BULL. AUSTRAL. MATH. SOC. 18A15, 18C10, 02E10, 68A10, 68A05
VOL. 17 (1977), 207-233.

Primitive recursive algebraic theories and program schemes

W. Kühnel, J. Meseguer, M. Pfender, and I. Sols

We introduce primitive recursion as a generation process for arrows of algebraic theories in the sense of Lawvere and carry over important results on algebraic theories and functorial semantics to the enriched setting of "primitive recursive algebra": existence of free primitive recursive theories and of theories presented by operations and equations on primitive recursive functions; existence of free models presented by generators and equations. Finally semantical correctness of translations is reduced to correctness for the basic operations. There is a connection to the theory of program schemes: program schemes involving primitive recursion correspond to arrows of a primitive recursive theory freely generated over a graph of basic operations. This theory T can be viewed as a programming language with "arithmetics" given by the basic operations and with DO-loops. A machine loaded with a compiler for T can be interpreted as a T-model in Lawvere's sense, preserving primitive recursion.

The simplest type of program schemes are calculation trees. These can be formalized as the arrows of a free algebraic theory in the sense of Lawvere [*14*] (see also [*6*]). Arrows of such a theory are obtained by substitution out of a generating set of (basic) operations. We introduce primitive recursion as a new generation-process for arrows of algebraic theories and carry over important results on algebraic theories and

Received 29 March 1977. Communicated by G.M. Kelly.

functorial semantics in the enriched setting of "primitive recursive algebra": existence of free primitive recursive theories and theories presented by operations, and equations possibly involving primitive recursion (Theorem 2.5), existence of free models of primitive recursive theories and of models presented by generators and equations (Theorem 3.8). In the last section (semantical) correctness of translations is reduced to correctness for the basic operations (Theorem 4.2).

Our work is to be seen in the context of other categorical approaches to the semantics of program schemes, in particular [*1*], [*8*], [*10*]. We try here to follow as closely as possible Lawvere's functorial semantics, because this approach seems to us to describe very well the program schemes and their interpretations: program schemes correspond to arrows of a primitive recursive theory T freely generated over a graph Σ of basic operations. T can be seen as a programming language with "arithmetic" given by Σ and with DO-loops. A machine loaded with a compiler for T is then a T-model in the Lawvere sense which preserves primitive recursion. The paper is based on general material on monadic and "algebraic" functors; see [*21*].

0. Many-sorted algebras

For reference in later sections we state here a generalization of two principal results of [*14*]: "algebraic theories are algebras" and "models of such theories in cartesian closed categories have algebraic forgetful functors of finite rank". Proofs are given in [*12*]. The case of one-sorted algebras in closed monoidal categories has been treated in [*21*]. As is usual, I^* denotes the free monoid on I.

DEFINITION 0.1. For a set I, $\Sigma \in \mathcal{S}ets^{I^* \times I}$ is called an I-sorted *algebraic type* or *scheme of operators*.

Σ may be seen as a directed graph with node set $I^* : \Sigma \in \mathcal{G}raph_{I^*}$, which has only arrows with codomain in I (coarity 1).

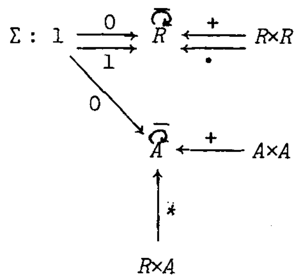
A Σ-*model* in a category $\mathcal{B}$ with 1 and given binary product $\times$ is a graph-morphism $M : \Sigma \to \mathcal{B}$ which preserves products:

$$I^* \ni A = i_1 \ldots i_n \mapsto MA = Mi_1 \times \ldots \times Mi_n =$$
$$= (\ldots (Mi_1 \times Mi_2) \times \ldots) \times Mi_n ,$$

$$I^* \ni 1 \mapsto 1 \ .$$

EXAMPLE.

$$I := \{R, A\} \ ,$$



(where we have written $R \times A$ instead of $RA \in I^*$) is the scheme of le modules (over varying rings). Modules are Σ-models satisfying the usua equations.

A *Σ-homomorphism* $f : M \to M'$ is a family

$$f = (fi)_{i \in I} \in Sets^I\big((Mi), (M'i)\big)$$

compatible with the operations: for each $A = i_1 \ldots i_n \xrightarrow{\omega} i$ in Σ ,

(1)
$$\begin{array}{ccc} MA & \xrightarrow{\prod_{j=1}^{n} fi_j} & M'A \\ {\scriptstyle M\omega}\downarrow & = & \downarrow{\scriptstyle M'\omega} \\ Mi & \xrightarrow[fi]{} & M'i \end{array} \ .$$

This defines a category $Mod(\Sigma, B)$ with forgetful functor $U : Mod(\Sigma, B) \to B^I$, $M \mapsto (Mi)_{i \in I}$.

DEFINITION 0.2. A functor $U : K \to B$ is *algebraic*, if

(i) U has a left adjoint F ,

(ii) U creates (inverse) limits,

(iii) U creates coequalizers of U-kernel pairs (K-pairs mapped by U into kernel pairs), that is to say, quotients by

https://doi.org/10.1017/S0004972700010431 Published online by Cambridge University Press

congruences can be calculated downstairs.

U is of *finite rank* if

(iv) U creates filtered colimits.

DEFINITION 0.3. An *I-sorted algebraic theory* is a category T with object set $|T| = I^*$ (free monoid over I), given products

$$\left[A = i_1 \ldots i_n \xrightarrow{p_j} i_j\right]_{j=1,\ldots,n}, \quad i_j \in I,$$

and $1 \in I^*$ terminal. A *morphism* $t : T \to T'$ *of such theories* is a functor which is the identity on objects and preserves the given projections. This defines a category Th_I with obvious forgetful functor $V : Th_I \to Graph_{I^*}$. For B with given finite product, a *T-model in* B is a functor $M : T \to B$ which preserves the given finite products. A T-homomorphism $f : M \to M'$ is a transformation compatible with $\times$, that is a family $f = (fi)_I \in B^I\big((Mi), (M'i)\big)$ satisfying (1) in Definition 0.1, for all $A \xrightarrow{\omega} i$ in T. This defines the category $Funct_\times(T, B)$ of T-models in B with forgetful functor $U : Funct_\times(T, B) \to B^I$.

THEOREM 0.4. *The forgetful functor* $V : Th_I \to Graph_{I^*} \cong Sets^{I^*\times I^*}$ *is algebraic of finite rank* (*cf.* Definition 0.2). *Moreover* Th_I *is complete and cocomplete.*

0.5 Extension of models and presentation of theories. (i) Each $M \in Mod(\Sigma, B)$ extends uniquely to $\overline{M} \in Funct_\times(F\Sigma, B)$, $F\Sigma \in Th_I$ being the free theory generated by $\Sigma \in Graph_{I^*}$. This defines an isomorphism $Mod(\Sigma, B) \cong Funct_\times(F\Sigma, B)$ of categories compatible with the forgetful functors.

(ii) Each *species* (Σ, G), $\Sigma, G \in Graph_{I^*} \cong Sets^{I^*\times I^*}$, $G \subseteq F\Sigma \times F\Sigma$, *presents* the theory $T = F\Sigma/\overline{G}$, and $Funct_\times(T, B)$ is isomorphic to $Mod\big((\Sigma, G), B\big)$: the full subcategory of $Mod(\Sigma, B)$ consisting of those

https://doi.org/10.1017/S0004972700010431 Published online by Cambridge University Press

$M : \Sigma \to \mathcal{B}$ *satisfying* G ; that is, $\overline{M} : F\Sigma \to \mathcal{B}$ equalizes $G \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} F\Sigma$.

(iii) Each $T \in Th_I$ has an *algebraic* presentation (Σ, G) ; that is $\Sigma, G \in \mathcal{S}ets^{I^* \times I}$. (Presentation is by arrows of coarity 1 .)

