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SOME RESULTS ON QUASI-UNIFORM SPACES

BY
KAREN S. CARTER AND T. L. HICKS

ABSTRACT. Constructions are made of a T_1 space which does not have a T_1 completion and of a quasi-uniform space which is complete, but not strongly complete. An example relating to a completion due to Popa is given. An alternate definition for Cauchy filter, called C-filter, is examined and a construction of a C-completion is given. We discuss quasi-pseudometrics over a Tikhonov semifield R^{Δ}. Every topological space is quasi-pseudometrizable over a suitable R^{Δ}. It is shown that if a quasi-pseudometric space over R^{Δ} is complete, the corresponding quasi-uniform structure is C-complete. A general method for constructing compatible quasi-uniform structures is given.

The topological concepts used in this paper are as defined in Gaal [7]. The basic definitions relating to quasi-uniform spaces are given in Murdeshwar and Naimpally [9].

DEFINITION 1.1. Let X be a nonvoid set. A *quasi-uniform structure*, $\mathscr{U}$, on X is a filter on $X\times X$ satisfying:

(1) $\Delta=\{(x, x):x\in X\}\subseteq U$ for each $U\in\mathscr{U}$;

(2) if $U\in\mathscr{U}$, then there is a $V\in\mathscr{U}$ such that $V\circ V\subseteq U$.

DEFINITION 1.2. If $(X,\mathscr{U})$ is a quasi-uniform space, we obtain a topology $\mathscr{t}_{\mathscr{U}}$ on X by taking as a base for the neighborhood system at $x\in X$, the collection $\mathscr{N}(x)=\{U[x]:U\in\mathscr{U}\}$ and we say that $\mathscr{U}$ *generates* $\mathscr{t}_{\mathscr{U}}$. If $\mathscr{t}$ is a topology on X and $\mathscr{t}_{\mathscr{U}}=\mathscr{t}$, then $\mathscr{t}$ is said to be *compatible with* $\mathscr{U}$. For each $O\in\mathscr{t}$, define

$$S(O) = O\times O \cup (X-O)\times X.$$

Pervin [10] showed that $\{S(O):O\in\mathscr{t}\}$ is a subbase for a quasi-uniform structure which is compatible with $\mathscr{t}$. We shall denote this structure by $\mathscr{P}$ and refer to it as the *Pervin* structure. A quasi-uniform structure on a set X is said to be *transitive* if there is a base $\mathscr{B}$ for the structure such that $B\in\mathscr{B}$ implies that $B\circ B=B$.

DEFINITION 1.3. Let $(X,\mathscr{U})$ be a quasi-uniform space and let $\mathscr{F}$ be a filter on X. If for each $U\in\mathscr{U}$ there is an $x\in X$ such that $U[x]\in\mathscr{F}$, we say that $\mathscr{F}$ is *$\mathscr{U}$-Cauchy*. We define $(X,\mathscr{U})$ to be *complete* (*strongly complete*) if every $\mathscr{U}$-Cauchy filter has nonempty adherence (limit).

DEFINITION 1.4. $(Y,\mathscr{V})$ is a *completion* of $(X,\mathscr{U})$ if $(Y,\mathscr{V})$ is complete and $(X,\mathscr{U})$ is quasi-uniformly isomorphic to a dense subset of $(Y,\mathscr{V})$. *Strong completion* is defined similarly.

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DEFINITION 1.5. Suppose $(X, \mathscr{t})$ is a topological space. If $x \in O \in \mathscr{t}$, *a cover of X about* (x, O) is an open cover $\mathscr{A}$ of X such that there is an $A \in \mathscr{A}$ with $x \in A \subseteq O$. Let $\mathscr{B}$ be an open cover of X such that for every $x \in X$, $\bigcap\{B \in \mathscr{B}: x \in B\}$ is open. Then $\mathscr{B}$ is called a *Q-cover of X*.

DEFINITION 1.6. Suppose that $(X, \mathscr{t})$ is a topological space and let $\mathscr{C}$ be a collection of Q-covers of X satisfying the following condition: for each $O \in \mathscr{t}$ and each $x \in O$, $\mathscr{C}$ contains a cover of X about (x, O). Define $\mathscr{U}_{\mathscr{C}}=\{U_{\mathscr{A}}: \mathscr{A} \in \mathscr{C}\}$ where $U_{\mathscr{A}}=\bigcup\{\{x\}\times O_x^{\mathscr{A}}: x \in X\}$ and $O_x^{\mathscr{A}}=\bigcap\{A_x: x \in A_x \in \mathscr{A}\}$.

Fletcher [5] proves that $\mathscr{U}_{\mathscr{C}}$ is a compatible transitive quasi-uniform structure called a *covering quasi-uniformity*. He also proves that a quasi-uniform structure is transitive if and only if it is a covering quasi-uniformity.

2. **Some examples concerning completeness.** One would like for separation properties of the original space to carry over to the completion or strong completion. Carlson and Hicks [3] give an example of a T_2 quasi-uniform space which does not have a T_2 completion and, therefore, does not have a T_2 strong completion. They also give an example of a discrete space which does not have a T_1 strong completion. As we show below, this same space is also an example of a discrete space which does not have a T_1 completion.

EXAMPLE 2.1. Let $N=\{1, 2, 3, \ldots\}$. Define $U_n=\{(x, y): x=y \text{ or } x \geq n\}$. The collection $\{U_n: n \in N\}$ is a base for a quasi-uniform structure $\mathscr{U}$. Moreover, the topology $\mathscr{t}$ generated by $\mathscr{U}$ is discrete. Let $\mathscr{F}$ be the filter on N consisting of all subsets of N which have finite complements. $\mathscr{F}$ is easily seen to be $\mathscr{U}$-Cauchy. Next suppose $(N^*, \mathscr{U}^*)$ is a T_1 completion for $(N, \mathscr{U})$. Now $\mathscr{F}$ generates a $\mathscr{U}^*$-Cauchy filter $\mathscr{F}^*$ on N^*. Since $(N^*, \mathscr{U}^*)$ is complete, there is an $x^* \in N^*$ such that $x^* \in \operatorname{adh} \mathscr{F}^*$. We first show that $x^* \notin N$. If $x^* \in N$, there is an open set $O^* \in \mathscr{t}^*$ such that $O^* \cap N=\{x^*\}$. Now $N-\{x^*\} \in \mathscr{F} \subseteq \mathscr{F}^*$. By the above, $O^* \cap (N-\{x^*\})=\varnothing$ and, thus, $x^* \notin \operatorname{cl}(N-\{x^*\})$, a contradiction. Next, let B^* be an open set containing x^*. There is a $U^* \in \mathscr{U}^*$ such that $x^* \in U^*[x^*] \subseteq B^*$. Let $W^* \in \mathscr{U}^*$ with $W^* \circ W^* \subseteq \mathscr{U}^*$. We claim that $W^*[x^*] \cap N$ is infinite. Suppose $A=W^*[x^*] \cap N$ is finite. Then $N-A \in \mathscr{F} \subseteq \mathscr{F}^*$. And, $W^*[x^*] \cap (N-A)=\varnothing$ which implies that $x^* \notin \operatorname{cl}(N-A)$. From this it follows that $x^* \notin \operatorname{adh} \mathscr{F}^*$, a contradiction. Thus, we see that if $k \in N$, there is an $s>k$ such that $s \in W^*[x^*] \cap N$. Now there is an $m \in N$ with $U_m \subseteq W^* \cap (N \times N)$ and an $s>m$ such that $s \in W^*[x^*] \cap N$. Therefore, $(x^*, s) \in W^*$ and $\{s\} \times N \subseteq W^*$ and, hence, $\{x^*\} \times N \subseteq W^* \circ W^* \subseteq U^*$. It follows that $N \subseteq U^*[x^*] \subseteq B^*$; that is, every open set containing x^* contains N. But, then $(N^*, \mathscr{t}^*)$ could not be T_1.

