Seventh Meeting, 14*th June* 1907.

J. ARCHIBALD, Esq., M.A., President, in the Chair.

Herbert Spencer and Mathematics.

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About forty years ago, when Spencer was rising into philosophic fame, it used often to be said by his admirers that he was an accomplished mathematician. This statement was accepted without demur, though it was known that he had not measured himself against rivals of his own age, or, what is more important, had not produced anything new in this old science.

Since his *Autobiography** has been published, an estimate can be formed from his own statements of what were his acquirements in this subject, and what were his contributions to it.

If the estimate which follows seems severe, it must be remembered that Spencer was an unsparing critic of others. Of his own character he has said :—

"No one will deny that I am much given to criticism. Along with exposition of my own views there has always gone a pointing out of defects in the views of others. And if this is a trait in my writing, still more is it a trait in my conversation. The tendency to fault-finding is dominant—disagreeably dominant." (II. 438)

He has also said :

"It has been remarked that I have an unusual faculty of exposition—set forth my data and reasonings and conclusions with a clearness and coherence not common." (II. 437)

We shall have examples of this clearness of exposition later on.

In summing up the results of his education till the age of

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* This consists of two huge volumes, the first containing 556, and the second 542 pages. The preface by himself is dated 27th April 1894, when he was 74 years of age ; he was born on 27th April 1820. The publication of the *Autobiography* by his trustees took place in 1904.

To save repetition of *Autobiography* when the book is referred to, the volume and the page have alone been noted.

thirteen, he says, among other instructive remarks about what he did not know, that

"I had merely the ordinary knowledge of arithmetic; and beyond that no knowledge of mathematics."

The ordinary knowledge of arithmetic would be an acquaintance with the simple and compound rules. Mathematics at that time meant Geometry and Algebra, more frequently Geometry alone.

Towards the end of June 1833 he was taken by his parents to visit his uncle Thomas at Hinton Charterhouse, near Bath, and was left there to be educated.

"I had supposed I was about to spend a month's summer holidays; but I was taken by my uncle one morning and set down to the first proposition in Euclid. Having no love of school or of books, this caused in me great disgust. However there was no remedy, and I took to the work tolerably well: my faculty lying more in that direction than in the directions of most subjects I had dealt with previously. This was significantly shown before the end of a fortnight; when I had reached perhaps the middle of the first book. Having repeated a demonstration after the prescribed manner up to a certain point, I diverged from it; and when my uncle interrupted me, telling me I was wrong, I asked him to wait a moment, and then finished the demonstration in my own way; the substituted reasoning being recognised by him as valid." (I. 93)

At Hinton he learned Euclid and Latin in the morning, and in the evening some Alegbra. Towards the end of October he says "there is mention of demonstrations made by myself of propositions in the fourth book of Euclid: not, however, approved by my uncle."

In a letter to his father dated January 28, 1834, there occurs the passage:

"I forgot to tell you in my last letter that I had made some problems in Algebra with which my uncle was much pleased, and as I want something to fill up I will tell you them all. My uncle was most pleased with the 5th of these which he thought was very original."

"Correspondence shows that in March I was learning... Trigonometry. With Trigonometry I speak as being delighted: sending my father some solutions of trigonometrical questions."

"Euclid was gone through again at this time; and mention is made of the fact that I was able to repeat some of the propositions without the figures: not, as might be supposed, by rote-learning, but by the process of mentally picturing the figures and their letters, and carrying on the demonstrations from the mental pictures." (I. 104–105)

What is meant by "Euclid" is very vague, and the remark that he was able to repeat some of the propositions without the figures is naive. Why it might be supposed that he did it by rote-learning is significant of some of the geometrical teaching which at that time prevailed.

Spencer does not seem ever to have been aware that, before his time, the constructions and demonstrations of several of Euclid's books "in general terms" had been published, the diagrams being left to the readers' imagination, and no letters being required.

"Before the end of May [1835] I had been through the eleventh book of Euclid, and also through 'Lectures on Mechanics'—either Wood's Mechanics, a text-book in my uncle's college days, which I certainly went through at some time, or else the Cambridge Lectures which he had written down, and which we studied from his MS."