THEOREM 0.6. *For an I-sorted algebraic theory* $T \in Th_I$ *(finitely presented) and* ($\mathcal{B}$ *an elementary topos with a natural numbers object or)* $\mathcal{B}$ *a well-powered cartesian closed category with coequalizer-mono factorization and coequalizers closed under composition, the category* $\mathcal{F}unct_\times(T, \mathcal{B})$ *of T-models in* $\mathcal{B}$ *has an algebraic forgetful functor* $U : \mathcal{F}unct_\times(T, \mathcal{B}) \to \mathcal{B}^I$ *of finite-rank.*

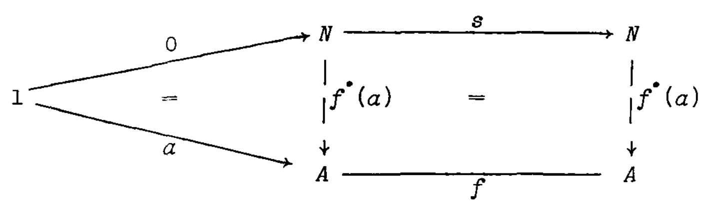
By the triangle-theorem, see [4] and [21], we get immediately:

COROLLARY 0.7. *For* $\mathcal{B}$ *as above and a morphism* $t : T' \to T$ *in* Th_I*, the "algebraic" functor* $\mathcal{F}unct_\times(t, \mathcal{B}) : \mathcal{F}unct_\times(T, \mathcal{B}) \to \mathcal{F}unct_\times(T', \mathcal{B})$ *is algebraic of finite rank.*

1. Natural numbers object and primitive recursion

This section gives a diagrammatic description of primitive recursive functions. It is based on work by Lawvere, Freyd, and Joyal.

DEFINITION 1.1 ([15]). A natural numbers object in a (cartesian closed) category C is an object N together with a "zero"-map $1 \xrightarrow{0} N$ and a "successor"-map $N \xrightarrow{s} N$ satisfying the following universal property of "simple recursion": for any diagram $1 \xrightarrow{a} A \xrightarrow{f} A$ there is a unique "sequence"-map $N \xrightarrow{f^\bullet(a)} A$ such that



In $\mathcal{S}ets$, $f^\bullet(a)$ is defined by $\mathbb{N} \ni n \mapsto f^n(a)$, and the universal property

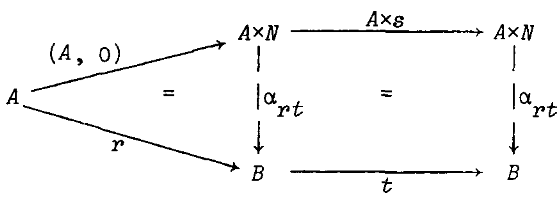
https://doi.org/10.1017/S0004972700010431 Published online by Cambridge University Press

of $\mathbb{N}$ is equivalent to Peano's axioms for the natural numbers; see [15] and [18] for a proof.

In cartesian closed categories this property of N guarantees the existence of functions defined by primitive recursion. Freyd's proof ([9], 5.22) uses the fact that - by cartesian closure (!) - the universal property of N is equivalent to the following one (Proposition 5.21).

For any $A \xrightarrow{r} B \xrightarrow{t} B$ in C there exists a unique $\alpha_{rt} : A\times N \to B$ satisfying

(pr)

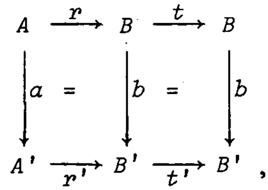


where $A \xrightarrow{0} N := A \to 1 \xrightarrow{0} N$.

The remainder of Freyd's construction of primitive recursive functions does not use cartesian closure. So, in order to make things independent of this additional structure, we use the latter universal property (pr) of "primitive recursion" for definition of the natural numbers object $N = 1 \xrightarrow{0} N \xrightarrow{s} N$. It clearly determines the natural numbers object up to isomorphism.

REMARKS 1.2. (i) Specialization $r = \mathrm{id}_A$ gives Joyal's definition of natural numbers objects as "free monoid relative to actions".

(ii) Uniqueness of the α_{rt} implies the following naturality condition. If



then

$$\begin{array}{ccc} A\times N & \xrightarrow{\alpha_{rt}} & B \\ {\scriptstyle a\times N}\downarrow & = & \downarrow{\scriptstyle b} \\ A'\times N & \xrightarrow[\alpha_{r't'}]{} & B' \end{array} .$$

(iii) For many (concrete) categories a natural numbers object is available. If the countable coproduct of the terminal object 1 exists and is preserved by $A \times -$ for all A, then

$$1 \xrightarrow{in_0} \coprod_{n\in\mathbb{N}} 1 \xrightarrow{[in_{n+1}]_{n\in\mathbb{N}}} \coprod_{n\in\mathbb{N}} 1$$

is a natural numbers object. This implies, by uniqueness of the natural numbers object, that for a natural numbers object $1 \xrightarrow{0} N \xrightarrow{s} N$,

$$\left[1 \xrightarrow{0} N \xrightarrow{s^n} N\right]_{n\in\mathbb{N}} \quad (\text{composite} := n)$$

is a coproduct preserved by $A \times -$.

For $A \xrightarrow{r} B \xrightarrow{t} B$, $\alpha_{rt} : A\times N \to B$ is then the morphism induced by $[A \xrightarrow{r} B \xrightarrow{t^n} B]_{n\in\mathbb{N}}$ out of the coproduct $[A \xrightarrow{(A,n)} A\times N]_{n\in\mathbb{N}}$.

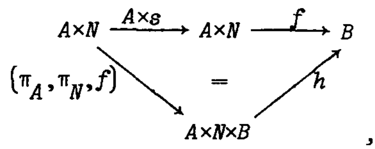
Using the (pr)-definition of natural numbers objects above, Freyd's Proposition 5.22 can be stated not only for topoi, but as follows.

LEMMA 1.3. *A category* $\mathbf{C}$ *with finite products and natural numbers objects (primitive recursive, see above) has primitive recursion, that is: given* $g : A \to B$, $h : A\times N\times B \to B$, *there exists a unique* $f : A\times N \to B$ *such that*

$$A \xrightarrow{(A,0)} A\times N \xrightarrow{f} B \; = \; A \xrightarrow{g} B ,$$

that is in **Sets** : $f(a, 0) = ga$ *and*

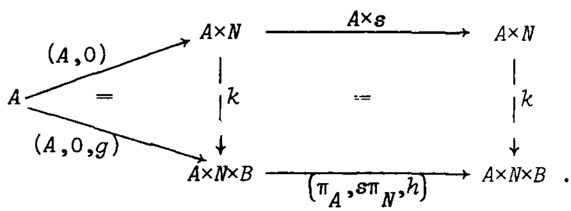
https://doi.org/10.1017/S0004972700010431 Published online by Cambridge University Press



that is in Sets : $f(a, sn) = h(a, n, f(a, n))$.

Abbreviation: $f = \mathrm{pr}(g, h)$.

The proof is exactly Freyd's $f := \pi_B k$, k given by



REMARK 1.4. The (pr)-definition of natural numbers objects is just a special instance of the schema for primitive recursion in the lemma

$$\alpha_{rt} = \mathrm{pr}\left(A \xrightarrow{r} B,\ A\times N\times B \xrightarrow{\pi_B} B \xrightarrow{t} B\right) .$$

In the later sections we will use the following notation:

DEFINITION 1.5. A category C with finite products (hence also with 1) and with natural numbers objects (here and later on always in the (pr)-sense) is called *category with primitive recursion* or pr-*category*. A functor between such categories which preserves both is called a *primitive recursive* or pr-*functor*.

