ON THE SEMIGROUP OF C^k SELFMAPS OF $\mathbb{R}^n$

Dedicated to the memory of Hanna Neumann

G. R. WOOD

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Introduction

Magill, Jr. and Yamamuro have been responsible in recent years for a number of papers showing that the property that every automorphism is inner is held by many semigroups of functions and relations on topological spaces. Following [9], we say a semigroup has the Magill property if every automorphism is inner. That the semigroup of Fréchet differentiable selfmaps $\mathscr{D}$, of a finite dimensional Banach space E, had the Magill property was shown in [10], while a lengthy result in [6] extended this to the semigroup of k times Fréchet differentiable selfmaps, $\mathscr{D}^k$, of a Fréchet Montel space (FM-space). In the latter paper it was noted that with a little additional effort the semigroup C^k, of k times continuously Fréchet differentiable selfmaps of FM-space, could be shown to possess the Magill property. It is the purpose of this paper to present a simpler proof of this result in the case where the underlying space is finite dimensional.

NOTATION: Notation and terminology are the same as in [10]. For basic definitions and a discussion of differential calculus in Banach spaces the reader is referred to [3, chapter 8]. We shall use Greek letters to denote elements of the reals, R, and Roman letters for the elements of the real finite dimensional Banach spaces E and F. The constant map, whose single value is $a \in E$, is denoted by c_a. $\mathscr{L}(E,F)$ denotes the space of linear maps from E into F with the norm topology. The dual of E is given by $\bar{E}$. If $\bar{a} \in \bar{E}$, $a \in E$ we define a map $a \otimes \bar{a}$ in $\mathscr{L}(E) = \mathscr{L}(E,E)$ by

$$(a \otimes \bar{a})(x) = \langle x, \bar{a} \rangle a, \text{ for } x \in E.$$

More generally we have $a \otimes^m \bar{a}$ in the space of m-linear maps $\mathscr{L}(E^m, E)$. If $h \in \mathscr{D}^m$ we write the mth Fréchet derivative of h at $x \in E$ as $h^{(m)}(x)$. After m evaluations at $a \in E$ we write this as $h^{(m)}(x)(a)^m$. We now prove

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THEOREM: *If E is a finite dimensional real Banach space, every automorphism ϕ of C^k is inner. That is, there exists a bijection $h \in C^k$ such that $h^{-1} \in C^k$ and*

$$\phi(f) = hfh^{-1}, \text{ for all } f \in C^k. \tag{1}$$

PROOF: 1. *There exists a bijection h such that* (1) *holds.*

Magill showed this for automorphisms of the semigroup of all differentiable selfmaps of the reals in [5]. For a proof in this more general instance see [7]. It was also shown we can assume $h(0) = 0$ in [8].

2. *$h(\xi a)$ is continuous in $\xi \in \mathbb{R}$, each $a \in E$.*

In [8] it was shown that if $a \in E$, $\langle h(\xi a), \bar{a}\rangle$ is continuous in ξ, for each $\bar{a} \in \bar{E}$. But the weak andstrong topologies in E coincide so the result follows.

Our aim is to show that $h \in C^k$, for since any property true of h is also true of h^{-1}, (1) will then hold. Whereas in [6] this was achieved by elementary methods, we show here that the problem can be arranged in such a way that a classical theorem concerning differentiability is applicable.

DEFINITION: A family $\{\psi(\xi) : \xi \in \mathbb{R}\}$ of selfmaps of E is said to be a one-parameter group if

$$\psi(\xi + \eta) = \psi(\xi)\psi(\eta), \text{ for any } \xi, \eta \in \mathbb{R}.$$

Bochner and Montgomery have proved the following in [2]: if E is finite dimensional, $\{\psi(\xi)\}$ a one-parameter group with $\psi(\xi)(x)$ separately continuous, and $\psi(\xi) \in C^k$ for each $\xi \in R$, then $\psi : \mathbb{R} \times E \to E$ is jointly C^k.