One question which naturally arises is does there exist a quasi-uniform space which is complete, but not strongly complete. As the following example shows, the answer is affirmative.

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EXAMPLE 2.2. Let X be the integers. For $n \in X$, define $U_n = \Delta \cup \{(x, y): x \geq n, y = 0 \text{ or } 1\}$. Now $\{U_n : n \in X\}$ is a base for a quasi-uniform structure $\mathscr{U}$ on $X \times X$ and the topology $\mathscr{t}$ generated by $\mathscr{U}$ is discrete. Suppose that $\mathscr{F}$ is a Cauchy filter. Now if $n \in X$, $U_n[x] = \{0, 1, x\}$ if $x \geq n$ and $U_n[x] = \{x\}$ if $x < n$. It easily follows that $\mathscr{F}$ must be generated by a finite set and that adh $\mathscr{F} \neq \phi$. Let $\mathscr{S}$ be the collection of all supersets of $\{0, 1\}$. Now lim $\mathscr{S} = \phi$ and, in fact, $\mathscr{S}$ is the only non-convergent $\mathscr{U}$-Cauchy filter.

Popa [11] gave a construction that yields the following result.

THEOREM. *Let $(X, \mathscr{U})$ be a T_2 quasi-uniform space. Then there exists a strongly complete quasi-uniform space $(\hat{X}, \hat{\mathscr{U}})$ and a uniformly continuous mapping $\phi: X \to \hat{X}$ having the following properties:*

(a) *For every uniformly continuous mapping $f: X \to Y$, Y being a strongly complete quasi-uniform space, there exists a unique uniformly continuous mapping $g: \hat{X} \to Y$ such that $f = g \circ \phi$.*

(b) *The pair $(\hat{X}, \phi)$ is unique up to an isomorphism of quasi-uniform spaces.*

Popa calls $(\hat{X}, \hat{\mathscr{U}})$ "the" completion of $(X, \mathscr{U})$. In a uniform space setting, one proves the above theorem and also proves:

(c) *$\phi: X \to \hat{X}$ is an isomorphism of X onto a dense subspace of $\hat{X}$.*

It seems reasonable to inquire about the status of (c) in this setting. In fact, most authors would not call $\hat{X}$ a completion unless X is isomorphic to a dense subspace of $\hat{X}$. The following example show that "Popa's completion" is not a good candidate for a completion of a quasi-uniform space.

EXAMPLE 2.3. The following construction gives a quasi-uniform structure $\mathscr{U}$ for the set N of natural numbers such that:

1. The topology $\mathscr{t}_{\mathscr{U}}$ generated by $\mathscr{U}$ is the discrete topology and
2. Popa's completion $(\hat{N}, \hat{\mathscr{U}})$ of $(N, \mathscr{U})$ is a single point.

Let $U_n = \{(x, y): x = y \text{ or } x \geq n\}$, $\mathscr{B} = \{U_n : n \in N\}$, and let $\mathscr{U}$ denote the quasi-uniform structure generated by the base $\mathscr{B}$. If $x < n$, $U_n[x] = \{y: (x, y) \in U_n\} = \{x\}$. Thus $\mathscr{t}_{\mathscr{U}}$ is the discrete topology. If $\mathscr{F} = \{N\}$, $\mathscr{F}$ is $\mathscr{U}$-Cauchy since $U_n[n] = N$. Let $\phi: (N, \mathscr{U}) \to (\hat{N}, \hat{\mathscr{U}})$ be the uniformly continuous mapping constructed by Popa. $\phi(\mathscr{F}) = \{\phi(N)\}$ is a base for a $\hat{\mathscr{U}}$-Cauchy filter $\mathscr{F}^* = \{A: \phi(N) \subseteq A\}$ so there exists $z \in \hat{N}$ such that $\mathscr{F}^*$ converges to z. It follows that $\phi(N)$ is the only neighborhood of z in the subspace $\phi(N)$. The space is T_2 so $\phi(N) = \{z\}$ and cl $\phi(N) = N$ gives $\hat{N} = \{z\}$.

3. **On the definition of Cauchy filter.** The present definition of Cauchy filter, proposed by Sieber and Pervin [12], is an extension of the concept of Cauchy filter for a uniform space and, moreover, convergent filters are clearly Cauchy.

We would like to have a definition of Cauchy filter which would allow us to construct completions which preserve more of the separation properties than is possible with the present definition.

DEFINITION 3.1. Let $(X, \mathscr{U})$ be a quasi-uniform space and let $\mathscr{F}$ be a filter on X. We say that $\mathscr{F}$ is a *C-filter* with respect to $\mathscr{U}$ if $\mathscr{F}$ satisfies either of the following two conditions:

(i) given $U \in \mathscr{U}$, there is an F in $\mathscr{F}$ such that $F \times F \subset U$;

(ii) $\lim \mathscr{F} \neq \phi$.

The concepts of C-complete, C-strong complete, C-completion, and C-strong completion are defined in the obvious manner.

It is clear that in the uniform space case the concepts of Cauchy filter and C-filter are precisely the same. One may easily show that if $\mathscr{F}$ is a filter satisfying condition (i) of the definition of C-filter, then $\text{adh}\, \mathscr{F} = \lim \mathscr{F}$; and thus, if $\mathscr{F}$ is a C-filter such that $\text{adh}\, \mathscr{F} \neq \phi$, then $\lim \mathscr{F} \neq \phi$. We see, therefore, that the concepts of C-complete and C-strong complete coincide. Although the concepts of complete and strongly complete do coincide for uniform spaces, we have shown in example 2.2 that they are not the same for quasi-uniform spaces.

