Canad. Math. Bull. Vol. **27** (3), 1984

ON AN OVAL WITH THE FOUR POINT PASCALIAN PROPERTY

BY
OLGA FERNANDES

ABSTRACT. In this paper it is proved that a finite translation plane of order $n \equiv 3 \pmod 4$ which contains an oval with the four point Pascalian property, or a finite dual translation plane of order $n \equiv 3 \pmod 4$ which contains an oval with the four point Pascalian property, can be coordinatized by a commutative semifield.

1. **Introduction.** The purpose of this paper is to prove that a finite translation plane of order $n \equiv 3 \pmod 4$ which contains an oval with the four point Pascalian property, or a finite dual translation plane of order $n \equiv 3 \pmod 4$ which contains an oval with the four point Pascalian property, can be coordinatized by a commutative semifield (Theorem 4.3).

To establish this result, preliminary theorems dealing with certain special properties of an oval are proved in Section 3. In Section 4, it is shown, in Theorem 4.2, that an oval with the four point Pascalian property in a projective plane of order $n \equiv 3 \pmod 4$ has these properties; and hence, by using a theorem of Ostrom [8, Theorem 2.8], that it determines an orthogonal polarity of the plane. This polarity and a theorem of Ganley [4, Theorem 3] are used to prove Theorem 4.3.

Throughout this paper it is assumed that the projective plane is finite and of odd order n. It may be mentioned, though, that some of the definitions and results are valid even in planes of infinite order.

2. **Background.** The basic theory of projective planes and of ovals in finite projective planes can be found in Dembowski [3], Hughes and Piper [6], and Ostrom [8].

An oval in a finite projective plane of order n is a set of $(n+1)$ points no three of which are collinear. Segre [10] has proved that in a finite Desarguesian plane of odd order every oval is a conic.

An oval is a Pascalian oval if every non-degenerate hexagon inscribed in it is a Pascalian hexagon. Definitions of Pascalian hexagons and Pascalian ovals can be found in Buekenhout [2], Hofmann [5], and Rigby [9]. It has been proved

Received by the editors October 23, 1982 and, in revised form, September 19, 1983.
1980 Mathematics Subject Classification: 51A40, 51E15.

https://doi.org/10.4153/CMB-1984-044-2 Published online by Cambridge University Press

by Buekenhout [3], Artzy [1], and Rigby [9] that if there is one Pascalian oval in a projective plane, then the plane is Pappian and the oval is a conic.

A 5 point (4 point, 3 point) hexagon is a non-degenerate hexagon having at most 5 (4, 3) distinct vertices.

An oval is said to have the 5 point (4 point, 3 point) Pascalian property if every 5 point (4 point, 3 point) hexagon inscribed in it is Pascalian. Hofmann has proved [5] that an oval with the 5 point Pascalian property is a Pascalian oval.

A line is called a secant, tangent, or exterior line of an oval if it contains 2, 1 or 0 points of the oval respectively. The unique tangent at a point L of the oval is denoted by LL; the line through two distinct points A, B of the projective plane by AB; and the point of intersection of the lines AB, CD, by $AB \cap CD$. If A and B are distinct points of the oval $\mathcal{O}$ and if M is another point on the secant AB, then $AM \cap \mathcal{O}$ denotes the second point of intersection, B, of the secant AM with $\mathcal{O}$.

It is well known that in a projective plane of odd order every point not on an oval $\mathcal{O}$ is on either exactly two tangents of $\mathcal{O}$ or on none at all. Such a point is called an exterior or interior point of $\mathcal{O}$ according as it is on two tangents or on no tangent of $\mathcal{O}$. The polar of an exterior point P is the secant AB through the points A, B in which the two tangents to $\mathcal{O}$ through P meet $\mathcal{O}$. P (which can be written $AA \cap BB$) is called the pole of AB [8, p. 417]. PAB is called a fundamental triangle of $\mathcal{O}$ [8, p. 422].

An ordered set of collinear points $\{AB; CD\}$ is said to be harmonic with respect to an oval $\mathcal{O}$ if (i) A and B belong to $\mathcal{O}$ (ii) C is an exterior point (iii) D is on the polar of C. The point D is called the harmonic conjugate or the fourth harmonic point of C [8, p. 420].

Ostrom has formulated certain assumptions relating to pairs of secants, harmonic sets, and fundamental triangles of an oval and called them Assumption A1 ([8, p. 420] and [7, p. 190]), Assumption A2 [8, p. 423], and Assumption B [8, p. 423]. If Assumption A1 (Assumption A2, Assumption B) is satisfied for the oval, then we find it convenient to say that the oval has Property $\mathcal{A}_1$ (Property $\mathcal{A}_2$, Property $\mathcal{B}$) and, using this phraseology, define the Properties by restating the definitions of the corresponding Assumptions. This does not affect Theorems 2.1, 2.2, 2.8 and Lemmas 2.1, 2.2 of [8] which we need later. The notation is the one we will be using in our proofs.

