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# PERIODIC POINTS AND CHAOS FOR EXPANDING MAPS OF THE INTERVAL

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Expanding maps of the interval with unique turning points have periodic points of period $2^n \cdot 3$ for some $n$ and therefore are chaotic.

## Introduction

In recent years there has been considerable interest in the dynamical properties of difference equations defined by endomorphisms of the unit interval. The complicated asymptotic behaviour which often arises has been emphasized by the use of the term "chaotic" to characterize certain dynamical properties of a large class of such equations [5], [6], [7]. This complexity can be dealt with statistically if the transformation admits an invariant measure [3] especially one which is absolutely continuous with respect to Lebesgue measure. Thus there has been much interest in proving the existence of such measures [4], [8].

However the connection between these two ideas has not yet been clarified. It is known [2] that transformations which admit periodic points of period $2^n \cdot 3$ must exhibit chaotic behaviour. On the other hand expanding maps admit absolutely continuous invariant measures [8]. In this note we show that expanding maps with unique turning point must have a periodic point of period $2^n \cdot 3$ . This generalizes the result of Butler

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and Pianigiani [1] for maps of constant slope.

## Discussion

A piecewise monotonic map $\tau : [0, 1] \to [0, 1]$ is *expanding* if there exists a constant $\lambda > 1$ such that $|\tau(x)-\tau(y)| \geq \lambda|x-y|$ whenever both $x$ and $y$ belong to some interval on which $\tau$ is monotonic. Call $\lambda$ an *expansion constant* for $\tau$ .

Our main result is the following.

**THEOREM.** *Suppose* $\tau : [0, 1] \to [0, 1]$ *is a continuous expanding map with expanding constant* $\lambda > 1$ *, which is increasing on* $[0, c]$ *and decreasing on* $[c, 1]$ *. If* $\lambda^{2^{n-1}} < 2 \leq \lambda^{2^n}$ *, then* $\tau$ *admits a periodic point of period* $2^m \cdot 3$ *for* $m \leq n$ *.*

The proof will depend on a sequence of lemmas.

**LEMMA 1** [1]. *Suppose* $f : [0, 1] \to [0, 1]$ *is continuous. If there exist closed intervals* $I, J$ *such that* $I \cap J$ *consists of at most one point and* $f(I) \cap f(J) \supseteq I \cup J$ *then* $f$ *has a point of period* 3 .

**LEMMA 2.** *Suppose* $f : [0, 1] \to [0, 1]$ *admits a point* $z$ *with*

$$f(z) < z < f^2(z) \leq f^3(z) .$$

*Then* $f$ *has a point of period* 3 .

Proof. Take $I = [f(z), z]$ and $J = [z, f^2(z)]$ in Lemma 1.

**LEMMA 3.** *Suppose the expanding map* $f$ *admits a point* $z$ *with* $f(z) < z < f^2(z)$ *and* $f|[f(z), z]$ *decreasing and* $f|[z, f^2(z)]$ *increasing. If* $f$ *has no point of period* 3 *and* $g = f \circ f$ *then either* $g$ *admits a point of period* 3 *or else* $g$ *admits a point* $w$ *with* $f(z) < w < z$ , $g(w) < w < g^2(w)$ , $g|[g(w), w]$ *decreasing and* $g|[w, g^2(w)]$ *increasing.*

Proof. Since $f|[f(z), z]$ is a homeomorphism onto the interval $[f(z), f^2(z)]$ , there exists a unique point $w$ , $f(z) < w < z$ , such that $f(w) = z$ or $g(w) = f(z)$ . Now $f|[f(z), w]$ is decreasing and $f|f[f(z), w] = f|[z, f^2(z)]$ is increasing, so $g|[f(w), w] = g|[g(w), w]$

is decreasing. Also since $g$ is expanding we have $g^2(w) > w$ .

Now suppose that $g^2(w) > z$ . Then

$$g[w, g^2(w)] \geq g[w, z] \geq [f(z), f^2(z)] .$$

If $f$ has no point of period 3 then $f^3(z) < f^2(z)$ (Lemma 2) and

$$[f(z), f^2(z)] \supseteq [f(z), f^3(z)] = [g(w), g^2(w)] .$$

But $g[g(w), w] \supseteq [g(w), g^2(w)]$ and Lemma 1 implies that $g$ has a point of period 3 .

If, on the other hand, $g^2(w) \leq z$ , then $g|[w, g^2(w)]$ is increasing since it is the composition of $f|[w, g^2(w)]$ which is decreasing and $f|f([w, z])$ where $f([w, z]) = [f(z), z]$ where $f$ is increasing.

We now proceed to the proof of the theorem.

Proof. Since $\tau$ is expanding, $\tau(c) > c$ and $\tau^2(c) < c$ . Suppose $\tau^3(c) < c$ . Then $\tau([\tau^2(c), c]) \supseteq [c, \tau(c)]$ . Thus there exists a point $x_0 < c$ such that $\tau(x_0) = c$ , that is, $\tau([x_0, c]) = [c, \tau(c)]$ . Note that $\tau|[x_0, c]$ can have no fixed point. Thus

$$\tau^3[x_0, c] = \tau^2[c, \tau(c)] = \tau[\tau^2(c), \tau(c)] = [\tau^2(c), \tau(c)] \supseteq [x_0, c]$$

contains a point fixed under $\tau^3$ which cannot be a fixed point of $\tau$ . Thus $\tau$ has a point of period 3 .