Using the current definition of Cauchy filter, Sieber and Pervin [12] obtain a generalization of the Niemytzki-Tychonoff theorem. If we use the definition of C-filter and replace the concept of precompactness by that of total boundedness, we may similarly derive the following:

THEOREM 3.1. *A topological space $(X, \mathscr{t})$ is compact if and only if it is C-complete with respect to every compatible quasi-uniformity.*

Proof. The proof of the theorem given by Sieber and Pervin [12] carries over with minor changes.

We remark that every finite space is C-complete. This follows from the fact that every finite space has a unique compatible quasi-uniform structure generated by a single set, as shown by Fletcher [4].

One may show that a C-filter is Cauchy in the usual sense. It then follows that if $(X, \mathscr{U})$ is complete, then $(X, \mathscr{U})$ is also C-complete. Hence, any completion or strong completion would also be a C-completion. One might, therefore, hope to be able to obtain better results with C-completions. The following example shows that a T_2, locally connected space may have a T_2, locally connected C-completion, but not have a T_2, locally connected strong completion. This illustrates that the concept of C-filter is an improvement over that of Cauchy filter.

EXAMPLE 3.1. Let $X = \{1, 2, 3, \ldots\}$. Define $U_n = \Delta \cup \{(x, y) : x \geq n \text{ and } y \geq x\}$. Now $\{U_n : n \in X\}$ is a base for a quasi-uniform structure $\mathscr{U}$ on X and $(X, \mathscr{t}_{\mathscr{U}})$ is discrete. Suppose that $\mathscr{F}$ is a C-filter. If $\lim \mathscr{F} = \phi$, then there is an $F \in \mathscr{F}$ such that $F \times F \subseteq U_n$. In this case, $\mathscr{F}$ must be the collection of all supersets of a singleton

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set $\{x\}$. Therefore, $(X, \mathscr{U})$ is easily seen to be a C-completion of itself. Let $\mathscr{G}$ be the filter generated by the collection $\{G_n : n \in X\}$ and $G_n = \{n, n+1, \ldots\}$. Then $\mathscr{G}$ is $\mathscr{U}$-Cauchy, but it does not converge. Therefore, $(X, \mathscr{U})$ is not strongly complete. Now let $(X^*, \mathscr{U}^*)$ be a T_2, locally connected strong completion for $(X, \mathscr{U})$. Now $\mathscr{G}$ generates a $\mathscr{U}^*$-Cauchy filter $\mathscr{G}^*$. Since $(X^*, \mathscr{U}^*)$ is strongly complete, there is an $x^* \in X^*$ such that $x^* \in \lim \mathscr{G}^*$. Therefore, if $U^* \in \mathscr{U}^*$, $U^*[x^*] \supseteq \{n, n+1, \ldots\}$ for some $n \in X$. We first show $x^* \notin X$. Suppose that $x^* = n \in X$. There is a U^* in $\mathscr{U}^*$ such that $U^* \cap (X \times X) = U_{n+1}$. Therefore, $U^*[n] \cap X = \{n\}$. Since $X \in \mathscr{G} \subseteq \mathscr{G}^*$, we obtain the contradiction $\{n\} \in \mathscr{G}^*$. Thus, $x^* \in X^* - X$. Next we show $\{x^*\} = X^* - X$. Let $y^* \in X^* - X$ with $y^* \neq x^*$. Since $(X^*, \mathscr{U}^*)$ is T_2, there is a $V^* \in \mathscr{U}^*$ such that $V^*[x^*] \cap V^*[y^*] = \phi$. Since there is a k in X such that $V^*[x] \supseteq \{k, k+1, \ldots\}$, $V^*[y^*] \cap X \subseteq \{1, \ldots, k-1\}$. Let $m \in X$. Since X is discrete, there is a $W_1^* \in \mathscr{U}^*$ such that $W_1^*[n] \cap X = \{n\}$. Since $(X^*, \mathscr{U}^*)$ is T_2, there is a $W_2^* \in \mathscr{U}^*$ such that $W_2^*[n] \cap W_2^*[y^*] = \phi$. Letting $W^* = W_1^* \cap W_2^*$, we have that $W^*[y^*] \cap X = \phi$. This implies that $\bar{X} \neq X^*$, a contradiction. Therefore $X^* = X \cup \{x^*\}$ where $x^* \notin X$. Now it is easily seen that a neighborhood basis at x^* is $\{O_n : n \in X\}$ where $O_n = \{x^*\} \cup \{n, n+1, \ldots\}$ and a neighborhood basis at $n \in X$ is $\{n\}$. Thus, $(X^*. \mathscr{U}^*)$ is not locally connected.

Carlson and Hicks [3] give a construction of a strong completion for a quasi-uniform space which possesses a transitive base. The following construction of a C-completion was motivated by their work, except that we do not require a transitive base. Whenever the proof is straightforward or the same as in the earlier construction, we omit details.

Let $(X, \mathscr{U})$ be a quasi-uniform space (not necessarily transitive). Let Ω be the collection of all nonconvergent ultrafilters on X which satisfy condition (i) in the definition of C-filter. Define an equivalence relation on Ω as follows:

If $\mathscr{M}_1, \mathscr{M}_2 \in \Omega$, then $\mathscr{M}_1 \cap \mathscr{M}_2$ if and only if (i) for each $U \in \mathscr{U}$ and $F \in \mathscr{M}_2$ with $F \times F \subseteq U$, then $F \in \mathscr{M}_1$; and, (ii) for each $U \in \mathscr{U}$ and $F \in \mathscr{M}_1$ with $F \times F \subseteq U$, then $F \in \mathscr{M}_2$.

Let $\Lambda = \{\hat{\mathscr{M}} : \mathscr{M} \in \Omega\}$, where $\hat{\mathscr{M}}$ denotes the equivalence class of $\mathscr{M}$ under the relation $\sim$. Let $X^* = X \cup \Lambda$. Let Δ^* denote the diagonal in $X^* \times X^*$. If $V \in \mathscr{U}$, define $S(V) = V \cup \Delta^* \cup \{(\hat{\mathscr{M}}, y) : y \in V[x]$ for some $x \in X$, where $V[x] \in \mathscr{M}$ for all $\mathscr{M} \in \hat{\mathscr{M}}\}$.