REMARK 1.6. By uniqueness of the α_{rt} and of $\mathrm{pr}(g, h)$ such a functor F preserves the recursion-schema:

$$F\alpha_{rt} = \alpha_{Fr,Ft} , \quad F\big(\mathrm{pr}(g, h)\big) = \mathrm{pr}(Fg, Fh) .$$

2. Primitive recursive algebraic theories

This section treats the construction of free primitive recursive theories and the presentation of such theories by (formal) operations and (recursive) equations.

DEFINITION 2.1. A primitive recursive I-sorted (algebraic) theory is an $\overline{I} := I \dot\cup \{N\}$-sorted (Lawvere-) theory T (see Definition 0.3) with given arrows $1 \xrightarrow{0} N \xrightarrow{s} N$ making N into a natural numbers object of the category T.

These theories form a category $PRTh_I$, morphisms being theory-morphisms preserving the natural numbers object; that is to say, which are primitive recursive functors.

$PRTh_I$ may alternatively be described as a full subcategory of the comma category $N \downarrow Th_{\overline{I}}$. Some formal precision seems to be necessary: $N \downarrow Th_{\overline{I}}$ stands for the category whose objects are graph-morphisms $N \xrightarrow{i} T$ with $i(N) = N$, $i(1) = 1 \in \overline{I}^*$, and whose morphisms are the $Th_{\overline{I}}$-morphisms compatible with the i's. We shall always write $0, s$ instead of $i(0), i(s)$. Now $PRTh_I$ is the full subcategory of $N \downarrow Th_{\overline{I}}$ consisting of those $N \to T$ for which N is a natural numbers object in T.

PROBLEMS A. Given a graph $\Sigma \in \mathcal{G}raph_{\overline{I}^*}$ of formal operations (with domains and codomains out of the free monoid $\overline{I}^*$ over $\overline{I} = I \dot\cup \{N\}$), construct the free pr-theory over Σ, that is construct a left adjoint for the forgetful functor

$$PRTh_I \hookrightarrow N \uparrow Th_{\overline{I}} \to N \downarrow \mathcal{G}raph_{\overline{I}^*} \to \mathcal{G}raph_{\overline{I}^*} .$$

Σ above already contains a prototype N of a natural numbers object and may contain "start"-arrows $g : A \to B$ and "induction-step"-arrows $h : A \times N \times B \to B$ for primitive recursion.

Part of Problem A is the following, and we will later reduce Problem A to it.

B. Given $T \in N \downarrow Th_{\overline{I}}$, construct the pr-closure RT of T, that is to say, construct a reflector R for the full inclusion $PRTh_I \hookrightarrow N \downarrow Th_{\overline{I}}$.

C. Presentation of pr-theories by operations and equations, that is if Problem A is solved: divide out a given $T \in PRTh_I$ by a given set G of equations (pairs of arrows in T).

https://doi.org/10.1017/S0004972700010431 Published online by Cambridge University Press

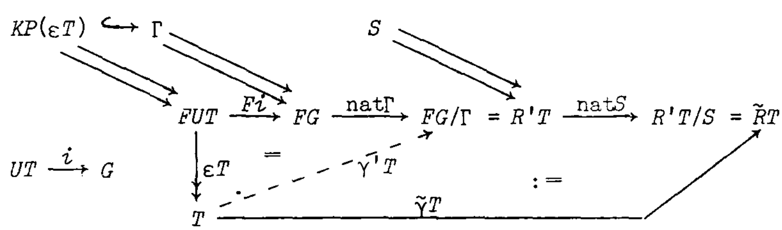
So let us first attack Problem B.

2.2. pr-closure of theories. Given: an $\bar{I}$-sorted theory $T = N \to T \in N \downarrow Th_{\bar{I}}$ with distinguished "natural numbers object" $1 \xrightarrow{0} N \xrightarrow{s} N$; wanted: the pr-closure $RT \in PRTh_{I}$ of T .

First Step. For each $A \xrightarrow{r} B \xrightarrow{t} B$ in T , add formally an arrow $A \times N \xrightarrow{\alpha_{rt}} B$ to the underlying graph UT of T . This gives a graph $G \in Graph_{\bar{I}*}$ and an inclusion $i : UT \hookrightarrow G$. Form the free theory $FG \in Th_{\bar{I}}$ over G and divide it out by the relations holding for T and by the "natural numbers object equations" (primitive recursive) for the α_{rt} ; formally: let $\Gamma \rightrightarrows FG$ be the congruence (kernel pair) generated by $KP(\varepsilon T) \rightrightarrows FUT \xrightarrow{Fi} FG$ (ε being the counit of the adjunction $F \dashv U$), and by the set of pairs

$$\{(r, \alpha_{rt}(A, 0)), (t\alpha_{rt}, \alpha_{rt}(A \times s)) : A \xrightarrow{r} B \xrightarrow{t} B \text{ in } T\} .$$

Γ exists - as an intersection of kernel pairs - and so does $R'T := FG|_{\Gamma} \in Th_{\bar{I}}$, by completeness and cocompleteness of $Th_{\bar{I}}$; *cf.* Theorem 0.4. Now force uniqueness of the α_{rt} adjoined so far: let $S \rightrightarrows R'T$ be the congruence generated by those pairs $h, h' : A \times N \to B$ in $R'T$ which both satisfy the natural numbers object equations for appropriate common $A \xrightarrow{r} B \xrightarrow{t} B$ in $R'T$. Define $\tilde{R}T := RT|_{S} \in Th_{\bar{I}}$, and the "unit" $\tilde{\gamma}T : T \to \tilde{R}T$ as the composition in



,

$\gamma'T$ being the unique morphism out of the coequalizer.

∞-step. Define $RT := \varinjlim (T \xrightarrow{\tilde{\gamma}T} \tilde{R}T \xrightarrow{\gamma\tilde{R}T} \tilde{R}^2T \to \ldots \to \tilde{R}^nT \to \ldots)$ and the unit $T \xrightarrow{\gamma T} RT$ as the zero-th colimit-injection. RT (as all the

https://doi.org/10.1017/S0004972700010431 Published online by Cambridge University Press

$\tilde{R}^n T$) trivially becomes an $N \downarrow Th_{\overline{I}}$-object, and γT an $N \downarrow Th_{\overline{I}}$-morphism by $N \to RT := N \to T \xrightarrow{\gamma T} RT$.

REMARK 2.3. Within the above construction, there are two different degrees of "constructiveness". $R'T$ can be recursively described, provided that such a description exists for T (for example if T is presented by finitely many operations and equations). This is not the case for S , because for given h, h' we would have to decide whether they come from a common $A \xrightarrow{r} B \xrightarrow{t} B$. So it would make sense to consider instead of RT the theory $\overline{R}T := \varinjlim R'^n T$, which has a weak natural numbers object (uniqueness of α_{rt} fails in general) and which is universal over T in this regard.

The above construction provides the desired universal extension.