"In a letter to my father dated July 28, I apologise for breaking off because 'I have to learn a quantity of Newton to keep up with the others this morning'; and there occurs the sentence—'But I am very proud of having got into Newton.' Reference to the MS. book, which I still possess, shows that I did not go very far." (I. 110)

"That which remained with me best was the mathematical knowledge I acquired; for though the details of this slipped, I readily renewed them. Thus in May 1836 I describe myself in a letter as going through six books of Euclid in a week and a half." (I. 115.)

In the appendices to each of the volumes of his *Autobiography* Spencer reproduces some of his articles to periodicals and some memoranda he made when he was a young man.

Part of those which refer to mathematical matters are here extracted.

"It was either during the autumn of 1836 or during that of 1837 that I hit upon a remarkable property of the circle, not, so far as I have been able to learn, previously discovered . . . . I did not then attempt a proof. This was not supplied until some two years later." (I. 119)

"When seventeen I hit on a geometrical theorem of some interest. This remained with me in the form of an empirical truth; but . . . responding to a spur from my father, I made a demonstration of it; and now that it had reached this developed form, it was published in *The Civil Engineer and Architect's Journal* for July 1840. . . . . I did not know, at the time, that this theorem belongs to that division of mathematics at one time included under the name *Descriptive Geometry*, but known in more recent days as *The Geometry of Position*—a division which includes many marvellous truths. Perhaps the most familiar of these is the truth that if to three unequal circles anywhere placed, three pairs of tangents be drawn, the points of intersection of the tangents fall in the same straight line—a truth which I never contemplate without being struck by its beauty at the same time that it excites feelings of wonder and of awe: the fact that apparently unrelated circles should in every case be held together by this plexus of relations, seemingly so utterly incomprehensible. The property of a circle which is enunciated in my own theorem has nothing like so marvellous an aspect, but is nevertheless sufficiently remarkable." (I. 164)

The fact that Spencer allowed his "remarkable property of the circle" to remain with him "in the form of an empirical truth" (!) implies (Spencer is very fond of implications) that he was at this time somewhat devoid either of geometrical ardour or of geometrical skill.

But before going farther it may be well to enunciate what Spencer calls "his own theorem."

"Geometrical Theorem

"Sir

"I believe that the following curious property of a circle has not hitherto been noticed; or if it has, I am not aware of its existence in any of our works on Geometry."

[The reader is requested to make the figure.]

"Let ABCDE [read clockwise] be a circle of which ACD is any given segment: Let any number of triangles ABD, ACD, etc., be drawn in this segment, and let circles be inscribed in these triangles; their centres F, G, etc., are in the arc of a circle, whose centre is at E, the middle of the arc of the opposite segment AED."

The theorem is undoubtedly true, but Spencer's diagram is unnecessarily complicated, and his demonstration rather verbose. The property however which Spencer signalises had been noticed long before his time.

This is how the theorem arises.

Let a circle be circumscribed about a triangle ABC (call its centre O), and another circle be inscribed in ABC (call its centre I); to find the expression for the distance OI in terms of the radii of the two circles.

The first mathematician to find this expression was William Chapple* in 1746, but in his demonstration the property discovered by Spencer does not appear. It does appear however in the solution given by John Turner in 1748 in *The Mathematician*, p. 311, to the problem:

One side of a triangle, together with the radii of its circumscribing and inscribed circles being given, to construct the triangle geometrically.

Spencer's property is also given by Nicolas Fuss in 1794. See the 10th volume of the *Nova Acta Academiae Scientiarum Imperialis Petropolitanae* (Petropoli, 1797).

In the third † part of "The Elements of Plane Geometry" by the Rev. J. Luby, p. 57, among the exercises on Loci occur

Given the base and vertical angle of a triangle, required the loci

(*a*) of the centre of the inscribed circle

(*b*) of the centre of the circle that touches the base and the two sides produced

(*c*) of the centre of the circle that touches one side and the productions of the base and other side.

Exercise (*a*) is Spencer's theorem under another guise, and (*b*) and (*c*) are extensions of it.

The remarkable property of the circle is, as far as Spencer is concerned, the fact that he remarked it. It is, if one wished to describe it more accurately, a property of the triangle, and has nothing whatever to do with "Descriptive Geometry," which Spencer confounds with "Geometry of Position" (*Die Geometrie der*

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* *Miscellanea Curiosa Mathematica*, p. 117–124.