Lemma 3.1. *$\{S(U) : U \in \mathscr{U}\}$ is a subbase for a quasi-uniform structure $\mathscr{U}^*$ on X^* and $\mathscr{U} = \mathscr{U}^* \cap (X \times X)$, where we understand $\mathscr{U}^* \cap (X \times X)$ to be $\{U^* \cap X \times X : U^* \in \mathscr{U}^*\}$.*

Lemma 3.2. *$(X^*, \mathscr{U}^*)$ is C-complete.*

Proof. Let $\mathscr{F}^*$ be a C-filter with respect to $\mathscr{U}^*$. If adh $\mathscr{F}^* \neq \phi$, we are through. Suppose that adh $\mathscr{F}^* = \phi$ and let $\mathscr{M}^*$ be an ultrafilter containing $\mathscr{F}^*$. Now $\mathscr{M}^*$

does not converge and $X \in \mathcal{M}^*$. Then $\mathcal{M}=\{n^* \cap X : n^* \in \mathcal{M}^*\}$ is a C-ultrafilter on X satisfying condition (i) of the definition. It is easily seen that $\mathcal{M}$ does not converge. Now we must have that $\mathcal{M}^*$ converges to $\hat{\mathcal{M}} \in \Lambda$, a contradiction.

LEMMA 3.3 *X is dense in X^*.*

Proof. Let $\hat{\mathcal{M}} \in \Lambda$ and $U^* \in \mathcal{U}^*$. Then $U^* \supseteq \bigcap \{S(V_i) : 1 \leq i \leq n\}$, where $V_i \in \mathcal{U}$, $1 \leq i \leq n$. Let $\mathcal{M} \in \hat{\mathcal{M}}$. Now, there is an $M \in \mathcal{M}$ such that $M \times M \subseteq V_i$ for all $1 \leq i \leq n$. Therefore, if $V=\bigcap\{V_i : 1 \leq i \leq n\}$ and $x \in M$, then $x \in V[x] \in \mathcal{M}$ for all $\mathcal{M} \in \hat{\mathcal{M}}$. Hence $x \in U^*[\hat{\mathcal{M}}]$.

THEOREM 3.2. *$(X^*, \mathcal{U}^*)$ is a C-completion for $(X, \mathcal{U})$.*

Proof. A consequence of lemmas 3.1, 3.2, and 3.3.

THEOREM 3.3. *Suppose that $(X, t_{\mathcal{U}})$ is a T_1 topological space and $\mathcal{U}$ is the Pervin structure. Then $(X^*, t_{\mathcal{U}^*})$ is T_1.*

Proof. Suppose that x^* and y^* are elements of X^*. If both x^* and y^* are members of either X or Λ, the result is obvious. Now suppose that $x^* \in X$ and $y^*= \hat{\mathcal{M}} \in \Lambda$. Let $\mathcal{M} \in \hat{\mathcal{M}}$. Since $\mathcal{M}$ does not converge to x^*, there is an open set O such that $x^* \in O$ and $O \notin \mathcal{M}$. Let $V=(O \times O) \cup (X-O) \times X$, $U=(X-\{x^*\}) \times (X-\{x^*\}) \cup \{x^*\} \times X$ and $W=U \cap V$. Now $x^* \notin S(W)[\hat{\mathcal{M}}]$ and $\hat{\mathcal{M}} \notin S(W)[x^*]$.

DEFINITION 3.2. A quasi-uniform space $(X, \mathcal{U})$ is R_3 if and only if given $x \in X$ and $U \in \mathcal{U}$, there is a symmetric $V \in \mathcal{U}$ such that $V \circ V[x] \subseteq U[x]$.

THEOREM 3.4. *Let $(X, \mathcal{U})$ be a T_1 and R_3 quasi-uniform space. Then $(X^*, \mathcal{U}^*)$ is T_1.*

Example 2.1 may be used to show that a discrete space need not have a T_1 C-completion. This shows that C-completeness is not a vast improvement over the standard concept of completeness. Theorem 4.5 of the next section supports the argument that it is an improvement.

4. **Quasi-pseudometrics over R^Δ.** Let Δ denote a non-empty set. R^Δ will denote the set of all functions from Δ into the set R of real numbers. Thus R^Δ is the product of m copies of R where m is the cardinal number of Δ. Give R^Δ the product topology If $f, g \in R^\Delta$, $f \leq g$ means $f(a) \leq g(a)$ for every $a \in \Delta$. Addition and multiplication in R^Δ are defined pointwise. R^Δ is called a Tikhonov semi-field.

DEFINITION 4.1. $d: X \times X \to R^\Delta$ is called a *quasi-pseudometric* (q.p. metric) *on X over R^Δ* provided:

(1) $d(x, y) \geq 0$ and $d(x, x)=0$ for every $x, y \in X$.

(2) $d(x, y) \leq d(x, z)+d(z, y)$ for every $x, y, z \in X$.

If d also satisfies $d(x, y)=0$ implies $x=y$, d is a quasi-metric over R^Δ and if in addition $d(x, y)=d(y, x)$ for every $x, y \in X$, d is a metric over R^Δ.

For a discussion of metric spaces over R^Δ and the theory of Topological Semi-Fields see [1] and its references. Given R^Δ, $\mathcal{N}(0)$ will denote the set of neighborhoods of 0 in R^Δ. Let

$$U^q_{a,b} = \{f \in R^\Delta : a < f(q) < b\}.$$

We recall that $\{U^q_{a,b} : a, b \in R, q \in \Delta\}$ is a subbase for the product topology and, if $U \in \mathcal{N}(0)$, there exists $q_i \in \Delta$, $\varepsilon > 0$, $i = 1, 2, \ldots, n$ such that $U \supseteq \bigcap_{i=1}^n U^{q_i}_{-\varepsilon,\varepsilon}$.

THEOREM 4.1. *Suppose (X, d) is a q.p. metric space over R^Δ. For $x \in X$ and $U \in \mathcal{N}(0)$,*

$$\Omega(x, U) = \{y \in X : d(x, y) \in U\}.$$

Then $\mathcal{N}(x) = \{\Omega(x, U) : U \in \mathcal{N}(0)\}$ is the set of neighborhoods in a topology t_d, called the natural topology for X.

Proof. Clearly, $x \in \Omega(x, U)$ and $\Omega(x, U) \cap \Omega(x, V) = \Omega(x, U \cap V)$. If $M \supseteq \Omega(x, U)$, put $V = U \cup \{d(x, y) : y \in M\}$. Then $M = \Omega(x, V)$.

To complete the proof, we must show that given $\Omega(x, U)$, there exists $V \in \mathcal{N}(0)$ such that $\Omega(x, V) \subseteq \Omega(x, U)$ and $y \in \Omega(x, V)$ implies $\Omega(x, U) \in \mathcal{N}(y)$. The result follows if we can prove it for $U^q_{-a,a}$. In this case, let $V = U^q_{-a/2,a/2}$ and note that if $y \in \Omega(x, V)$, then $\Omega(y, V) \subseteq \Omega(x, U)$.

THEOREM 4.2. *Let (X, d) be a q.p. metric space over R^Δ. If $U \in \mathcal{N}(0)$,*

$$S(U) = \{(x, y) \in X \times X : d(x, y) \in U\}.$$

Then $\mathcal{B} = \{S(U) : U \in \mathcal{N}(0)\}$ is a base for a quasi-uniform structure $\mathcal{U}$ and $t_u = t_d$.