We define a one-parameter group of C^k selfmaps of E, $\{\psi(\xi)\}$, by $\psi(\xi) = \phi(e^\xi)$, $\xi \in R$. Continuity with respect to the parameter follows readily from the continuity of $h(\xi a)$ with respect to ξ. We show in steps 3 to 6 that the k times continuous differentiability with respect to the parameter suffices to show $h \in C^k$.

3. *For $\alpha \in \mathbb{R}, \alpha > 0$, $(d^k/d\alpha^k)h(\alpha x)$ exists and is continuous in α.*

Tedious differentiation shows that if $\alpha = e^\xi$, $y \in E$, and $m \in \mathbb{N}$, the set of natural numbers, then

$$(d^m/d\xi^m)h(e^\xi y) = \sum_{r=1}^{m} C_r^m e^{r\xi}(d^r/d\alpha^r)h(\alpha y) \tag{2}$$

provided we assume that these derivatives exist. The coefficients $C_r^m \in \mathbb{N}$ are given inductively by $C_1^1 = 1$, $C_r^m = rC_r^{m-1} + C_{r-1}^{m-1}$, for $r, m > 1$.

We show the result using the above and complete induction. When $k = 1$, since $\psi(\xi)(x)$ is continuously differentiable in ξ, we have the existence of

$$(d/d\xi)\, he^\xi h^{-1}(x) = e^\xi(d/d\alpha)h(\alpha y)$$

where $y = h^{-1}(x), \alpha = e^\xi$. Hence for $\alpha > 0$, $(d/d\alpha)h(\alpha y)$ exists. Since $(d/d\xi)h(e^\xi y)$ is continuous in ξ,

$$(d/d\alpha)h(\alpha y) = e^{-\xi}(d/d\xi)h(e^{\xi}y)$$

is also continuous in ξ. Theorems 8 and 9 of [4, page 95] ensure the continuity in α.

Assuming now that the result holds for all natural numbers less than some $m \in \mathbb{N}$, $m \leqq k$, we may use (2) and the existence and continuity in ξ of $(d^m/d\xi^m)h(e^{\xi}y)$ to give the existence and continuity in α of $(d^m/d\alpha^m)h(\alpha y)$ for all $\alpha > 0$ in an entirely parallel manner.

Let $\{e_i\}$ be the standard basis for $E = \mathbb{R}^n$, and $(\{e_i\}, \{\bar{e}_i\})$ a biorthogonal pair. Let $h_i = h(e_i \otimes \bar{e}_i)$.

4. $$h_i \in C^k, i = 1, \cdots, n$$

The proof is by induction on k. That $h_i \in \mathscr{D}$ follows as in [10]. The continuity of h_i' follows as in the general case so we omit it here. For the purposes of this section the index i may be fixed so we shall simplify the notation by referring to e_i as e.

Suppose $h_i \in \mathscr{D}^m$, some m, $1 \leqq m < k$. We can readily show $(d^m/d\alpha^m)h_i(\alpha e) = h_i^{(m)}(\alpha e)(e)^m$, and since $(d^{m+1}/d\alpha^{m+1})h(\alpha e) = (d^{m+1}/d\alpha^{m+1})h_i(\alpha e)$ exists and is continuous for $\alpha > 0$ we deduce the limit

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}[h_i^{(m)}(\alpha e + \varepsilon e)(e)^m - h_i^{(m)}(\alpha e)(e)^m]$$

exists and is continuous for such α. We call this limit $(h_i^{(m)})^*(\alpha e)(e)^{m+1}$.