† The title-page is not dated, but the preface to the first part ends with Trinity College [Dublin] Sept. 7, 1833.

Lage of the Germans) or what is now called "Projective Geometry." It is a metrical property, like those expounded by Euclid.

The familiar truth which Spencer never contemplates without being struck by its beauty, etc., is expressed by him in terms of singular inexactitude.

"If to three unequal circles" [They needn't be unequal] "anywhere placed" [For example, in different planes?] "three pairs of tangents be drawn" [Millions of pairs of tangents may be drawn to three circles. What is implied and should have been stated is that the tangents must be common to every pair of the three circles. Furthermore, Spencer does not seem to have known that six pairs of common tangents can be drawn to three circles, taken two by two, and that the points of intersection of these six pairs give rise to four straight lines. From the phrase "anywhere placed" Spencer does not seem to have imagined any position of the three circles in which pairs of common tangents were impossible; for example when the first circle is inside the second, the second inside the third, and there is no mutual contact.]

The feelings of wonder and awe excited by "the fact that apparently unrelated circles should in every case be held together by this plexus of relations seemingly so utterly incomprehensible" probably arose from Spencer's unfamiliarity with any geometrical truths outside the first six books of Euclid. The circles are not "apparently unrelated." To any fairly well read geometer the following relations are evident at a glance:

(1) The circles are in the same plane.

(2) Being circles they are similar figures.

(3) Every pair of them may be regarded as similarly situated, that is, as having an external centre of similitude.

(4) Every pair of them may be regarded as oppositely situated, that is, as having an internal centre of similitude.

The application of a few geometrical theorems to the relations just stated soon removes any utter incomprehensibility.

The following extract from the last book Spencer published, *Facts and Comments* (1902) pp. 203–4, is not easy to characterise.

"In youth we pass without surprise the geometrical truths set down in our Euclids. It suffices to learn that in a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides: it is demonstrable, and that is enough. Concerning the multitudes of remarkable relations among lines and among spaces very few ever ask—Why are they so? Perhaps the question may in later years be raised, as it has been in myself, by some of the more conspicuously marvellous truths now grouped under the title of 'the Geometry of Position.' Many of these are so astounding that but for the presence of ocular proof they would be incredible; and by their marvellousness, as well as by their beauty, they serve, in some minds at least, to raise the unanswerable question—How come there to exist among the parts of this seemingly-structureless vacancy we call Space, these strange relations? How does it happen that the blank form of things presents us with truths as incomprehensible as do the things it contains?"

The phrases "very few ever ask" and "they serve in some minds at least" betray Spencer's consciousness of his superiority to other people; and the remark that "but for the presence of ocular proof" certain marvellous truths "would be incredible" indicates a very slight acquaintance with the properties of geometrical figures, as well as a very humble standard by which to judge of mathematical truths.

What truths incomprehensible or not are presented to us by a "seemingly-structureless vacancy" or by a "blank form of things" I am unable to conceive.

In the description of his visit to America, Spencer gives us a glimpse into his knowledge of Mathematical Geography.

"While sitting on a ledge of rock facing the East, and looking over the wide country stretching away to the horizon below the Hudson, it was interesting to think that here we were in a land we had read about all our lives—interesting, and a little difficult, to think of it as some three thousand miles from the island on the other side of the Atlantic whence we had come. Not easy was it either, and indeed impossible in any true sense, to conceive the real position of this island on that vast surface which slowly curves downward beyond the horizon: the impossibility being one which I have vividly felt when gazing sea-ward at the masts of a vessel below the horizon, and trying to conceive the actual surface of the Earth, as slowly bending round till its meridians met eight thousand miles beneath my feet: the attempt producing what may be figuratively called a kind of mental choking, from the endeavour to put into the intellectual structure a conception immensely too large for it." (II. 390)

Many pupils in a Geography class know that meridians meet at the North and the South Poles, and that any diameter of the Earth, that is, a straight line passing through its centre and terminated both ways by its surface, is approximately eight thousand miles. Spencer was never south of the Equator; hence he must have been at the North Pole to conceive even imperfectly what he "vividly felt."

Here is a glimpse into his knowledge of Mathematical Astronomy.