Proof. $d(x, x) = 0 \in U$ implies $S(U) \supseteq \{(x, x) : x \in X\}$. $S(U_1) \cap S(U_2) = S(U_1 \cap U_2)$. Given $S(U)$, there exists $q_i \in \Delta$ and $\varepsilon > 0$ such that

$$U \supseteq \bigcap_{i=1}^n U^q_{-\varepsilon,\varepsilon}. \qquad \text{Let} \quad V = \bigcap_{i=1}^n U^q_{-\varepsilon/2,\varepsilon/2}.$$

Since $S(U^q_{-\varepsilon/2,\varepsilon/2}) \circ S(U^q_{-\varepsilon/2,\varepsilon/2}) \subseteq S(U^q_{-\varepsilon,\varepsilon})$, $S(V) \circ S(V) \subseteq U$. Also, $S(U)[x] = \{y : (x, y) \in S(U)\} = \{y : d(x, y) \in U\} = \Omega(x, U)$. Thus $\mathcal{B}$ is a base for a structure $\mathcal{U}$ such that $t_{\mathcal{U}} = t_d$.

REMARK 4.1. If we set

$$\Omega'(x, U) = \{y \in X : d(y, x) \in U\}$$

we get another topology $t_{d'}$ for X. This is just the natural topology for the q.p. metric d' over R^Δ where $d'(x, y) = d(y, x)$. If d is a pseudometric over R^Δ, $d = d'$ and the quasi-uniform structure in theorem 4.2 is a uniform structure.

REMARK 4.2. d is a q.p. metric on X over R^Δ. Then (1) t_d is T_0 if and only if $x \neq y$ implies $d(x, y) \neq 0$ or $d(y, x) \neq 0$, and (2) t_d is T_1, if and only if $d(x, y) = 0$ implies $x = y$. Thus t_d is T_1 if and only if d is a quasi-metric over R^Δ.

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THEOREM 4.3. *Suppose $\mathcal{U}$ is a quasi-uniform structure with a base $\mathcal{B}$ such that $B\circ B=B$ for every $B\in\mathcal{B}$. If $B\in\mathcal{B}$, let $\rho_B(x,y)=0$ if $(x,y)\in B$ and $\rho_B(x,y)=1$ if $(x,y)\notin B$. For $\varepsilon>0$ and $B\in\mathcal{B}$,*

$$U_{\rho_B,\varepsilon}=\{(x,y):\rho_B(x,y)<\varepsilon\}.$$

Then ρ_B is an ordinary q.p. metric and $\{U_{\rho_B,\varepsilon}:B\in\mathcal{B},\ \varepsilon>0\}$ is a base for $\mathcal{U}$.

Proof. Clearly, $\rho_B(x,x)=0$ and $\rho_B(x,y)\geq 0$. Is $\rho_B(x,y)\leq\rho_B(x,z)+\rho_B(z,y)$? Yes, since $\rho_B(x,z)=\rho_B(z,y)=0$ implies (x,z), $(z,y)\in B$ which gives $(x,y)\in B\circ B=B$ or $\rho_B(x,y)=0$. Note that $U_{\rho_B,\varepsilon}=X\times X$ if $\varepsilon>1$ and equals B if $\varepsilon\leq 1$.

REMARK 4.3. Quasi-uniform structures that have a base $\mathcal{B}$ such that $B\circ B=B$ for every $B\in\mathcal{B}$ are called transitive structures and they have been characterized by Fletcher [5]. The Pervin structure is transitive. The ρ_B's in theorem 4.3 are quasi-uniformly upper semi-continuous with respect to the structure $\mathcal{U}\times\mathcal{U}^{-1}$ Thus a transitive quasi-uniform structure is determined by a nice family of ordinary q.p. metrics. If $\mathcal{U}$ is a uniform structure and $\mathcal{U}$ has a base $\mathcal{B}$ such that for every $B\in\mathcal{B}$, $B=B^{-1}=B\circ B$, then the family $\{\rho_B:B\in\mathcal{B}\}$ will give back the structure and each ρ_B is uniformly continuous with respect to $\mathcal{U}\times\mathcal{U}$.

In [1], it is shown that a topological space is metrizable over some Tikhonov semi-field if and only if it is T_2 and uniformizable (completely regular). The following theorem was proved independently by Boltjanskii [2]; however, the proof given below is different.

THEOREM 4.4. *Every topological space $(X,\mathscr{t})$ is q.p. metrizable over some Tikhonov semi-field R^Δ. Every T_1 topology is quasi-metrizable over some R^Δ.*

Proof. If $\mathcal{U}$ is the Pervin structure, $\mathscr{t}=\mathscr{t}_{\mathcal{U}}$ and $\mathcal{U}$ has a base $\mathcal{B}$ such that $B\circ B=B$ for each $B\in\mathcal{B}$. Consider the family $\{\rho_B:B\in\mathcal{B}\}$ of ordinary q.p. metrics defined in theorem 4.3. Let $\Delta=\mathcal{B}$ and define $d:X\times X\to R^\Delta$ as follows: If $(x,y)\in X\times X$, $d(x,y)(B)=\rho_B(x,y)$. Now $d(x,y)(B)=\rho_B(x,y)\geq 0$ for every $B\in\mathcal{B}$ implies $d(x,y)\geq 0$. $(x,x)\in B$ for every $B\in\mathcal{B}$ gives $d(x,x)=0$. Also,

$$d(x,y)(B)=\rho_B(x,y)\leq\rho_B(x,z)+\rho_B(z,y)=d(x,z)(B)+d(z,y)(B).$$

Thus d is a q.p. metric over R^Δ. By theorem 4.1, d gives rise to a quasi-uniform structure $\mathcal{V}$ such that $\mathscr{t}_d=\mathscr{t}_{\mathcal{V}}$. We show that $\mathcal{U}=\mathcal{V}$ and then we have $\mathscr{t}_d=\mathscr{t}_{\mathcal{V}}=\mathscr{t}_{\mathcal{U}}=\mathscr{t}$.

$\mathcal{C}=\{S(V):V\in\mathcal{N}(0)\}$ is a base for $\mathcal{V}$ where $S(V)=\{(x,y):d(x,y)\in V\}$ and $\mathcal{A}=\{U_{\rho_B,\varepsilon}:B\in\mathcal{B},\ \varepsilon>0\}$ is a base for $\mathcal{U}$. Note that $U_{\rho_B,\varepsilon}=S(U^B_{-\varepsilon,\varepsilon})$ and it follows that $\mathcal{U}=\mathcal{V}$.

If $\mathscr{t}$ is T_1, the q.p. metric d generates a T_1 topology and d is a quasi-metric by remark 4.2.