PROPERTY $\mathcal{A}_1$. An oval has Property $\mathcal{A}_1$ if, for every exterior point U and each pair of distinct secants AB, $A'B'$ through U, where A, B, A', B' are points of the oval, it follows that the points $AA' \cap BB'$ and $AB' \cap A'B$ are on the polar of U. An oval which has Property $\mathcal{A}_1$ is called a harmonic oval. (See Ostrom [7, pp. 190, 191].)

PROPERTY $\mathcal{A}_2$. An oval in a projective plane of order $n \equiv 3 \pmod 4$ has Property $\mathcal{A}_2$ if, for every interior point I and each pair of distinct secants AB,

https://doi.org/10.4153/CMB-1984-044-2 Published online by Cambridge University Press

$A'B'$ through I, where A, B, A', B' are points of the oval, it follows that

(1) there exist exterior points E, E' such that $\{AB; EI\}$ and $\{A'B'; E'I\}$ are harmonic sets, and

(2) the points $AA' \cap BB'$ and $AB' \cap A'B$ are on the line EE'. (See Ostrom [8, p. 423].)

REMARK. We shall see later, in Result 2.2, that (when $n \equiv 3 \pmod 4$) a sufficient condition for the existence of the points E, E' in the definition of Property $\mathcal{A}_2$ is that the oval should have Property $\mathcal{A}_1$.

PROPERTY $\mathcal{B}$. An oval has Property $\mathcal{B}$ if, for each pair of fundamental triangles PAB, $P'A'B'$ such that A, B, A', B' are distinct points of the oval, it follows that the points $AA' \cap BB'$ and $AB' \cap A'B$ are on the line PP'. (See Ostrom [8, p. 423].)

We now introduce another definition, motivated by Ostrom's Theorem 2.1 in [8].

PROPERTY $\mathcal{U}-\mathcal{V}$. An oval has Property $\mathcal{U}-\mathcal{V}$ if, for each pair of exterior points U, V such that U is on the polar of V, it follows that V is on the polar of U.

The following two known results are due to Ostrom:

RESULT 2.1. An oval which has Property $\mathcal{A}_1$ (that is, a harmonic oval) has Property $\mathcal{U}-\mathcal{V}$ [8, Theorem 2.1, p. 421].

The next result, Result 2.2, is contained in the first paragraph of the proof of Lemma 2.2, p. 424 in [8]. The projective plane is assumed to be of order $n \equiv 3 \pmod 4$. The Lemma requires the oval to have Properties $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}$. For Result 2.2 it is sufficient, as we shall see from the details of the proof, for the oval to have Property $\mathcal{A}_1$.

RESULT 2.2. If an oval in a projective plane of order $n \equiv 3 \pmod 4$ has Property $\mathcal{A}_1$, then each interior point on any secant is the harmonic conjugate of a unique exterior point on that secant.

Proof. The harmonic conjugate of an exterior point on a secant is a unique interior point on it [8, Theorem 2.2, p. 421]. Distinct exterior points on a secant cannot have the same harmonic conjugate because their polars pass through the pole of the secant (Result 2.1) and are distinct lines. The fact that the number of exterior points on a secant is the number of interior points on it [8, p. 419] completes the proof.

3. **Preliminary theorems**

THEOREM 3.1. *If an oval has both Property $\mathcal{B}$ and Property $\mathcal{U}-\mathcal{V}$, then it has Property $\mathcal{A}_1$.*

https://doi.org/10.4153/CMB-1984-044-2 Published online by Cambridge University Press

Proof. Let AB and $A'B'$ be any two distinct secants through an arbitrary exterior point U, where A, B, A', B' are points of the oval. Let P, P' be the poles of $AB, A'B'$ respectively. Since the oval has Property $\mathcal{B}$, the points $AA' \cap BB'$ and $AB' \cap A'B$ are on the line PP'. But, since the oval also has Property $\mathcal{U}-\mathcal{V}$, the line PP' is the polar of U. The theorem follows from the definition of Property $\mathcal{A}_1$.

THEOREM 3.2. *If an oval in a projective plane of order* $n \equiv 3 \pmod 4$ *has both Property* $\mathcal{B}$ *and Property* $\mathcal{U}-\mathcal{V}$, *then it has Property* $\mathcal{A}_2$.