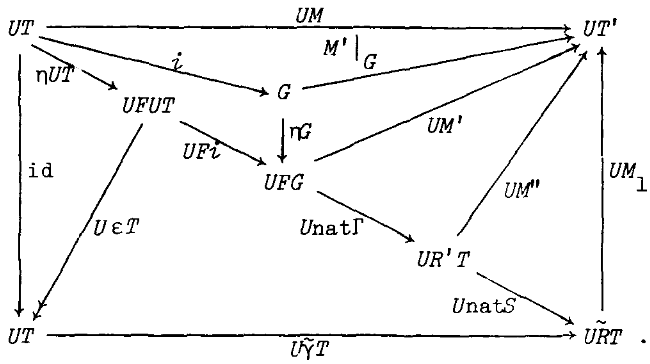
THEOREM 2.4. *The category* $PRTh_I$ *of primitive recursive* I*-sorted theories is a full reflective subcategory of the category* $N \downarrow Th_{\overline{I}}$ *of* $I \dot{\cup} \{N\}$*-sorted theories with distinguished successor-algebra* N *. The reflection-morphism of* T *into* $PRTh_I$ *is* $\gamma T : T \to RT$ *above.*

Proof. First we have to show $RT \in PRTh_I$. We use the fact that $U : Th_{\overline{I}} \to Graph_{\overline{I}^*} \cong Sets^{\overline{I}^* \times \overline{I}^*}$ creates filtered colimits by Theorem 0.4. So $\varinjlim \tilde{R}^n T$ is constructed separately in $Sets$ for each $RT(A, B)$. Therefore, given $A \xrightarrow{r} B \xrightarrow{t} B$ in RT , r is represented by some $A \xrightarrow{r'} B$ in $\tilde{R}^n T$, t by some $B \xrightarrow{t'} B$ in $\tilde{R}^m T$, hence both by $A \xrightarrow{r''} B \xrightarrow{t''} B$ in $\tilde{R}^p T$ $(p = \max(n, m))$. By definition of $\tilde{R}$, there is a unique induced $\alpha_{r''t''} : A \times N \to B$ in $\tilde{R}^{p+1} T$. So the equivalence class α_{rt} of $\alpha_{r''t''}$ in RT satisfies the natural numbers equations for r, t . α_{rt} is hereby uniquely determined, because for other representants r''', t''' one can find an $\tilde{R}^q T$ containing all the data, hence $\alpha_{r''t''}, \alpha_{r'''t'''}$ have the same image in $\tilde{R}^{q+1} T$, hence in RT .

https://doi.org/10.1017/S0004972700010431 Published online by Cambridge University Press

Let us now prove the universal property of $T \xrightarrow{\gamma T} RT$. For $T' \in PRTh_I$, $M : T \to T'$ a theory-morphism in $N \downarrow Th_{\bar{I}}$, M is extended stepwise into $\bar{M} : RT \to T'$ as follows.

First step. Consider the following diagram (*cf*. 2.2):



For $A \xrightarrow{r} B \xrightarrow{t} B$ in T , there is a unique T-arrow $\alpha_{Mr,Mt} : MA \times N \to MB$ induced by $MA \xrightarrow{Mr} MB \xrightarrow{Mt} MB$. Hence, for $M' : FG \to T'$ extending M , $M'|_G$ is necessarily the graph-morphism extending M and mapping α_{rt} into $\alpha_{Mr,Mt}$. So there exists a unique $M' : FG \to T'$ extending M (and therefore preserving N), namely the unique extension of that graph-morphism into a morphism of theories. Next we show that M' factors uniquely through nat Γ . By definition of M on the α_{rt} and by $T' \in PRTh_I$, M' identifies the components of the pairs $(r, \alpha_{rt}(A, 0))$ and $(t\alpha_{rt}, \alpha_{rt}(A \times s))$. On the other hand, $M' \circ Fi$ is the unique extension of M to FUT , and is therefore, by a general property of adjoint situations, equal to $\varepsilon T' \circ FUM = M \circ \varepsilon T$, so M' equalizes $KP(\varepsilon T)$ followed by Fi . Hence, by definition of Γ , there is a unique $M'' : R'T \to T'$ with $M'' \circ \text{nat}\,\Gamma = M'$.

Because of the uniqueness of the $\alpha_{Mr,Mt}$, M'' equalizes S , so it uniquely factors through nat $S : R'T \to \tilde{R}T$, say by a theory-morphism $M_1 : \tilde{R}T \to T'$. M_1 is the wanted extension of M , because the whole diagram above commutes and U is faithful. M_1 is unique with that property because of the uniqueness of each of the extensions M', M'', M_1 .

n+1-st step. By the first step, the (already constructed) theory-morphism $M_n : \tilde{R}^n T \to T'$ uniquely extends to an $M_{n+1} : \tilde{R}^{n+1} T \to T'$ satisfying $M_{n+1} \circ \tilde{\gamma}\tilde{R}^n T = M_n$.

∞-step. So M defines a unique cone $[M_n]_{n \in \mathbb{N}_0}$ with $M_0 = M$ out of the linear diagram $T \xrightarrow{\gamma T} \tilde{R}T \xrightarrow{\tilde{\gamma}\tilde{R}T} \tilde{R}^2 T \to \ldots$ into T' , which in turn induces a unique $\overline{M} : RT \to T'$ out of the colimit, satisfying $\overline{M} \circ \gamma T = M$ ($\overline{M}$ trivially preserves N , because M does and $N \to RT := N \to T \xrightarrow{\gamma T} RT$). $\overline{M}$ is hereby clearly uniquely determined, hence also by M . So $T \xrightarrow{\gamma T} RT$ is a reflection of T into the full subcategory $PRTh_I$, hence $PRTh_I$ is a reflective subcategory of $N \downarrow Th_{\overline{I}}$. This completes the proof.

Problems A and C (*cf.* corollary below) are solved by the following main theorem of this section.

THEOREM 2.5. *The inclusion* $PRTh_I \hookrightarrow N \downarrow Th_{\overline{I}}$ *and the forgetful functor* $PRTh_I \to Graph_{\overline{I}*}$ *into the category of graphs with node-set* $\overline{I}* = (I \dot{\cup} \{N\})*$ *are* algebraic *(that is they are monadic and they create quotients of kernel pairs)* of finite rank *(that is they create filtered colimits)*.

Proof. $PRTh_I$ is in $N \downarrow Th_{\overline{I}}$ closed under limits (construct the α_{rt} componentwise), under quotients by kernel pairs (construct the α_{rt} for representants) and under filtered colimits (take α_{rt} in a component where you can map both r and t , see the proof of Theorem 2.4). Combined with full reflexivity this is to say that the inclusion is algebraic of finite rank, see Definition 0.2.

The forgetful functor $Th_{\overline{I}} \to Graph_{\overline{I}*}$ is algebraic of finite rank by Theorem 0.4. This implies trivially that $N \downarrow Th_{\overline{I}} \to N \downarrow Graph_{\overline{I}*}$ is algebraic as well. An easy verification shows that $N \downarrow Graph_{\overline{I}*} \to Graph_{\overline{I}*}$

https://doi.org/10.1017/S0004972700010431 Published online by Cambridge University Press

has the same property (left adjoint; adjoin formally 0 and s). Hence, by the composition theorem,

$$PRTh_I \to Graph_{\bar{I}*} = PRTh_I \hookrightarrow N \downarrow Th_{\bar{I}} \to N \downarrow Graph_{\bar{I}*} \to Graph_{\bar{I}*}$$

is algebraic of finite rank (see [21], p. 3; for a proof see [12], 1.6).

REMARK. The Birkhoff Inclusion Theorem ([21], p. 3) does not apply for proving reflexivity, since $PRTh_I$ is not closed under subobjects in $N \downarrow Th_{\bar{I}}$.

COROLLARY 2.6. *For any* $T \in PRTh_I$ *(especially for* $RF\Sigma$ *freely generated over "operations"* $\Sigma \in Graph_{\bar{I}*}$ *) and "equations"* G *on* T *, that is an equivalence relation* $G \rightrightarrows UT$ *on* UT *in* $Graph_{\bar{I}*}$ *, there is a "realization", namely the coequalizer* T/G *of* T *by* G *relative* U *; see [17]. This coequalizer is the coequalizer of the intersection* $\bar{G}$ *of those kernel pairs on* T *which contain* G *and is therefore created by* $PRTh_I \to Graph_{\bar{I}*}$ *algebraic. This solves Problem C.*

REMARK 2.7. By the theorem, two more problems are solved: free adjunction of primitive recursively defined operations to $T \in Th_I$, that is construction of a left adjoint to $PRTh_I \to Th_I$, and construction of a free pr-theory over $\Sigma \in Graph_{I*}$. Consider the commutative diagram

$$\begin{array}{ccccc}
PRTh_I & \longrightarrow & N{\downarrow}Th_{\bar{I}} & \longrightarrow & N{\downarrow}Graph_{\bar{I}*} \\
 & \searrow^{\text{truncation}} & \downarrow & & \downarrow \\
 & & Th_{\bar{I}} & & Graph_{\bar{I}*} \\
 & & \downarrow{\scriptstyle\text{truncation}} & & {\scriptstyle\text{truncation}}\downarrow \\
 & & Th_I & \longrightarrow & Graph_{I*}
\end{array} .$$

$Graph_{\bar{I}*} \to Graph_{I*}$ clearly has a left adjoint: adjoin formally new nodes; hence also the composition $PRTh_I \to Graph_{I*}$ (second problem), and therefore $PRTh_I \to Th_I$ has a left adjoint, by Dubuc's triangle theorem in [4]. We shall discuss the latter truncation in more detail at the end of the next section.