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(X, d) is a q.p. metric space over R^{Δ}. Sets of the form $U_\epsilon(q_1, \ldots, q_n) = \{f \in R^{\Delta}: -\varepsilon < f(q_i) < \varepsilon \text{ for } 1 \leq i \leq n\}$, $\varepsilon > 0$, make up a base for the neighborhoods of $\bar{0}$ in R^{Δ}, where $\bar{0}(g) = 0$, for all $g \in \Delta$. Let Λ denote the collection of sets of the above form. If Λ is ordered by inverse inclusion, Λ is a directed set. Let $\{X_0 : O \in \Lambda\}$ be a net in X.

DEFINITION 4.2. We say that $\{X_0 : O \in \Lambda\}$ *converges* to $y \in X$ if and only if for any $U \in \mathcal{N}(\bar{0})$, there exists $O^U \in \Lambda$ such that $O > O^U$ implies $d(x_0, y) \in U$ and $d(y, x_0) \in U$.

DEFINITION 4.3. $\{x_0 : O \in \Lambda\}$ is a *Cauchy net* if and only if for any $U \in \mathcal{N}(\bar{0})$, there exists an $O^U \in \Lambda$ such that $O_1, O_2 > O^U$ implies $d(x_{0_1}, x_{0_2}) \in U$ and $d(x_{0_2}, x_{0_1}) \in U$.

DEFINITION 4.4. (X, d) is *complete* if and only if every Cauchy net $\{x_0 : O \in \Lambda\}$ converges.

(X, d) quasi-pseudometrizes $(X, \mathcal{U})$ if and only if $\{S(U) : U \in \Lambda\}$ is a base for $\mathcal{U}$, where $S(U) = \{(x, y) \in X \times X : d(x, y) \in U\}$. The proof of theorem 4.4 shows that for every quasi-uniform structure $\mathcal{U}$ for X, there exists a q.p. metric d over some R^{Δ} such that (X, d) quasi-pseudometrizes $(X, \mathcal{U})$.

THEOREM 4.5. *Suppose that (X, d) quasi-pseudometrizes $(X, \mathcal{U})$. Then (X, d) complete implies that $(X, \mathcal{U})$ is C-complete.*

Proof. Let $\mathcal{F}$ be a C-filter in $(X, \mathcal{U})$. If $\mathcal{F}$ converges, we are done. Suppose that $\mathcal{F}$ does not converge. Let $O = U_\varepsilon(q_1, \ldots, q_n) \in \Lambda$. By the definition of C-filter, there is an $F^0 \in \mathcal{F}$ such that $F^0 \times F^0 \subseteq S(O)$. Let $x_0 \in F^0$. Then $\{x_0 : O \in \Lambda\}$ is a Cauchy net with respect to (X, d). Since $\{x_0 : O \in \Lambda\}$ is Cauchy and (X, d) is complete, $\{x_0 : O \in \Lambda\}$ converges to some point $a \in \Lambda$. We claim that $\mathcal{N}_X(a) \subseteq \mathcal{F}$. Now a base for the neighborhood system $\mathcal{N}_X(a)$ is given by the collection $\{\Omega(a, O) : O \in \Lambda\}$ where $\Omega(a, O) = \{y \in X : d(a, y) \in O\}$. Suppose that $U_\varepsilon(q_1, \ldots, q_n) \in \Lambda$. We wish to show that $\Omega(a, U_\varepsilon(q_1, \ldots, q_n)) \in \mathcal{F}$ and hence that $\mathcal{N}_X(a) \subseteq \mathcal{F}$. By definition of convergence, there is $\hat{O} \in \Lambda$ such that $O > \hat{O}$ implies that $d(a, x_0) \in U_{\varepsilon/2}(q_1, \ldots, q_n)$ and $d(x_0, a) \in U_{\varepsilon/2}(q_1, \ldots, q_n)$. It is easily seen that we can choose $\hat{O}$ to be of the form $\hat{O} = U_r(q_1, \ldots, q_n, q_{n+1}, \ldots, q_m)$ where $r < \varepsilon/4$. Let $U = U_{r/2}(q_1, \ldots, q_n, \ldots, q_m)$ and $z \in F^U$. Now, $d(a, z)(q_i) \leq d(a, x_U)(q_i) + d(x_U, z)(q_i)$. Also $(x_U, z) \in F^U \times F^U \subseteq S(U)$ and hence $d(x_U, z)(q_i) < r/2 < \varepsilon/4$ for $1 \leq i \leq n$. Since $U > \hat{O}$, by definition of convergence we obtain that $d(a, x_U) \in U_{\varepsilon/2}(q_1, \ldots, q_n)$ and hence $d(a, X_U)(q_i) < \varepsilon/2$ for $1 \leq i \leq n$. Thus $d(a, z)(q_i) < \varepsilon/2 + \varepsilon/4 < \varepsilon$.

5. **Unsolved problems and related results.** One of the more interesting questions concerning quasi-uniform structures which remains unanswered is whether or not every topological space has a compatible strongly complete quasi-uniform structure. Fletcher [4] showed that a finite space possesses a unique compatible

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quasi-uniformity generated by a single transitive subset of $X \times X$. This structure is easily seen to be strongly complete. For infinite T_1 spaces we obtain the following result.

THEOREM 5.1. *Let $(X, \mathcal{t})$ be an infinite T_1 topological space; let $\mathcal{FT}$ be the fine transitive quasi-uniform structure; and, let $\mathcal{G}=\{A \subseteq X: X-A$ is finite$\}$. Then $(X, \mathcal{FT})$ is strongly complete provided every $\mathcal{FT}$-Cauchy filter containing $\mathcal{G}$ converges.*

Proof. Let $\mathcal{F}$ be a non-convergent filter on X. If $\mathcal{F}$ contains $\mathcal{G}$, $\mathcal{F}$ is not $\mathcal{FT}$-Cauchy. Suppose $\mathcal{F}$ does not contain $\mathcal{G}$. Then there exists $\{x_i \in X: 1 \leq i \leq n\}$ such that $X-\{x_i: 1 \leq i \leq n\} \notin \mathcal{F}$. It follows that $X-\{x_j\} \notin \mathcal{F}$ for some $1 \leq j \leq n$. And since X is T_1, $X-\{x_j\}$ is open. Also, since $\lim \mathcal{F}=\phi$, there exists an open set O such that $x_j \in O$ and $O \notin \mathcal{F}$. Suppose that $y \in X$. Let $O_y=O-\{x_j\}=O \cap (X-\{x_j\})$ if $y \in O-\{x_j\}$; $O_y=X-\{x_j\}$ if $y \in X-O$; and $O_{x_j}=O$. Let $U(\mathcal{F})=\{(x, y): y \in O_x\}$. Clearly $U(\mathcal{F})$ contains the diagonal. If $(x, y) \in U(\mathcal{F})$ and $(y, z) \in U(\mathcal{F})$, then $(x, z) \in U(\mathcal{F})$. Thus, the set $U(\mathcal{F})$ generates a transitive quasi-uniform structure $\mathcal{U}(\mathcal{F})$ and the resulting topology will be weaker than $\mathcal{t}$. It easily seen that the least upper bound $\mathcal{V}$ of $\{\mathcal{P}\} \cup \{\mathcal{U}(\mathcal{F})\}$ is a compatible transitive quasi-uniform structure and that $\mathcal{F}$ is not $\mathcal{V}$-Cauchy. Thus $\mathcal{F}$ is not $\mathcal{FT}$-Cauchy.