Proof. Let $\mathcal{O}$ be the oval. By Theorem 3.1, it has Propery $\mathcal{A}_1$. Let AB, CD be any two distinct secants through an arbitrary interior point I of $\mathcal{O}$, where A, B, C, D are points of $\mathcal{O}$. There exist exterior points E, E' such that $\{AB; EI\}$ and $\{CD; E'I\}$ are harmonic sets (Result 2.2). Let O, T be the poles of AB, CD respectively. By Property $\mathcal{B}$, the points $AC \cap BD$ and $AD \cap BC$ are on the line OT.

Thus, to prove that the oval has Property $\mathcal{A}_2$, it is sufficient to show that the line OT is the same as the line EE'.

If AB, CD are conjugate lines with respect to $\mathcal{O}$ (that is, lines which are such that each contains the pole of the other), then $O = E'$ and $T = E$; so $OT = EE'$ follows immediately.

If AB, CD are not conjugate lines with respect to $\mathcal{O}$, then because $\mathcal{O}$ has Property $\mathcal{U}-\mathcal{V}$, neither line contains the pole of the other. Thus O is not a point of CD. Let $OC \cap \mathcal{O} = C'$. Hence $C' \neq D$ and $CC' \neq OI$. Now, the polar of E passes through O (Property $\mathcal{U}-\mathcal{V}$) and through I (since $\{AB; EI\}$ is a harmonic set), and is therefore OI. Consequently, the lines EC and EC' are not tangents to $\mathcal{O}$. Furthermore, E is not a point of CC'; for, since $n \equiv 3 \pmod 4$, OE is an exterior line of $\mathcal{O}$ [8, Lemma 2.1, p. 421]. Let $EC \cap \mathcal{O} = D'$ and $EC' \cap \mathcal{O} = D_1$. It is easy to see that the points C, D', C', D_1 are distinct.

The oval has Property $\mathcal{A}_1$; so, since CD' and $C'D_1$ are secants through E, it follows that the points $CC' \cap D'D_1$ and $CD_1 \cap C'D'$ are on OI. Now OI and CC' pass through O. Hence D_1D' also passes through O. Accordingly, we can further deduce (Property $\mathcal{A}_1$) that the points $CD' \cap C'D_1$ and $CD_1 \cap C'D'$ are on AB. Thus the point $CD_1 \cap C'D' = OI \cap AB = I$. Therefore C, D_1, I are collinear points. Recalling that C, D, I are also collinear points, we infer that C, D, D_1, which are points of the oval $\mathcal{O}$, cannot all be distinct. We know that $C \neq D$; and also (since E is not a point of CC') that $C \neq D_1$. It follows that $D = D_1$. Consequently, $CC' \cap DD' = O$ and $CD' \cap C'D = E$.

Let $T' = C'C' \cap D'D'$. Consider the fundamental triangles TCD and $T'C'D'$. Since the oval has Property $\mathcal{B}$, the points $O = CC' \cap DD'$ and $E = CD' \cap C'D$ are on TT'. Thus O, E, T, T' are collinear points. In particular, O, E, T are collinear points.

By interchanging the roles of the secants AB and CD, and using an

https://doi.org/10.4153/CMB-1984-044-2 Published online by Cambridge University Press

argument similar to the one detailed above, it can be proved that T, E', O are collinear points. Since O, E, T are also collinear points, it follows that the line EE' is the same as the line OT. This completes the proof.

4. **The main theorem.** An oval with the four point Pascalian property has already been defined to be one for which every inscribed four point hexagon is Pascalian.

We introduce two more definitions. The successive vertices of inscribed hexagons are written in order, as in [9].

DEFINITION 4.1. An oval has the first type of four point Pascalian property if every inscribed four point hexagon for which two non-adjacent distinct vertices arise out of pairs of coincident vertices (e.g., $AABCCD$ would be such a hexagon) is Pascalian.

DEFINITION 4.2. An oval has the second type of four point Pascalian property if every inscribed four point hexagon for which at least two adjacent distinct vertices arise out of pairs of coincident vertices (e.g. $AABBCD$ would be such a hexagon) is Pascalian.

Result 2.2 in [9, p. 1463], which is also derived in [5, Proposition 2, p. 147], can now be re-stated thus: An oval which has the second type of four point Pascalian property has Property $\mathcal{U}-\mathcal{V}$. This is easily verified.

Clearly, an oval with the four point Pascalian property has both the first and the second types of four point Pascalian properties.

THEOREM 4.1. *An oval has the first type of four point Pascalian property if and only if it has Property* $\mathcal{B}$.