We show that $h_i^{(m)}$ is differentiable at $\alpha e, \alpha > 0$, and that

$$h_i^{(m+1)}(\alpha e) = (h_i^{(m)})^*(\alpha e)(e)^{m+1} \otimes {}^{m+1}\bar{e}.$$

Consider

$$\|y\|^{-1} \|h_i^{(m)}(\alpha e + y) - h_i^{(m)}(y) - (h_i^{(m)})^*(\alpha e)(e)^{m+1} \otimes {}^{m+1}\bar{e}(y)\|.$$

For y such that $\langle y, \bar{e} \rangle = 0$ the expression is zero so we may neglect such values. Further, $h_i^{(m)}(x) = h_i^{(m)}(x)(e)^m \otimes {}^m\bar{e}$, so the expression is

$$\|y\|^{-1} |\langle y,\bar{e}\rangle| \, \|\langle y,\bar{e}\rangle^{-1} (h_i^{(m)}(\alpha e + \langle y,\bar{e}\rangle e)(e)^m \otimes {}^m\bar{e}$$

$$- h_i^{(m)}(\langle y,\bar{e}\rangle e)(e)^m \otimes {}^m\bar{e}) - (h_i^{(m)})^*(\alpha e)(e)^{m+1} \otimes {}^m\bar{e}\|$$

$$\leqq \sup_{\|y_1\|=1} \cdots \sup_{\|y_m\|=1} |\langle y_m, \bar{e}\rangle| \cdots |\langle y_1, \bar{e}\rangle| \, \|\langle y, \bar{e}\rangle^{-1}$$

$$(h_i^{(m)}(\alpha e + \langle y,\bar{e}\rangle e)(e)^m - h_i^{(m)}(\alpha e)(e)^m) - (h_i^{(m)})^*(\alpha e)(e)^{m+1}\|$$

which converges to zero as $\|y\|$ converges to zero since

$$\sup_{\|y_j\|=1} |\langle y_j, \bar{e}\rangle| = \|\bar{e}\| = 1, \quad j = 1, \cdots, m.$$

We can now show $h_i^{(m)}$ is differentiable at $x \in E$ with $h_i^{(m)}(x) = [\phi(1 + (\langle x, \bar{e}\rangle - \alpha) c_e) h_i]^{(m+1)}(\alpha e)$. We use the expansion for the higher deriva-

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tive of a composition function as found in [1, page 3]. Consider

$$\|y\|^{-1}\|h_i^{(m)}(x+y)-h_i^{(m)}(x)-[\phi(1+(\langle x,\bar{e}\rangle-\alpha)c_e)h_i]^{(m+1)}(\alpha e)(y)\|$$
$$\leqq \|\langle y,\bar{e}\rangle^{-1}[[\phi(1+(\langle x,\bar{e}\rangle-\alpha)c_e)h_i]^{(m)}(\alpha e+\langle y,\bar{e}\rangle e)$$
$$-[\phi(1+(\langle x,\bar{e}\rangle-\alpha)c_e)h_i]^{(m)}(\alpha e)]-[\phi(1+(\langle x,\bar{e}\rangle-\alpha)c_e)h_i]^{(m+1)}(\alpha e)(e)\|$$

which converges to zero as $\|y\|$ converges to zero, since $[\phi(1+(\langle x,\bar{e}\rangle-\alpha)c_e)h_i]^{(m+1)}(\alpha e)(e)$ exists.

Now choose an x_0 such that $\langle x_0,\bar{e}\rangle$ is positive. Let $\{x_n\}$ converge to x_0. Standard calculations show

$$\|h_i^{(m+1)}(x_n)-h_i^{(m+1)}(x_0)\| = \|h_i^{(m+1)}(\langle x_n,\bar{e}\rangle e)(e)^{m+1}-h_i^{(m+1)}(\langle x_0,\bar{e}\rangle e)(e)^{m+1}\|.$$