DEFINITION 5.1. If $(X, \mathcal{U})$ is a quasi-uniform space, $\mathcal{U}^{-1}=\{U^{-1}: U \in \mathcal{U}\}$ is a quasi-uniform structure and $\mathcal{U}$ and $\mathcal{U}^{-1}$ are called *conjugate quasi-uniform structures.*

THEOREM 5.2. *Suppose that $\mathcal{U}$ is a transitive quasi-uniform structure for X. Then there is a base $\mathcal{B}$ for $\mathcal{U}$ such that;*

(i) *$\mathcal{B}$ is transitive;*
(ii) *if $B \in \mathcal{B}$, then $B[x] \in \mathcal{t}_{\mathcal{U}}$ for each $x \in X$;*
(iii) *if $B \in \mathcal{B}$, then $B^{-1}[x] \in \mathcal{t}_{\mathcal{U}^{-1}}$ for each $x \in X$; and*
(iv) *if $B \in \mathcal{B}$, then $B^{-1}[x]^c \in \mathcal{t}_{\mathcal{U}}$ for each $x \in X$.*

Proof. It is a well-known fact that any transitive base will suffice for (i), (ii), or (iii). Since $\mathcal{U}$ is transitive, it is a covering quasi-uniformity [6]. Therefore, there is a collection of Q-covers $\mathcal{A}$ such that $\{U_{\mathcal{C}}: \mathcal{C} \in \mathcal{A}\}$ is a subbase, $\mathcal{S}$, for $\mathcal{U}$; where $U_{\mathcal{C}}=\bigcup\{\{x\} \times A_x^{\mathcal{C}}: x \in X\}$ and $A_x^{\mathcal{C}}=\bigcap\{C \in \mathcal{C}: x \in C\} \in \mathcal{t}_{\mathcal{U}}$. Let $y \in X$. We claim that $U_{\mathcal{C}}^{-1}[y]^c \in \mathcal{t}_{\mathcal{U}}$. Let $z \in U_{\mathcal{C}}^{-1}[y]^c$. Then $z \notin U_{\mathcal{C}}^{-1}[y]$ implies that $(y, z) \notin U_{\mathcal{C}}^{-1}$ which in turn implies that $(z, y) \notin U_{\mathcal{C}}$. Hence $y \notin A_z^{\mathcal{C}}$. We wish to show $z \in A_z^{\mathcal{C}} \in U_{\mathcal{C}}^{-1}[y]^c$. Suppose that $t \in A_z^{\mathcal{C}}$. Then $A_t^{\mathcal{C}} \subseteq A_z^{\mathcal{C}}$ implies that $y \notin A_t^{\mathcal{C}}$. Therefore $(y, t) \notin U_{\mathcal{C}}^{-1}$. Thus, $t \notin U_{\mathcal{C}}^{-1}[y]^c$ which implies that $t \in U_{\mathcal{C}}^{-1}[y]$. It follows that $A_z^{\mathcal{C}} \subseteq U_{\mathcal{C}}^{-1}[y]^c$ or $U_{\mathcal{C}}^{-1}[y]^c \in \mathcal{t}_{\mathcal{U}}$. Now, let $\mathcal{B}$ be the base generated by $\mathcal{S}$. Then $\mathcal{B}$ is transitive and $B \in \mathcal{B}$ implies that $B=\bigcap\{U_{\mathcal{C}_i}: 1 \leq i \leq n\}$. If $x \in X$, then $B^{-1}[x]^c=\bigcup\{U_{\mathcal{C}_i}^{-1}[x]^c: 1 \leq i \leq n\} \in \mathcal{t}_{\mathcal{U}}$.

COROLLARY. *Let $(X, \mathcal{U})$ be a transitive quasi-uniform structure. Suppose that $\mathcal{t}_{\mathcal{U}^{-1}}$ is the discrete topology. Then $\mathcal{t}_{\mathcal{U}}$ is T_1.*

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Fletcher and Lindgren [6] give an example of a quasi-uniform space which has neither a symmetric nor a transitive base. As we show below, theorem 5.2 gives another proof that this space does not have a transitive base.

EXAMPLE 5.1. Let X be the reals. Define $U_n=\{(x,y):y-x<(1/2^{n-1})\}=\{(x,y):y<(1/2^{n-1})+x\}$ where $1\leq n<\infty$. $\{U_n:1\leq n<\infty\}$ is a base for a quasi-uniform structure, $\mathscr{U}$, on X. $t_{\mathscr{U}}$ consists of all sets of the form $(-\infty,a)$ where $a\in X$. A base for $\mathscr{U}^{-1}$ is $\{U_n^{-1}:1\leq n<\infty\}$ where $U_n^{-1}=\{(y,x):y-(1/2^{n-1})<x\}$. Then $t_{\mathscr{U}^{-1}}$ consists of all sets of the form (b,∞) where $b\in X$. This means $[a,\infty)=(-\infty,a)^c\notin t_{\mathscr{U}^{-1}}$. Clearly, then $\mathscr{U}$ cannot be transitive.

An interesting problem is that of determining when the fine structure is the same as the fine transitive structure. Fletcher [6] has derived a general method for constructing any compatible transitive quasi-uniform structure for a topological space.

There is a simple method for construcing the fine transitive quasi-uniform structure. As a subbase we simply take the collection $\{U\subseteq X\times X:U[x]\in\mathscr{N}(x)$ for all $x\in X$, and $U\circ U=U\}$. One might then ask whether the set $\mathscr{S}(\Delta)=\{U\subseteq X\times X:U[x]\in\mathscr{N}(x)$ for all $x\in X\}$ is a subbase for the fine quasi-uniform structure. If (X,t) is a topological space with X finite, $\mathscr{S}(\Delta)$ is easily seen to generate the fine quasi-uniform structure. Fletcher [4] has shown that such a space has a unique compatible quasi-uniform structure generated by a single transitive set W. As the following example shows, $\mathscr{S}(\Delta)$ does not, in general, form a subbase for the quasi-uniform structure.

