\infty}$ such that $\hat{K}_{m p_\infty} \cap (k_{ab} \cdot K) = K$.

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