But $h_i^{(m+1)}(\langle x_0,\bar{e}\rangle e)(e)^{m+1} = (h_i^{(m)})^*(\langle x_0,\bar{e}\rangle e)(e)^{m+1}$ which is known to be continuous at x_0. Thus $h_i^{(m+1)}$ is continuous at x_0. Letting x be an arbitrary element in E, and $\{x_n\}$ converge to x it may readily be shown that

$$\|h_i^{(m+1)}(x_n)-h_i^{(m+1)}(x)\|$$
$$= \|[\phi(1+\langle x-x_0,\bar{e}\rangle c_e)h_i]^{(m+1)}(x_0+x_n-x)$$
$$-[\phi(1+\langle x-x_0,\bar{e}\rangle c_c)h_i]^{(m+1)}(x_0)\|,$$

so $h_i\in C^{m+1}$. Inductively we have now shown that for each i, $h_i\in C^k$.

5. $$(e_i\otimes\bar{e}_i)h\in C^k,\quad i=1,\cdots,n.$$

Since $h_i\in C^k$ it follows that

$$\phi^{-1}[h_i] = h^{-1}(h(e_i\otimes\bar{e}_i))h = (e_i\otimes\bar{e}_i)h\in C^k.$$

6. $$h\in C^k.$$

Again the proof is by induction. Differentiability of h follows as in [10], while the continuity of h' follows as in the general case. Suppose $h\in C^m$, some m, $1\leqq m<k$. Then $[(e_i\otimes\bar{e}_i)h]^{(m+1)}(x)(y)$ exists, and for a null sequence $\{\varepsilon_n\}$ equals,

$$\lim_{n\to\infty}(e_i\otimes\bar{e}_i)[\varepsilon_n^{-1}(h^{(m)}(x+\varepsilon_n y)-h^{(m)}(x))].$$

Since the weak and strong convergence of a sequence of operators in $\mathscr{L}(E^m,E)$ coincide, we have that $\lim_{n\to\infty}\varepsilon_n^{-1}[h^{(m)}(x+\varepsilon_n y)-h^{(m)}(x)]$ exists.

If we call this $(h^{(m)})^*(x)(y)$ then for each i,

$$[(e_i\otimes\bar{e}_i)h]^{(m+1)}(x)(y) = (e_i\otimes\bar{e}_i)(h^{(m)})^*(x)(y). \tag{3}$$

Because $\{\bar{e}_i\}$ is a basis for $\bar{E}$ and $e_i\neq 0$, each i, it follows that $(h^{(m)})^*(x)\in\mathscr{L}(E^{m+1},E)$. Assume now that this is not the Fréchet derivative of $h^{(m)}$ at x. Then there is a null sequence $\{y_n\}$, sequences $\{z_n^1\},\cdots,\{z_n^m\}$ in the

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closed unit sphere S of E, and an $\bar{e}_i$ such that

$$\langle \|y_m\|^{-1}[h^{(m)}(x+y_n)(z_n^1)\cdots(z_n^m) - h^{(m)}(x)(z_n^1)\cdots(z_n^m) - (h^{(m)})^*(x)(y_n)(z_n^1)\cdots(z_n^m)], \bar{e}_i\rangle$$

does not converge to zero with n. But this contradicts (3).

We now show $h \in C^{m+1}$, again by contradiction. Suppose $h^{(m)}$ is discontinuous at $x \in E$. Then there is a sequence $\{x_n\}$ converging to x, sequences $\{y_n^1\}, \cdots, \{y_n^{m+1}\}$ in S, and an $\bar{e}_i$ such that $\langle [h^{(m+1)}(x_n)\{y_n^1\}\cdots(y_n^{m+1}) - h^{(m+1)}(x)\{y_n^1\}\cdots(y_n^{m+1})], \bar{e}_i\rangle$ does not go to zero with n. This contradicts the continuity of $[(e_i \otimes \bar{e}_i)h]^{(m+1)}$ at x. Hence $h \in C^k$ and the theorem is proved.

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Department of Mathematics
Institute of Advanced Studies
Australian National University
Canberra, ACT 2600.

Present address
University of Canterbury
Christchurch 1
New Zealand.

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