3. Models of primitive recursive algebraic theories

As expected in this Lawvere-type framework, semantics of our theories will be defined functorially. This section extends principal results of functorial semantics to the case of pr-theories and their functorial models.

DEFINITION 3.1. A *model* of a primitive recursive theory $T \in PRTh_I$ in a pr-category $\mathcal{B}$ is a pr-functor (see Definition 1.5) $M : T \to \mathcal{B}$, that is to say, M preserves the given products and the natural numbers object $1 \xrightarrow{0} N \xrightarrow{s} N$. A *homomorphism* $f : M \to M'$ is just a natural transformation. This gives us the category $Mod(T, \mathcal{B}) = Funct_{\times, N}(T, \mathcal{B})$ of T-models in $\mathcal{B}$ and the forgetful functor

$$U : Mod(T, \mathcal{B}) \to \mathcal{B}^I ,$$
$$M \longmapsto (Mi)_{i \in I} .$$

We want to show that the pr-closure RT of $T \in N \downarrow Th_{I \dot{\cup} \{N\}}$ is a conservative extension of T, that is to say, it gives rise to an isomorphism of the corresponding model-categories. To this end we make models into morphisms of theories by means of the following construction.

3.2 **Full-image-factorization.** Let $M : T \to \mathcal{B}$ be an N-preserving model of $T \in N \downarrow Th_{\bar{I}}$ $(\bar{I} = I \dot{\cup} \{N\})$ in a pr-category $\mathcal{B}$. Then M admits a unique full-image-factorization $T \xrightarrow{M_0} T_M \xrightarrow{M^0} \mathcal{B}$ with $T \xrightarrow{M_0} T_M$ in $N \downarrow Th_I$, T_M a pr-theory and M^0 a pr-functor which is identity on mor-sets.

Proof. T_M is defined by $|T_M| = |T| = \bar{I}^*$, $T_M(A, B) = \mathcal{B}(MA, MB)$, these sets made disjoint. $T_M \in Th_{\bar{I}}$ and $M_0 \in Th_{\bar{I}}(T, T_M)$ for the obvious M_0 is trivial. $1 \xrightarrow{0} N \xrightarrow{s} N$ of $\mathcal{B}$ makes T_M into $T_M \in N \downarrow Th_I$ and $M_0 \in N \downarrow Th_I(T, T_M)$. Clearly $1 \xrightarrow{0} N \xrightarrow{s} N$ is a natural numbers object for T_M, and M^0 defined by $A \xrightarrow{f} B \longmapsto MA \xrightarrow{f} MB$ preserves it.

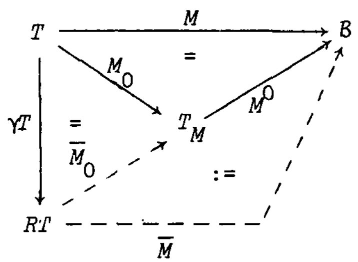
https://doi.org/10.1017/S0004972700010431 Published online by Cambridge University Press

EXTENSION-THEOREM 3.3. *(i) The pr-closure* $T \xrightarrow{\gamma T} RT$ *for* $T \in N \downarrow Th_{\overline{I}}$ *is universal with respect to arbitrary product- and -preserving functors* $M : T \to B$ *into arbitrary pr-categories.*

(ii) γT *induces an isomorphism* $Mod(\gamma T, B) : Mod(RT, B) \xrightarrow{\sim} Mod(T, B)$ *compatible with the forgetful functors to* B^I *and natural in* T *and* B .

(iii) For $\Sigma \in Graph_{\overline{I}*}$ *there is such an isomorphism* $Mod(RF\Sigma, B) \xrightarrow{\sim} Mod(\Sigma, B)$, $Mod(\Sigma, B)$ *consisting of the product- (on objects) and* N*-preserving graph-morphisms into* B .

Proof. *(i)* Consider



,

the upper triangle being the full-image factorization of M with $M_0 : T \to T_M$ in $N \downarrow Th_{\overline{I}}$, γT being the reflection of Theorem 2.4. M_0 extends uniquely to $\overline{M}_0$ in $PRTh_I$. So $\overline{M} := M^0 \circ \overline{M}_0$ is primitive recursive and makes the outer triangle commute. It is thereby uniquely determined, because it necessarily has M^0 as second factor of its full-image factorization.

(ii) follows from *(i)*.

(iii) by composition of the isomorphism in *(ii)* and $Mod(F\Sigma, B) \xrightarrow{\sim} Mod(\Sigma, B)$ induced by the universal embedding $\eta\Sigma : \Sigma \to F\Sigma$; see 0.5.

Again by full-image factorization we now show that "dividing out a theory by equations" is universal not only with respect to other theories, but also with respect to models.

DEFINITION 3.4. For $T \in N \downarrow Th_{\overline{I}}$, G a subgraph of $T \times T$

(projections $p_1, p_2 : G \rightrightarrows T$), a model $M : T \to B$ *satisfies* G if $Mu = Mv$ for all $(u, v) \in G$, that is if $Mp_1 = Mp_2$. These models define a full subcategory $Mod((T, G), B)$ of $Mod(T, B)$.

THEOREM 3.5. *(i) For* $M \in Mod((T, G), B)$ *there is a unique* $T/\overline{G}$-*model* $\overline{M} : T/\overline{G} \to B$ *satisfying* $\overline{M}\,\mathrm{nat}_{\overline{G}} = M$, *where* $\overline{G}$ *denotes the congruence on* T *generated by* G.

(ii) $\mathrm{nat}_{\overline{G}}$ *defines an isomorphism*

$$Mod(\mathrm{nat}_{\overline{G}}, B) : Mod(T/\overline{G}, B) \xrightarrow{\sim} Mod((T, G), B)$$

compatible with the forgetful functors and natural in B.

Proof. Use full-image factorization of M and the fact that $T \to T/\overline{G}$ is the coequalizer of p_1, p_2 relative to the forgetful functor $N + Th_{\overline{I}} \to Graph_{\overline{I}*}$; see [17].

The most important special case of this theorem is the following: take a (finite) graph $\Sigma \in Graph_{\overline{I}*}$ of operators (for example names in programming language for standard algebraic operations, as addition, multiplication, and so on). Then take G to be a (finite) set of equations (for example commutativity of addition, but also, that an operation can be defined recursively by others, equations between primitive recursively defined operations, for instance matrix-operations) which intuitively hold for any "reasonable" model (computer equipped with a compiler) of the pr-theory $F_r^{\cdot}\Sigma = RF\Sigma$ freely generated by Σ (programming-language with DO-loops generated by Σ). Then the theorem says that any common property of all the "reasonable" models can be found in a purely syntactic construction, namely in $\overline{G} \subseteq RF\Sigma \times RF\Sigma$ (but take into account Remark 2.3).

Let us now discuss the general properties of the model-categories for primitive recursive theories. Bearing in mind the above motivation, the following restriction for our theories does not exclude the important cases. It is necessary in order to obtain the main results of this section.

