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Abstract

We show that the image of a properly embedded Legendrian submanifold under a homeomorphism that is the C^0-limit of a sequence of contactomorphisms supported in some fixed compact subset is again Legendrian, if the image of the submanifold is smooth. In proving this, we show that any closed non-Legendrian submanifold of a contact manifold admits a positive loop and we provide a parametric refinement of the Rosen–Zhang result on the degeneracy of the Chekanov–Hofer–Shelukhin pseudo-norm for properly embedded non-Legendrians.

1. Terminology and notation

Let (M, ξ) be a $(2n+1)$-dimensional, possibly non-compact, contact manifold with contact distribution $\xi \subset TM$. We will assume M to be co-orientable and so we can chose a contact one-form α where $\xi = \ker\{\alpha\}$. We will denote by $\Lambda \subset M$ a connected properly embedded (not necessarily closed) *Legendrian* (submanifold), which means $\dim(\Lambda) = n$ and $T\Lambda \subset \xi$. We will denote by $K \subset M$ a connected properly embedded smooth submanifold with $\dim(K) \leqslant n$. Usually K will be a *non-Legendrian* connected properly embedded submanifold, which means either $\dim(K) < n$, or $\dim(K) = n$ and there exists $x \in K$ such that $T_xK \not\subset \xi_x$. We will sometimes consider these non-Legendrians (and Legendrians) as parameterized; i.e. K (and Λ) is equipped with an embedding into M. One canonical example is the inclusion $\mathrm{Id}_M|_K : K \to K \subset M$. Also for any contactomorphism $\Phi \in \mathrm{Cont}(M, \xi)$ such that $\Phi(K) = K$ and $\Phi|_K \neq \mathrm{Id}_M|_K$, $\Phi|_K$ is another parameterization of K.

Recall that the contact isotopies $\Phi^t \colon M \to M$ of a co-orientable contact manifold are in bijective correspondence with the time-dependent smooth functions, so-called contact Hamiltonians $H_t \colon M \to \mathbb{R}$. Note that this bijection depends on the choice of contact one-form α. Given a contact isotopy, we can recover the contact Hamiltonian by

$$\alpha(\dot{\Phi}^t) = H_t \circ \Phi^t.$$

Conversely, a contact Hamiltonian determines an isotopy via the equation

$$\alpha(\dot{\Phi}^t) = H_t \circ \Phi^t, \quad d\alpha(\dot{\Phi}^t, \cdot)|_{\ker \alpha} = -dH_t|_{\ker \alpha}$$

for each t. We say that H_t *generates* the contact isotopy Φ^t. The contact isotopy generated by $H_t \equiv 1$ is called the *Reeb flow of* α.

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Note that the *sign* of the contact Hamiltonian at a point only depends on the choice of co-orientation. We say that a (parameterized) submanifold $A \subset (M, \xi)$ *admits a positive loop* if there exists a contact isotopy Φ^t for which $\Phi^1(A) = A$ (respectively, $\Phi^1 \circ \mathrm{Id}_A = \mathrm{Id}_A$) for which the generating contact Hamiltonian H_t satisfies $H_t(x) > 0$ for all $x \in \Phi^t(A)$ and t. We similarly say that A admits a *somewhere positive non-negative loop* if $H_t(x) \geqslant 0$ for all $x \in \Phi^t(A)$ and t, where the latter inequality, moreover, is strict for some point x and time t. Note that a contact Hamiltonian that vanishes along a properly embedded Legendrian submanifold induces a flow that fixes that Legendrian. Furthermore, any somewhere positive non-negative loop of a closed Legendrian can be generated by a contact Hamiltonian that is non-negative on the entire ambient manifold M.

In this paper all contactomorphisms, homeomorphisms, and isotopies are implicitly assumed to have support contained inside some fixed compact subset, even though the ambient contact manifold M and the connected properly embedded submanifolds Λ, K sometimes need not be compact.

2. Statements of results

In this section, we describe a number of results contrasting flexibility and rigidity, for Legendrians (loose or not) and non-Legendrians: C^0-limits, positive loops, and pseudo-metrics.

2.1 C^0-topology

We start with our main result, which is about Legendrians under homeomorphisms that are C^0-limits of contactomorphisms.

Theorem A. *Consider a sequence*

$$\Phi_k \colon (M, \xi) \xrightarrow{\cong} (M, \xi)$$

of contactomorphisms supported in a fixed compact set, and let $\Lambda \subset M$ be a properly embedded Legendrian. If $\Phi_k \xrightarrow{C^0} \Phi_\infty$, where Φ_∞ is a homeomorphism, and if $\Phi_\infty(\Lambda)$ is smooth, then $\Phi_\infty(\Lambda)$ is also Legendrian.