"When, many centuries after, Kepler discovered that the planets moved round the Sun in ellipses, and described equal areas in equal times . . ." *First Principles*, p. 103 (3rd ed. 1870)

I need not quote Kepler's three laws, but I may draw attention to the inadequacy of Spencer's statement of two of them. He omits to say what was the position of the Sun, namely, in one of the foci of the ellipses, and the description of equal areas in equal times ought to be attributed, not to the planets, but to the radii vectores of the planets, that is, to the straight lines drawn from the Sun to the planets.

Spencer's attitude towards the proposal to adopt the Metric System was one of uncompromising opposition, and in 1896 he published an ill-digested pamphlet entitled "Against the Metric System." I do not intend to discuss Spencer's arguments showing that the Metric is "a very imperfect system." The most important of these arguments have been answered time after time, and the others are puerile in the extreme. It is well known that the Metric System is not perfect, but the great difficulty is to get a better one. Here is Spencer's solution of the difficulty. I shall give it as far as possible in his own words, only remarking that the language he sometimes employs is veiy loose. He speaks of numeration and notation as if they were identical, and does not seem to know that while the decimal system of numeration has been adopted almost exclusively since the dawn of civilisation, or at any rate for several thousands of years, the systems of notation have varied considerably.

"We agree in condemning the existing arrangements under which our scheme of numeration and our modes of calculation based on it proceed in one way, while our various measures of length, area, capacity, weight, value proceed in other ways. Doubtless, the two methods of procedure should be unified; but how? You [addressing an opponent] assume that, as a matter of course, the measure-system should be made to agree with the numeration-system; but it may be contended that, conversely, the numeration-system should be made to agree with the measure-system—with the dominant measure-system, I mean."

If the British tables of measures be consulted it will be found that there is no dominant system. Of the tables of measures in use among the peoples of the continent of Europe before the introduction of the Metric System, Spencer says, probably because he knew, nothing.

The following quotation from memoranda which Spencer made more than 50 years before he issued his pamphlet will show what he means by the dominant measure-system.

"The fact that 12 has been so generally chosen [by whom?] as a convenient number for enumeration of weights and measures, is presumptive proof that it must have many advantages. We have 12 oz. = 1 pound in Troy weight and Apothecaries weight, 12 pence = 1 shilling, 12 months in the year, 12 signs to the Zodiac, 12 lines to the inch, 12 inches to the foot, 12 sacks one last, and 12 digits. Of multiples of 12 we have 24 grains one pennyweight, 24 sheets one quire, 24 hours one day, 60 minutes one hour, 360 degrees to the circle." (I. 531)

In reference to the preceding it may be asked, Who ever uses the Troy or Apothecaries' pound, or talks of the twelfth of an inch as a line? I have never heard any one speak of a last as containing 12 sacks (a sack may be almost any size), and the 12 digits I confess to be beyond me. All this ludicrous parade of 12's is intended to help in showing that 12 is a convenient number for a base of numeration. But to continue:

"During previous years [that is, before 1842] I had often regretted the progress of the decimal system of numeration; the universal adoption of which is by many thought so desirable. That it has sundry conveniences is beyond question; but it has also sundry inconveniences, and the annoyance I felt was due to a consciousness that all the advantages of the decimal system might be obtained along with all the advantages of the duodecimal system, if the basis of our notation were changed—if instead of having 10 for its basis, it had 12 for its basis: two new digits being introduced to replace 10 and 11, and 12 times 12 being the hundred. Most people are so little able to emancipate themselves from the conceptions which education has established in them, that they cannot understand that the use of 10 as a basis is due solely to the fact that we have five fingers on each hand and five toes on each foot. If mankind had had six instead of five, there never would have been any difficulty."

As regards the statement "most people . . . cannot understand that the use of 10 as a basis is due solely to the fact that we have five fingers on each hand," it may be remarked that nearly all pupils above the most elementary stage in all the schools of the world understand this. The statement that if mankind had had six fingers instead of five no difficulty would ever arise in calculation shows that Spencer's acquaintance with the properties of numbers was not very profound. He evidently did not know that whatever number be taken as the base of the numeration system certair difficulties would arise, and he evidently did not know that if 10 were displaced, other bases such as 6, 8, 16, 24, 60, . . . might be put forward as successors.