DEFINITION 3.6. $T \in PRTh_I$ is *generated by N-actions*, if it is

https://doi.org/10.1017/S0004972700010431 Published online by Cambridge University Press

generated by a graph $\Sigma \in \mathit{Graph}_{\overline{I}*}$ containing only arrows of the form $A \to i$ with $i \in I$.

This includes (basic) operations not involving N and operations of the form $A\times N\times B \to B$ (basic datum for a function defined by primitive recursion). Excluded are operations like "integer part" $\text{int} : \mathbf{R} \to N$, $\mathbf{R} \in I$ stands for the type of reals.

LEMMA 3.7. *If* $T \in PRTh_I$ *is generated by* N*-actions, then each* $A\times N^k \xrightarrow{f} N^{k'}$ *in* T $(A \in I^*)$ *admits a factorization* $A\times N^k \xrightarrow{\text{pr}} N^k \xrightarrow{\hat{f}} N^{k'}$ *with* $\hat{f}$ *out of the free (0-sorted) pr-theory generated by* N.

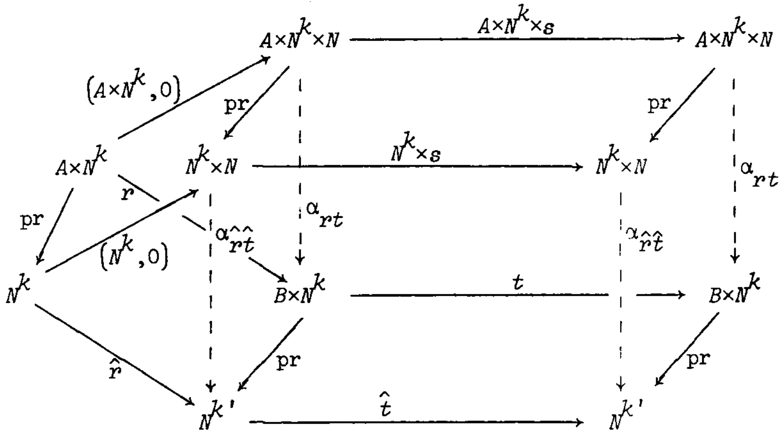
Proof. Let T be generated by a graph $\Sigma \in \mathit{Graph}_{\overline{I}*}$ of N-actions. Then trivially each arrow in $\Sigma \cup \mathsf{N}$ satisfies the assertion of the lemma. T being the "induced into product"-, composition-, and pr-closure of $\Sigma \cup \mathsf{N}$ (see Section 2), it is sufficient to show:

(i) the assertion is stable under forming induced morphisms into products and under composition;

(ii) it is stable under primitive recursion, that is to say, if it holds for $r : A \to B$ and $t : B \to B$, then also for $\alpha_{rt} : A\times N \to B$.

(i) is proved straightforwardly; see also the more general Lemma 3.2 in [12].

(ii) Consider the following diagram:



https://doi.org/10.1017/S0004972700010431 Published online by Cambridge University Press

By hypothesis, r and t give rise to factorizations through $\hat{r}$, $\hat{t}$ in the free pr-theory generated by N.

Now $\mathrm{pr} \circ \alpha_{rt} = \alpha_{\hat{r}\hat{t}} \circ \mathrm{pr}$, since all the other faces of the diagram commute and hence both sides of the equation equal $\alpha_{\hat{r}\circ \mathrm{pr}, \hat{t}}$. So again, the N-part $\mathrm{pr} \circ \alpha_{rt}$ of α_{rt} admits a factorization through the projection on the N-part and $\alpha_{\hat{r}\hat{t}}$ in the free pr-theory generated by N. This completes the proof.

MAIN THEOREM 3.8 (Primitive recursive algebras are algebras). *For B = Sets or B an elementary topos with natural numbers objects or B any complete, cocomplete, well-powered cartesian closed primitive recursive category with regular epi-mono factorization and with regular epimorphisms closed under composition and T an I-sorted primitive recursive theory generated by N-actions, the model-category of T has an algebraic forgetful functor $U : \mathrm{mod}(T, B) \to B^I$ of finite rank (see Definition* 0.2*).*

Proof. The theorem follows by specialization to the case $J = \{N\}$ from Theorem 3.3 in [12], and on the other hand from Theorem 3.9 below.

By the Lawvere Characterization Theorem (see [16]) monadic functors (of finite rank) into *Sets* come from algebraic theories. What is this theory in our case?

THEOREM 3.9. *For B, T as above, let $\check{T} \in Th_I$ be the full subcategory of T with object set $|\check{T}| = I^*$ (cut out the N). Then $\check{T} \hookrightarrow T$ defines, by restriction, an isomorphism $J : Mod(T, B) \xrightarrow{\sim} \mathrm{Funct}_\times(\check{T}, B)$, trivially compatible with the forgetful functors to B^I.*

Proof. By hypothesis, T is presented by (Σ, G), Σ a graph of N-actions. We have only to show that J is bijective on objects, full, and faithful. Let $M \in Mod(T, B)$ be an extension of a given $\check{M} : \check{T} \to B$. For $A \times N^k \xrightarrow{\omega} i$ in Σ and $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$, $MA \times N^k \xrightarrow{M\omega} Mi$ is a morphism induced out of the coproduct by the family

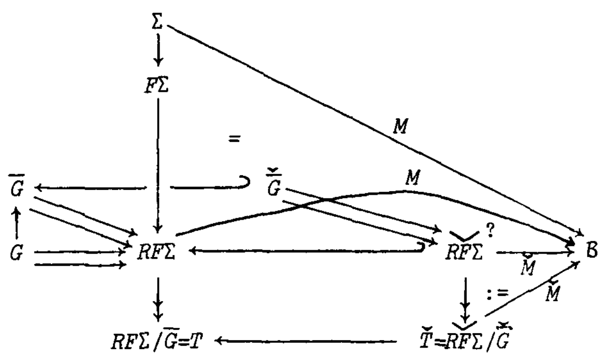
https://doi.org/10.1017/S0004972700010431 Published online by Cambridge University Press

$$\left(MA \xrightarrow{(MA, n_1, \ldots, n_k)} MA \times N^k \xrightarrow{M\omega} Mi\right)_{(n_1,\ldots,n_k)\in\mathbb{N}^k}$$

$$= \left(\breve{M}\left(A \xrightarrow{(A,n)} A \times N^k \xrightarrow{\omega} i\right)\right)_{n\in\mathbb{N}^k}$$

and therefore uniquely determined by $\breve{M}$. (Here $n := s^n \circ 0 : 1 \to N$.) For proving surjectivity, take this as a definition of M on Σ. Then M extends freely into $M \in \mathrm{mod}(RF\Sigma, \mathcal{B})$. We have to show that M equalizes $G \rightrightarrows RF\Sigma$, so gives rise to $M : T \to \mathcal{B}$ and that the latter restricts to $\breve{M}$ on $\breve{T}$. Consider the diagram

(1)



For proving the above assertions, we describe M completely in terms of $\breve{M}$.