Partial results of Theorem A have appeared elsewhere: Nakamura assumed that there was a uniform lower bound on the length of Reeb chords, as well as some small technical conditions [Nak20a, Theorem 3.4]; Rosen and Zhang assumed C^0-convergence for the smooth $f_k : M \to \mathbb{R}$ defined by $\Phi_k^* \alpha = e^{f_k} \alpha$ (often called conformal factors) [RZ20, Theorem 1.4]; Usher relaxed Rosen and Zhang's hypothesis to certain lower bounds on the conformal factors [Ush21, Theorem 1.2]; we proved the general case in dimension 3 [DRS21, Theorem D]; and Stokić made no assumptions in [Sto22, Proposition 6.1], but concluded the limiting submanifold could not be nearly Reeb invariant [Sto22, Definition 1.3]. Stokić showed that not being nearly Reeb invariant implies being Legendrian in the case when $\dim(\Lambda) = 1$ (in higher dimensions we do not know if the analogous result is true). Some of these results assumed the Legendrians were compact.

In [DRS21, Theorem D] we proved that Λ and $\Phi_\infty(\Lambda)$ are contactomorphic Legendrians when $\dim(\Lambda) = 1$. This equivalence, and many other weaker connections, are still unknown for $\dim(\Lambda) > 1$. For example, if $\Phi_\infty(\Lambda)$ is loose, then need Λ be loose as well? Only the following theorem is known to us.

Theorem 2.1. *Consider the set-up as in Theorem A. Assume Λ is closed,* $\dim(\Lambda) > 1$ *and* $k \gg 0$.

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(i) *There exists a standard one-jet neighborhood $\Lambda \subset U$ such that the Legendrian $\Phi_\infty(\Lambda)$ is contained in the one-jet neighborhood $\Phi_k(U)$. Furthermore, inside the one-jet neighborhood $\Phi_k(U)$, the Legendrian $\Phi_k(\Lambda)$ can be squeezed into a one-jet neighborhood of $\Phi_\infty(\Lambda)$ and $\Phi_\infty(\Lambda)$ can be squeezed into a one-jet neighborhood of $\Phi_k(\Lambda)$, in the sense of [DRS20, § 1.2].*

(ii) (a) *If Λ and $\Phi_\infty(\Lambda)$ are diffeomorphic, then $\Phi_k(\Lambda)$ and $\Phi_\infty(\Lambda)$ are smoothly isotopic inside $\Phi_k(U)$.*

(b) *$\Phi_\infty(\Lambda)$ is not loose inside the one-jet neighborhood $\Phi_k(U)$.*

(iii) *Suppose some stabilization $(M \times T^*X, \ker\{\alpha + p\,dq\})$ of M admits an open contact embedding into $J^1\mathbb{R}^N$ and the Legendrian $\Lambda \times \mathbf{0}_{T^*X}$ admits an augmentation for its Chekanov–Eliashberg differential graded algebra as a Legendrian in $J^1\mathbb{R}^N$. Then $\Phi_\infty(\Lambda)$ is not loose in M. (See [DRS20, § 1] for a review of Chekanov–Eliashberg differential graded algebras and augmentations in this context.)*

Remark 2.2. The assumption in Theorems A and 2.1 that Φ_∞ is a homeomorphism can be dropped if we instead assume $\Phi_\infty(U)$ is a neighborhood of $\Phi_\infty(\Lambda)$ for some standard Legendrian one-jet neighborhood $\Lambda \subset U$. See the proof of Theorem A and then use Theorem 2.1(i) in the rest of the proof of Theorem 2.1.

2.2 Positive loops

The strategy of the proof of Theorem A is inspired by Stokić's proof of [Sto22, Proposition 6.1]. However, instead of producing a Reeb invariant neighborhood of arbitrary non-Legendrians (it is unclear if they always exist), we show that non-Legendrians admit positive loops in Theorem C. We then allude to the classical theorem of non-existence of C^0-small positive loops of Legendrians proven by Colin, Ferrand and Pushkar [CFP17] (see Theorem 2.5).

In order to produce small positive loops of any non-Legendrian, we first prove the following flexibility when it comes to the choice of contact Hamiltonian for a contact isotopy of any non-Legendrian submanifold.