As regards his proposal to change, from 10 to 12, the basis of the system of numeration which prevails through the world, Spencer says

"I fully recognise the difficulties that stand in the way of making such changes as those indicated—difficulties greater than those implied by the changes which the adoption of the metric system involves. The two have in common to overcome the resistance to altering our tables of weights, measures, and values; and they both have the inconvenience that all distances, quantities, and values, named in records of the past, must be differently expressed. but there would be further obstacles in the way of a 12-notation system. To prevent confusion different names and different symbols would be needed for the digits, and to acquire familiarity with these, and with the resulting multiplication-table would, of course, be troublesome: perhaps not more troublesome, however, than learning the present system of numeration and calculation as carried on in another language. There would also be the serious evil that, throughout all historical statements, the dates would have to be differently expressed; though this inconvenience, so long as it lasted, would be without difficulty met by enclosing in parenthesis in each case the equivalent number in the old notation. But, admitting all this, it may still be reasonably held that it would be a great misfortune were there established for all peoples and for all time a very imperfect system, when with a little more trouble a perfect system might be established."

Nearly every sentence in the preceding paragraph calls for comment, but it would be tiresome to go into complete detail. We know what the metric system of measures is, and the nomenclature by which the various denominations in any table are connected together; but no man knows what Spencer's system (to call it so) of measures would be. It couldn't be duodecimal, and have a nomenclature that would be pronounceable.

Spencer says that "to prevent confusion different names and different symbols would be needed for the digits." The digits which now exist are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and these are common to nearly all civilised peoples. Why should their forms be changed, and why should the names given to them in the various languages be changed? What body of men could devise a better set of symbols, and what set of symbols would be agreed upon by all nations? The fact is that if a duodecimal system of numeration could be established, two new symbols for ten and eleven would be all that is necesssary for the notation (how they could be fixed upon it is impossible to conjecture), but all the words in *every* language denoting numbers higher than twelve would have to be altered. Is it conceivable that all civilised nations would agree to make this stupendous change, aud how could the change be carried out? No government in the world could impose on its subjects such a modification of their language.

It is significant of Spencer's ignorance of any language but his own that he should put the learning of the present system of numeration and calculation as carried on in another language on a par with the learning of the names of his new "digits" and with the multiplication-table which would thence result.

Think, again, of some of the consequences of setting up a duodecimal system of numeration. Every arithmetical and algebraical book in existence whether ancient or modern would be rendered useless, so would all the logarithmic tables in the world, and the hundreds of other tables of all sorts which save labour to the modest computer as well as to the profoundest mathematician.

Think of the vast body of statistics of every kind which every nation possesses, from the records of the observations of its scientific men down to the records of the population of the humblest villages. No one would think of converting these hundreds of millions of numbers from the decimal to the duodecimal scale, and reprinting them, and if the trouble of conversion had to be undergone each time a table was consulted the table might almost as well be non-existent.

Spencer remarks that throughout all historical statements the dates would have to be differently expressed. There does not appear to be any evidence that he had made a study of the Calendar, else he would have known what an amount of confusion and trouble has been caused all the world over (and indeed is still caused) by the change of dates. The Calendar plays a great part in the life of every one of us, and his proposal if it could be carried out would render every printed calendar in existence utterly useless.

It is abundantly clear from the *Autobiography* that Spencer's outfit of mathematical (or indeed any other) knowledge was both slender and scrappy, but it might have been thought that a man with even a small degree of insight into practical affairs would have hesitated before laying before his countrymen, in his mature age, the lucubrations of his youth, the memoranda of which had lain unused among his papers for more than half a century. A proposal so gigantic in its aims and so preposterous in its results could only have been conceived in ignorance and begotten of self-conceit.

Spencer asks, "Do I think this system will be adopted? Certainly not at present—certainly not for generations . . . . But it is, I think, not an unreasonable belief that further intellectual progress may bring the conviction that since a better system would facilitate both the thoughts and actions of men, and in so far diminish the friction of life throughout the future, the task of establishing it should be undertaken."

If any prophecy concerning human affairs in the long distant future is likely to be fulfilled, that surely is the one which predicts that mankind will never cut themselves off from the past by abandoning the decimal system of numeration.