PROPOSITION 3.10. *For arbitrary* $A \times N^k \xrightarrow{f} B \times N^{k'}$ *in* $RF\Sigma$, $A, B \in I^*$ *(up to isos each arrow in* $RF\Sigma$ *is of this form) we have:*

(i) $f = \left(A \times N^k \xrightarrow{\tilde{f}} B,\ A \times N^k \xrightarrow{\mathrm{pr}} N^k \xrightarrow{\hat{f}} N^{k'}\right)$ *with* $\hat{f}$ *out of the free pr-theory generated by* N *;*

(ii) $M\tilde{f}$ *is the induced morphism* $\left[\breve{M}\left(A \xrightarrow{(A,n)} A \times N^k \xrightarrow{\tilde{f}} B\right)\right]_{n\in\mathbb{N}^k}$, $\left[MA \xrightarrow{(MA,n)} MA \times N^k\right]_{n\in\mathbb{N}^k}$, *being a coproduct in* $\mathcal{B}$ *by Remark* 1.6 *and* $\mathcal{B}$ *cartesian closed.*

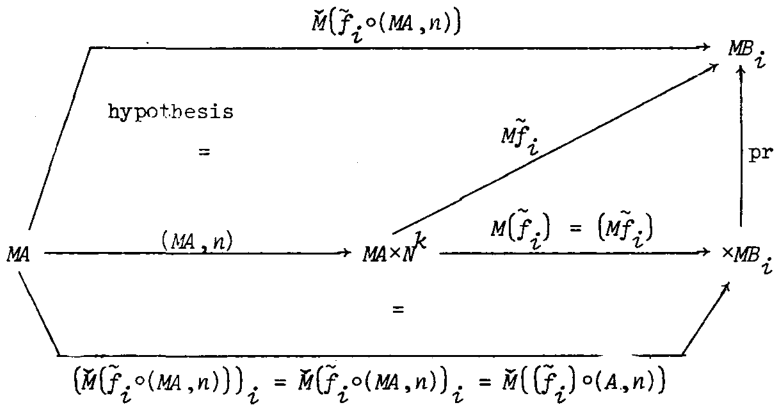
Hence $Mf = (M\tilde{f},\ M\hat{f} \circ \mathrm{pr}) = \left(\left[\breve{M}\left(A \xrightarrow{(A,n)} A \times N^k \xrightarrow{\tilde{f}} B\right)\right]_{n\in\mathbb{N}^k},\ \hat{f} \circ \mathrm{pr}\right)$.

We prove this by induction on the (depth of) expressions in $RF\Sigma$.

https://doi.org/10.1017/S0004972700010431 Published online by Cambridge University Press

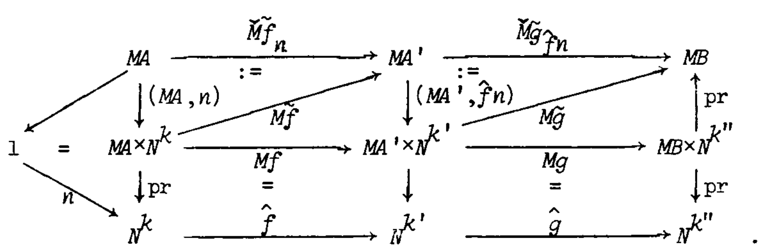
For $f : A\times N^k \to i$ in Σ, *(i)* is trivial, *(ii)* is true by definition of M. For $f = 0 : 1 \to N$ or $f = s : N \to N$, $\hat{f}$ is trivial and therefore the assertion holds.

Now, given a finite family $\left\{A\times N^k \xrightarrow{f_i} B_i\times N^{k_i} : i \in I'\right\}$, $f_i = \left\{A\times N^k \xrightarrow{\tilde{f}_i} B_i,\ A\times N^k \xrightarrow{\text{pr}} N^k \xrightarrow{\hat{f}_i} N^{k_i}\right\}$ by Lemma 3.7 (or by the induction hypothesis), $\hat{f}_i$ in the free pr-theory generated by N, $M\tilde{f}_i$ induced by $\left[M\left(A \xrightarrow{(A,n)} A\times N^k \xrightarrow{\tilde{f}_i} B_i\right)\right]_{n\in\mathbf{N}^k}$ (induction hypothesis), then (up to iso) $f = (f_i) = \left\{A\times N^k \xrightarrow{(\tilde{f}_i)} \times B_i,\ A\times N^k \xrightarrow{\text{pr}} N^k \dashrightarrow{(\hat{f}_i)} \times N^{k_i}\right\}$ and



so the induced f "splits" again into "I^*-part" and "N-part", and $M\tilde{f}$ is induced by its $\breve{M}(\tilde{f}\circ(A,n))$, $n \in \mathbf{N}^k$.

Next, let $f : A\times N^k \to A'\times N^{k'}$ and $g : A'\times N^{k'} \to B\times N^{k''}$ satisfy the proposition. For the composition consider



Then again $g \circ f = (\tilde{g}f, \hat{\hat{g}}f\mathrm{pr})$. The left upper rectangle commutes, because both ways equal $(\tilde{M}f_n, \hat{f}n)$; hence the I^*-part $M(\tilde{g}f)$ of the composition is again induced by $[\tilde{M}(\tilde{g}_{\hat{f}n} \circ \tilde{f}_n)]_{n \in \mathbb{N}^k}$.

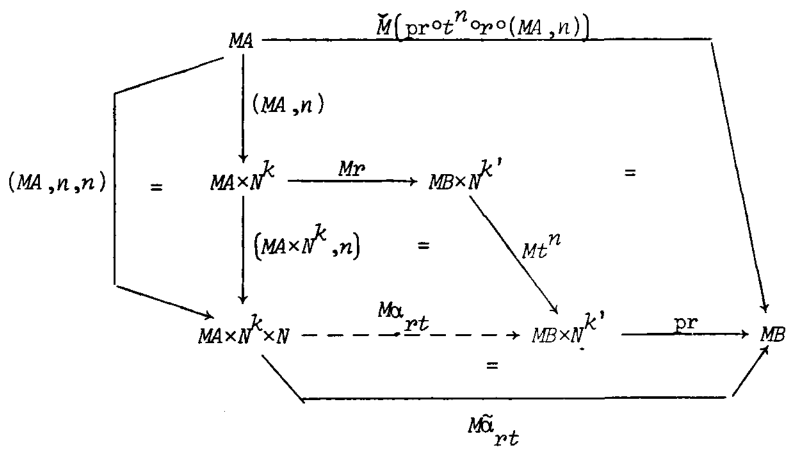
Up to now, we have proved the proposition to be stable under the generating procedures for algebraic theories.

Now let us prove that it is stable under primitive recursion: given

$$A \times N^k \xrightarrow{r=(\tilde{r},\hat{r}\mathrm{pr})} B \times N^{k'} \xrightarrow{t=(\tilde{t},\hat{t}\mathrm{pr})} B \times N^{k'} ,$$

$\alpha_{rt} = (\tilde{\alpha}_{rt}, \alpha_{\hat{r}\hat{t}} \circ \mathrm{pr}_{N^k \times N}) : A \times N^k \times N \to B \times N^{k'}$ with $\alpha_{\hat{r}\hat{t}}$ in the free pr-theory over N (Lemma 3.7, proof).

Now consider



.

$M\alpha_{rt}$ is induced by the $M(t^n \circ r)$ by Remark 1.2 (iii). By the hypothesis on r and t and by the composition step above,

$M(\widetilde{t^n \circ r}) = M(\mathrm{pr} \circ t^n \circ r)$ is induced by $[\tilde{M}(\mathrm{pr} \circ t^n \circ r \circ (MA, n))]_{n \in \mathbb{N}^k}$,

hence $M\alpha_{rt}$ is induced by $[\tilde{M}(\mathrm{pr} \circ t^n \circ r \circ (MA, n))]_{(n,n) \in \mathbb{N}^k \times \mathbb{N}}$. This proves the proposition.

Now take simply $k = k' = 0$ in the proposition to show that $M|_{\widetilde{RF\Sigma}} = \tilde{M}$; that is commutativity of ? in (1).

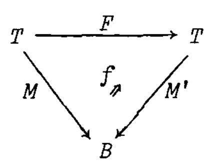
https://doi.org/10.1017/S0004972700010431 Published online by Cambridge University Press

We need the proposition in an (apparently) deeper way in order to show that M satisfies G : for $A \times N^k \overset{w_1}{\underset{w_2}{\rightrightarrows}} B$ in G (all pairs in G are of this form by hypothesis on T), the pairs $A \xrightarrow{(A,n)} A \times N^k \overset{w_1}{\underset{w_2}{\rightrightarrows}} B$ are in $\overline{G}$, hence in $\check{\overline{G}}$, therefore equalized by $\check{M} : \widecheck{RF\Sigma} \to B$. The proposition then implies $Mw_1 = Mw_2$. This shows M to be a T-model, since, by definition, it preserves N . Commutativity of (1) shows $M|_{\check{T}} = \check{M}$ as required. This proves J in the theorem to be bijective on objects. As first factor of a faithful functor J is faithful. It follows from the proposition that J is full.