Theorem B. *Let $K \subset (M^{2n+1}, \xi)$ be a properly embedded non-Legendrian and $\Phi^t \colon M \to M$ be a contact isotopy. There exists a contact isotopy $\Psi^t \colon M \to M$ such that the following statements hold.*

- *Ψ^t is generated by a contact Hamiltonian H_t that vanishes when restricted to $\Psi^t(K)$.*
- *$\Psi^t(K)$ is contained inside an ϵ-neighborhood of $\Phi^t(K)$ for an arbitrary choice of $\epsilon > 0$ and all $t \in [0, 1]$.*
- *Ψ^1 and Φ^1 agree in a small neighborhood of K.*

In contrast, a contact Hamiltonian that vanishes along a properly embedded Legendrian submanifold generates a contact isotopy that fixes the Legendrian setwise.

We continue by establishing some consequences of Theorem B, starting with the existence of positive loops.

By Theorem B there is a flexibility in the choice of contact Hamiltonian for a contact isotopy of a non-Legendrian submanifold; namely, it shows that we can C^0-deform the isotopy to one whose generating contact Hamiltonian vanishes along the image of the submanifold. The following direct consequence shows that there also is flexibility for the behavior of contact isotopies that are positive along the Legendrian.

Theorem C. *Any closed non-Legendrian $K \subset (M^{2n+1}, \xi)$ sits in a positive loop. Equivalently, there exists a contact isotopy Φ^t with generating Hamiltonian H_t such that the following statements hold:*

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– $\Phi^0|_V = \Phi^1|_V = \mathrm{Id}_V$ *is satisfied in some neighborhood* V *of* K*;*
– H_t *is positive on the image of* K *under* Φ^t *for all* $t \in [0, 1]$.

Remark 2.3. With the ideas behind Theorems B and C, it should be possible to prove that any contact isotopy of a closed non-Legendrian can be C^0-perturbed to an isotopy for which the contact Hamiltonian is C^0-close to any arbitrary function.

Consider a closed Legendrian $\Lambda \subset M$ that is loose in the sense of Murphy [Mur12]. Since a non-Legendrian push-off K of Λ admits a contractible positive loop, and since Λ can be squashed onto K by a contact isotopy (which automatically preserves the positivity) in the sense of [DRS22], Theorem C gives a new proof of the following result by Liu.

COROLLARY 2.4 [Liu20]. *Any closed loose Legendrian admits a contractible positive loop.*

This flexibility of non-Legendrians and loose Legendrians stands in contrast to the rigidity of certain non-loose Legendrians. Colin, Ferrand and Pushkar used generating functions to prove the non-existence of positive loops for the zero-section in a one-jet space $0_N \subset (J^1N, \xi_{\mathrm{st}})$ of a closed manifold N in [CFP17, Theorem 1]; in the case of $N = S^n$ the result was obtained independently by Chernov and Nemirovski in [CN10b]. The latter authors generalized the result to non-negative isotopies of $0_N \subset (J^1N, \xi_{\mathrm{st}})$:

THEOREM 2.5 [Corollary 5.5, [CN10a]]. *The zero-section in the one-jet space* $0_N \subset (J^1N, \xi_{\mathrm{st}})$ *of a (not necessarily closed) manifold* N *does not admit a somewhere positive non-negative loop supported in a compact subset.*

The original sources assume N is compact, but their arguments apply to our set-up, since the loop is assumed to have compact support. Take a double of a large pre-compact open $X \subset N$ with smooth boundary such that $J^1(X)$ contains the support of the purported loop. There is an induced non-negative loop of the zero-section in the jet-space $J^1(X \sqcup_{\partial X} X)$ of the double of X, i.e. the manifold obtained by gluing X to itself along its boundary.

2.3 The Chekanov–Hofer–Shelukhin pseudo-norm

Fix a properly embedded submanifold K and consider its orbit space under the action of $\mathrm{Cont}_0(M, \xi)$, the identity component of the space of contactomorphisms. The (unparameterized) *Chekanov–Hofer–Shelukhin pseudo-metric* on this orbit space is defined via

$$\delta_\alpha^{\mathrm{unp}}(K_0, K_1) \coloneqq \inf\{\|\Phi^1\|_\alpha; \Phi^t \in \mathrm{Cont}_0(M, \xi), \Phi^1(K_0) = K_1\},$$

where

$$\|\Phi^1\|_\alpha = \inf_{\Phi_H^1 = \Phi^1} \int_0^1 \max_{x \in M} |H_t(x)|\, dt$$

is the Shelukhin–Hofer norm on $\mathrm{Cont}_0(M, \xi)$ [She17].