Thus we may assume $\tau^3(c) > c$ . Then

$$|\tau^4(c)-\tau^2(c)| > \lambda|\tau^3(c)-\tau(c)| > \lambda^2|\tau^2(c)-c|$$

and $\tau^2(c) < \tau^4(c)$ since $c < \tau^3(c) < \tau(c)$ . Thus $\tau^2(c) < c < \tau^4(c)$ and $\tau^2|[\tau^2(c), c]$ is decreasing. If $\tau^6(c) > c$ then the above argument applied to $\tau^2$ will imply the existence of a periodic point of order 3 for $\tau^2$ or one of period 6 for $\tau$ .

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Thus we assume that $\tau^6(c) < c$ . Now if $\tau$ has no points of period 3 , then $c < \tau^4(c) < \tau(c)$ and thus $\tau^2(c) < \tau^5(c)$ . If $\tau^5(c) < c$ then $\tau^3(c) < \tau^6(c)$ which is impossible since $\tau^3(c) > c$ and $\tau^6(c) < c$ . Thus either $\tau^5(c) = c$ and by [10] we have a point of period $2 \cdot 3$ or else $\tau^5(c) > c$ . The latter case implies that $\tau^2 | [c, \tau^4(c)]$ is the composition of $\tau | [c, \tau^4(c)]$ where it is decreasing and $\tau | \tau([c, \tau^4(c)]) = \tau | [\tau^5(c), \tau(c)]$ where it also decreases. Thus $\tau^2 | [c, \tau^4(c)]$ is increasing and thus $f = \tau^2$ satisfies the hypotheses of Lemma 3 with $z = c$ . Suppose we have not point of period $2^k \cdot 3$ for $k = 0, \ldots, n-1$ . Then $f = \tau^{2^{n-1}}$ satisfies the hypotheses of Lemma 3 and so $g = \tau^{2^n}$ admits a point $w$ with $g(w) < w < g^2(w)$ and $g([g(w), w]) = [g(w), g^2(w)]$ . Thus

$$|g^2(w)-g(w)| \geq \lambda^{2n}|w-g(w)| \geq 2|w-g(w)| .$$

Now the length of $g([w, g^2(w)])$ is greater or equal to $2|g^2w-w| \geq |g^2(w)-g(w)|$ and so

$$g([w, g^2w]) \supseteq [g(w), g^2(w)] .$$

Thus Lemma 1 implies that $g$ has a point of period 3 and so $\tau$ has a point of period $2^n \cdot 3$ .

COROLLARY. *An expanding map with unique turning point is chaotic.*

*Proof.* If $\tau$ has a point of period $2^n \cdot 3$ , then $\tau^{2^n}$ has a point of period 3 . Thus $\tau^{2^n}$ is chaotic [7], and so $\tau$ is chaotic [9].

## References

[1] G.J. Butler and G. Pianigiani, "Periodic points and chaotic functions in the unit interval", *Bull. Austral. Math. Soc.* 18 (1978), 255-265.

[2] Frederick J. Fuglister, "A note on chaos", *J. Combin. Theory Ser. A* **26** (1979), 186-188.

[3] J. Guckenheimer, G. Oster and A. Ipaktchi, "The dynamics of density dependent population models", *J. Math. Biol.* **4** (1977), 101-147.

[4] М.В. Янобсон [M.V. Jakobson], "Топологичесние и метричесние свойства одномерных эндоморфизмов" [Topological and metric properties of one-dimensional endomorphisms], *Dokl. Akad. Nauk SSSR* **243** (1978), 866-869. English Transl: *Soviet Math. Dokl.* **19** (1978), 1452-1456.

[5] James L. Kaplan and Frederick R. Marotto, "Chaotic behaviour in dynamical systems", *Nonlinear systems and applications*, 199-210 (Proc. Internat. Conf. on Nonlinear Systems and Applications, University of Texas, Arlington, July 1976. Academic Press [Harcourt Brace Jovanovich], New York, San Francisco, London, 1977).

[6] Peter E. Kloeden, "Chaotic difference equations are dense", *Bull. Austral. Math. Soc.* **15** (1976), 371-379.

[7] Tien-Yien Li and James A. Yorke, "Period three implies chaos", *Amer. Math. Monthly* **82** (1975), 985-992.

[8] A. Lasota and James A. Yorke, "On the existence of invariant measures for piecewise monotonic transformations", *Trans. Amer. Math. Soc.* **186** (1973), 481-488.

[9] Melvyn B. Nathanson, "Permutations, periodicity, and chaos", *J. Combin. Theory Ser. A* **22** (1977), 61-68.

[10] А.Н. Шарновсной [A.N. Sharkovsky], "Сосуществование цинлов непрерывного преобразования прямой в себя" [Co-existence of the cycles of a continuous mapping of the line into itself], *Ukrain. Math. Ž.* **16** (1964), 61-71.

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