EXAMPLE 5.2. Let (X,t) be the real numbers with the co-finite topology. Let N be the natural numbers and let $U=[(X-N)\times X]\cup[\{1\}\times X]\cup[\bigcup\{\{i+1\}\times(X-\{1,2,\ldots,i\}):1\leq i<\infty\}]$. If $x\in X-N$, then $U[x]=X\in\mathscr{N}(x)$ and if $n\in N$, then $U[n]=X-\{1,2,\ldots,n-1\}\in\mathscr{N}(n)$. Clearly, then $U\supseteq\Delta$ and $U[x]\in\mathscr{N}(x)$ for each $x\in X$. Lindgren [8] has shown that (X,t) is uniquely quasi-uniformizable. We will show that U is not a member of the Pervin structure $\mathscr{P}$, and, therefore, not a member of the fine structure. Suppose $U\in\mathscr{P}$. Then there are nonempty open sets $O_1,\ldots,O_n$ such that $U\supseteq\bigcap\{O_i\times O_i\cup(X-O_i)\times X:1\leq i\leq n\}$. From this it follows that $U\supseteq[\bigcap\{O_i:1\leq i\leq n\}]\times[\bigcap\{O_i:1\leq i\leq n\}]$. Now, $\bigcap\{O_i:1\leq i\leq n\}$ is open in X and, therefore, has finite complement. Suppose $n\in N\cap[\bigcap\{O_i:1\leq i\leq n\}]$. If, also, $n-1\in\bigcap\{O_i:1\leq i\leq n\}$, then the point $(n,n-1)$ must be in $[\bigcap\{O_i:1\leq i\leq n\}]\times[\bigcap\{O_i:1\leq i\leq n\}]$ and, hence in U, a contradiction. Therefore, $n\in N\cap[\bigcap\{O_i:1\leq i\leq n\}]$ implies that $n-1\notin N\bigcap[\cap\{O_i:1\leq i\leq n\}]$. Since there must be infinitely many elements in $N\cap[\bigcap\{O_i:1\leq i\leq n\}]$, there must also be infinitely many elements in $N\cap[X-\bigcap\{O_i:1\leq i\leq n\}]$. But, this is a contradiction.

Let (X,t) be a topological space. Suppose there is a partially ordered set L such that for each $x\in X$, there is a base $\{N(x,\alpha):\alpha\in L\}$ for the neighborhood

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system at x. Suppose further that the following conditions are satisfied:

(i) If β is not a maximal element of L, there is an $\alpha > \beta$ such that for any given $N(x, \alpha)$ and $y \in N(x, \alpha)$, we have $N(y, \alpha) \subseteq N(x, \beta)$; and

(ii) If β is a maximal element of L, then $N(y, \beta) \subseteq N(x, \beta)$ for $y \in N(x, \beta)$.

We form a subbase for a compatible quasi-uniform structure $\mathscr{U}_L$ as follows: Let $U_\alpha = \bigcup\{\{x\} \times N(x, \alpha) : x \in X\}$ where $\alpha \in L$. As we show below the collection $\mathscr{S}_L = \{U_\alpha : \alpha \in L\}$ is the subbase. Clearly $\Delta \subseteq U_\alpha$ for each α in L and clearly the system will be compatible. Let $U_\beta \in \mathscr{S}_L$. We wish to find $U_\alpha \in \mathscr{S}_L$ such that $U_\alpha \circ U_\alpha \subseteq U_\beta$. First suppose that β is not a maximal element of L. Choose α as in (i). If $(x, y) \in U_\alpha \circ U_\alpha$, there is a $z \in X$ such that $(x, z) \in U_\alpha$ and $(z, y) \in U_\alpha$. Then $z \in U_\alpha[x] \subseteq N(x, \alpha)$ and $y \in U_\alpha[z] \subseteq N(z, \alpha)$. By (i), $N(z, \alpha) \subseteq N(x, \beta)$ which implies that $y \in N(x, \beta)$. This in turn implies that $(x, y) \in U_\beta$ and, hence $U_\alpha \circ U_\alpha \subseteq U_\beta$. Next, suppose β is a maximal element. Then, we claim $U_\beta \circ U_\beta \subseteq U_\beta$. If $(x, y) \in U_\beta \circ U_\beta$, then there is a $z \in X$ such that $(x, z) \in U_\beta$ and $(z, y) \in U_\beta$. Hence $z \in N(x, \beta)$ and $y \in N(z, \beta)$. By (ii) $N(z, \beta) \subseteq N(x, \beta)$. Therefore, $y \in N(x, \beta) = U_\beta[x]$ or $(x, y) \in U_\beta$. Thus $U_\beta \circ U_\beta \subseteq U_\beta$.

Next, let $\mathscr{A}$ be a collection of Q-covers of a topological space $(X, \mathscr{t})$ such that if $x \in O \in \mathscr{t}$, then there is a $\mathscr{C} \in \mathscr{A}$ for which $A_x^{\mathscr{C}} \subseteq O$, where $A_x^{\mathscr{C}} = \bigcap\{A_x : x \in A_x \in \mathscr{C}\}$. In our general method, let $L = \mathscr{A}$ and partially order $\mathscr{A}$ by inclusion; that is, $\mathscr{C} < \mathscr{D}$ if and only if $\mathscr{C} \subseteq \mathscr{D}$, but $\mathscr{C} \neq \mathscr{D}$. Let $N(x, \mathscr{C}) = A_x^{\mathscr{C}}$ for each $x \in X$ and $\mathscr{C} \in \mathscr{A}$. Then $\{N(x, \mathscr{C}) : \mathscr{C} \in \mathscr{A}\}$ is an open base for the neighborhood system at x. To see that (i) and (ii) are satisfied, suppose that $\mathscr{D} > \mathscr{C}$. If $y \in N(x, \mathscr{D})$, it is easily seen that $N(y, \mathscr{D}) \subseteq N(x, \mathscr{D}) \subseteq N(x, \mathscr{C})$. We remark that the subbase obtained by our general method is the same as that of the covering quasi-uniformity. Hence our general method can be used to construct any compatible transitive quasi-uniformity.

As we show below the general method applies to quasi-uniform structures which are not transitive also. Consider, again, example 5.1. Following the general method given at the beginning of this section, let L be the positive integers. For each $n \in L$ and $x \in X$, let $N(x, n) = \{y : y - x < (1/2^{n-1})\}$. It is easily seen that $\{N(x, n) : n \in L\}$ is a base for the neighborhood system of x and that (i) and (ii) are satisfied. It is also easily seen that $\mathscr{U}_L$ is the same as the quasi-uniform structure of example 5.1, which was a non-transitive structure.

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Department of Mathematics
University of Missouri-Rolla
Rolla, Missouri 65401
U.S.A.

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