Proof. Let A, B, A', B' be any four distinct points of an oval. Let $P = AA \cap BB$, $P' = A'A' \cap B'B'$. Consider the fundamental triangles PAB, $P'A'B'$ and the inscribed hexagons $AAA'BBB'$, $A'A'AB'B'B$ of the oval. The following statements are clearly equivalent: The points $AA' \cap BB'$ and $AB' \cap A'B$ are on the line PP'. The points P, P', $AA' \cap BB'$, $AB' \cap A'B$ are collinear. The points $AA \cap BB$, $AA' \cap BB'$, $AB' \cap A'B$ are collinear and the points $A'A' \cap B'B'$, $AA' \cap BB'$, $AB' \cap A'B$ are collinear. The hexagons $AAA'BBB'$ and $A'A'AB'B'B$ are Pascalian. The theorem follows.

THEOREM 4.2. *If a projective plane of order* $n \equiv 3 \pmod 4$ *contains an oval with the four point Pascalian property then the plane admits an orthogonal polarity.*

Proof. An oval with the four point Pascalian property has Property $\mathcal{B}$ (Theorem 4.1) and Property $\mathcal{U}-\mathcal{V}$ [5, Proposition 2, p. 147]. Therefore it has Property $\mathcal{A}_1$ (Theorem 3.1) and, since $n \equiv 3 \pmod 4$, it also has Property $\mathcal{A}_2$ (Theorem 3.2). Thus it has Properties $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}$. Hence, by a theorem of Ostrom [8, Theorem 2.8, p. 424], the plane admits a polarity. Furthermore,

$n \equiv 3 \pmod 4$ implies that n is not a square. So this polarity has exactly $(n+1)$ absolute points (see, for example, [6, Theorem 12.7, p. 242]); and is thus an orthogonal polarity (see [4, p. 104]).

THEOREM 4.3 *A translation plane of order* $n \equiv 3 \pmod 4$ *which contains an oval with the four point Pascalian property, or a dual translation plane of order* $n \equiv 3 \pmod 4$ *which contains an oval with the four point Pascalian property, can be coordinatized by a commutative semifield.*

Proof. By Theorem 4.2, if a translation plane or a dual translation plane is of order $n \equiv 3 \pmod 4$ and contains an oval with the four point Pascalian property, then it admits an orthogonal polarity. A translation plane which admits a polarity is a semifield plane [6, Theorem 12.13, p. 247]. Dually, a dual translation plane which admits a polarity is a semifield plane. Furthermore, by a result of Ganley [4, Theorem 3, p. 110], a finite semifield plane which admits an orthogonal polarity can be coordinatized by a commutative semifield. The theorem follows.

ACKNOWLEDGEMENTS. Most of this research is contained in my Ph.D. thesis submitted to the University of Poona. I wish to thank the referees of the thesis, Professor N. S. Gopalkrishnan, University of Poona; Professor P. J. Lorimer, University of Auckland, New Zealand; and Professor M. N. Vartak, Indian Institute of Technology, Bombay; and also the referee of this paper, for their careful reading of the material and for their valuable comments.

REFERENCES

1. R. Artzy, *Pascal's theorem on an oval*, Amer. Math. Monthly **75** (1968), 143–146.
2. F. Buekenhout, *Plans projectifs à ovoïdes Pascaliens*, Arch. Math. **17** (1966), 89–93.
3. P. Dembowski, *Finite Geometries*, Springer-Verlag, Berlin, Heidelberg, New York, 1968.
4. M. J. Ganley, *Polarities in translation planes*, Geom. Dedicata **1** (1972), 103–116.
5. Charles E. Hofmann, III, *Specializations of Pascal's theorem on an oval*, J. Geom. **1** (1971), 143–153.
6. D. R. Hughes and F. C. Piper, *Projective Planes*, Graduate Texts in Mathematics 6, Springer-Verlag, New York, Heidelberg, Berlin, 1973.
7. T. G. Ostrom, *Conicoids: conic-like figures in non-Pappian planes*, Geometry—von Staudt's point of view, pp. 175–196, NATO Adv. Study Inst. Ser., Ser. C: Math. and Phys. Sci., 70, Reidel, Dordrecht, 1981.
8. T. G. Ostrom, *Ovals, dualities and Desargues's theorem*, Canad. J. Math. **7** (1955), 417–431.
9. J. F. Rigby, *Pascal ovals in projective planes*, Canad. J. Math. **21** (1969), 1462–1476.
10. Beniamino Segre, *Ovals in a finite projective plane*, Canad. J. Math. **7** (1955), 414–416.

110 ST. PATRICK'S TOWN,
SHOLAPUR ROAD,
POONA 411013, INDIA