REMARK 3.11. The theorem shows that algebras with primitive recursion can be described by a suitable theory $\check{T}$ with no additional sort N . But it is horribly complicated to give a direct description of T realizing given operations and equations. Furthermore, $\check{T}$, as a Lawvere theory, is no longer of finite presentation, even if T is, as a pr-theory. This last point is very important for applications in computer science, because in that theory one of the fundamental principles is construction of "things" out of a finite number of basic "things" by means of a finite number of finitely describable generation procedures.

4. Correctness of translations

DEFINITION 4.1. For "languages" $T \in PRTh_I$, $T' \in PRTh_J$, a *syntactical translation* is a pr-functor $F : T \to T'$. For $M : T \to B$, $M' : T' \to B$ models ("compilers") a family $f_i : Mi \to M'Fi$ $(i \in I)$ (translation of the basic data) makes F into a *semantical translation* $(F, f) : (T, M) \to (T', M')$. (F, f) is *correct*, if it defines (by $f_{uv} := f_u \times f_v$, $f_N = \mathrm{id}_N$) a functor transformation



This means that executing a program by M (on input-data appropriate for M) and then translating the result by f (transforming it into another code) is the same as translating the input data by f and then executing the F-translated program.

THEOREM 4.2. *Let* (Σ, G) *be a presentation of* T ($\Sigma \in Graph_I$, *see Corollary* 2.6). *Then the correctness of a translation* $(F, f) : (T, M) \to (T', M')$ *reduces to the correctness for the (generating) arrows in* Σ .

Proof. Correctness is clearly closed under composition of arrows and by the universal property of the product - under forming induced morphisms into products. So correctness extends to the algebraic theory generated by Σ within T . It remains to show that the schema of primitive recursion preserves correctness, that is to say, for f correct on $A \xrightarrow{r} B \xrightarrow{t} B$, f is correct on α_{rt} , that is

$$\begin{array}{ccc} M(A\times N) & f_{A\times N} & \\ \| & \| & \\ MA\times N & \xrightarrow{\quad f_A\times N \quad} & M'FA\times N \\ \downarrow{\scriptstyle M\alpha_{rt}=\alpha_{Mr,Mt}} & = & \downarrow{\scriptstyle M'F\alpha_{rt}=\alpha_{M'Fr,M'Ft}} \\ MB & \xrightarrow[\quad f_B \quad]{} & M'FB \end{array}$$

This follows immediately from Remark 1.2 (ii).

The most important case of theories are the finitely presented ones (only these are considered in programming). The theorem reduces correctness checks to a finite number of basic operations.

References

[1] Suad Alagić, Michael A. Arbib, Robert W. Taylor, "Algebraic models of syntax, semantics and data structures" (Computer and Information Science Technical Report 74B-1. University of Massachusetts at Amherst, Amherst, 1974).

[2] M.A. Arbib and Y. Give'on, "Algebra automata I: Parallel programming as a prolegomena to the categorical approach", *Information and Control* 12 (1968), 331-345.

[3] Jean Bénabou, "Structures algébriques dans les catégories", *Cahiers Topologie Géom. Différentielle* 10 (1968), 1-126.

[4] Eduardo Dubuc, "Adjoint triangles", *Reports on the Midwest Category Seminar II*, 69-91 (Lecture Notes in Mathematics, 61. Springer-Verlag, Berlin, Heidelberg, New York, 1968).

[5] Hartmut Ehrig, Wolfgang Kühnel, Michael Pfender, "Characterization of recursion", *Category theory applied to computation and control*, 137-143 (Proc. First Internat. Sympos., San Francisco, 1974. Lecture Notes in Computer Science, 25. Springer-Verlag, Berlin, Heidelberg, New York, 1975).

[6] Samuel Eilenberg, Jesse B. Wright, "Automata in general algebras", *Information and Control* 11 (1967), 452-470.

[7] Samuel Eilenberg, Calvin C. Elgot, *Recursiveness* (Academic Press, New York and London, 1970).

[8] C.C. Elgot, "Monadic computation and iterative algebraic theories" (Report RC4564. IBM Research, Yorktown Heights, New York; San Jose, California; Zurich, Switzerland; 1973).

[9] Peter Freyd, "Aspects of topoi", *Bull. Austral. Math. Soc.* 7 (1972), 1-76.

[10] J.A. Goguen and J.W. Thatcher, "Initial algebra semantics" (Report RC4865. IBM Research, Yorktown Heights, New York; San Jose, California; Zurich, Switzerland; 1974).

[11]* M. André Joyal, "Arithmetic universes", unpublished talk at Conference on Category Theory, Oberwolfach, July 1973.

[12] W. Kühnel, J. Meseguer, M. Pfender, I. Sols, "Algebras with actions and automata" (Preprint Reihe Mathematik, 9. Technische Universität, Berlin, 1975).

* [Unverified reference . Editor].

[13] W. Kühnel, J. Meseguer, M. Pfender, I. Sols, "Primitive recursive algebraic theories with applications to program schemes", *Cahiers Topologie Géom. Différentielle* 16 (1975), 271-273.

[14] Francis William Lawvere, Jr., "Functorial semantics of algebraic theories" (PhD thesis, Columbia University, New York, 1963); see also: F. William Lawvere, "Functorial semantics of algebraic theories", *Proc. Nat. Acad. Sci. USA* 50 (1963), 869-873.

[15] F. William Lawvere, "An elementary theory of the category of sets", *Proc. Nat. Acad. Sci. USA* 52 (1964), 1506-1511.

[16] F.E.J. Linton, "Some aspects of equational categories", *Proc. Conf. Categorical Algebra*, La Jolla, California, 1965, 84-95 (Springer-Verlag, Berlin, Heidelberg, New York, 1966).

[17] F.E.J. Linton, "Coequalizers in categories of algebras", *Seminar on triples and categorical homology theory*, 75-90 (Lecture Notes in Mathematics, 80. Springer-Verlag, Berlin, Heidelberg, New York, 1969).

[18] Saunders Mac Lane, Garrett Birkhoff, *Algebra* (Macmillan, New York; Collier-Macmillan, London; 1967).

[19] José Meseguer Guaita, "Algebra universal y recursion primitiva en categorias monoidales" (Departamento de Electricidad y Electronica, Universidad de Zaragoza, Spain, 1975).

[20] A.R. Meyer, D.M. Ritchie, "Computational complexity and program structure" (Report RC1817. IBM Research, Yorktown Heights, New York; San Jose, California; Zurich, Switzerland; 1967).

[21] Michael Pfender, *Universal algebra in S-monoidal categories* (Algebra-Berichte, 20. Mathematisches Institut der Universität München, Verlag Uni-Druck, München, 1974).

https://doi.org/10.1017/S0004972700010431 Published online by Cambridge University Press

[22] Ignacio Sols Lucia, "Aportaciones a la teoria de topos, al algebra universal y a las matematicas fuzzy" (Departamento de Geometria y Topologia, Universidad de Zaragoza, Spain, 1975).

Fachbereich Mathematik,
Technische Universität Berlin,
Berlin,
Germany;

Departamento de Algebra y Fundamentos,
Facultad de Ciencias,
Universidad,
Santiago de Compostela,
Spain;

Fachbereich Mathematik,
Technische Universität Berlin,
Berlin,
Germany;

Departamento de Matematica,
Universidad de Zaragoza,
Zaragoza,
Spain.