We define the *parameterized Chekanov–Hofer–Shelukhin pseudo-metric* on the orbit space of parameterized embeddings $\phi_i : K \hookrightarrow M$ by

$$\delta_\alpha(\phi_0, \phi_1) \coloneqq \inf\{\|\Phi^1\|_\alpha;\ \Phi^1 \in \mathrm{Cont}_0(M, \xi),\ \Phi^1 \circ \phi_0 = \phi_1\}.$$

Given any parameterized submanifold $\phi : K \hookrightarrow M$, we get an induced pseudo-metric on $\mathrm{Cont}_0(M, \xi)$ by setting

$$\delta_{\alpha,\phi}(\Phi_0, \Phi_1) \coloneqq \delta_\alpha(\Phi_0 \circ \phi, \Phi_1 \circ \phi).$$

We typically consider this pseudo-metric defined by a choice of submanifold $K \subset M$ with the canonical parameterization $\phi = \mathrm{Id}_M|_K : K \hookrightarrow M$. Rosen and Zhang showed that the unparameterized pseudo-metric $\delta_\alpha^{\mathrm{unp}}$ identically vanishes on the orbit space of any closed non-Legendrian

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submanifold [RZ20, Theorem 1.10]. (This was independently proved later in Nakamura's MS thesis [Nak20b, Corollary D.16].) In contrast, $\delta_\alpha^{\mathrm{unp}}$ is non-degenerate on the orbit space of any Legendrian submanifold. The non-degeneracy of $\delta_\alpha^{\mathrm{unp}}$ for Legendrians was first proved by Usher [Ush21, Corollary 3.5] when there are no contractible Reeb orbits or relatively contractible Reeb chords, and then by Hedicke [Hed21, Theorem 5.2] when the Legendrian does not sit in a positive loop. We then proved the non-degeneracy of $\delta_\alpha^{\mathrm{unp}}$ for arbitrary closed Legendrians of closed contact manifolds in [DRS21, Theorem 1.5]. The analogous results for the parameterized Chekanov–Hofer–Shelukhin pseudo-metric δ_α follow readily from Theorem B.

Corollary 2.6. *The parameterized Chekanov–Hofer–Shelukhin pseudo-metric δ_α vanishes identically for any non-Legendrian K. For any Legendrian, δ_α is degenerate.*

Remark 2.7. When the Legendrian is closed, [DRS21, Theorem 1.5] implies that the degenerate δ_α does not vanish identically. But to apply [DRS21, Theorem 1.5], we need the contact manifold M to either be closed, or to have a codimension-0 contact embedding into a closed contact manifold $\tilde{M}$. In this latter case, moreover, we require the contact form α of M to be a restriction of a contact form $\tilde{\alpha}$ for $\tilde{M}$.

3. Proofs of results

3.1 Basic results for contact Hamiltonians

We start with some preliminary standard computations for contact Hamiltonians that will be useful. In the following we fix a contact form α on M for the correspondence between contact Hamiltonians and contact isotopies.

Lemma 3.1. *If $\Phi_i^t\colon M \to M$, $i = 0, 1$, are contact isotopies generated by time-dependent contact Hamiltonians $H_t^i\colon M \to \mathbb{R}$ then $\Phi_0^t \circ \Phi_1^t$ is a contact isotopy that is generated by*

$$G_t = H_t^0 + e^{f_t \circ (\Phi_0^t)^{-1}} H_t^1 \circ (\Phi_0^t)^{-1}$$

where the time-dependent function $f_t\colon M \to \mathbb{R}$ is determined by $(\Phi_0^t)^*\alpha = e^{f_t}\alpha$. In particular, $(\Phi_1^t)^{-1}$ is generated by $-e^{f_t \circ \Phi_1^t} H_t^1 \circ \Phi_1^t$ (which can be seen by setting $\Phi_0^t := (\Phi_1^t)^{-1}$).

Here and throughout, composition occurs at each t. For example, $\Phi_0^t \circ \Phi_1^t$ is an isotopy with the same time parameter as Φ_0^t and Φ_1^t.

Proof. The chain rule implies

$$\begin{aligned} G_t(\Phi_0^t \circ \Phi_1^t) &= \alpha\left(\frac{d}{dt}(\Phi_0^t \circ \Phi_1^t)\right) = \alpha\left(\left(\frac{d}{dt}\Phi_0^t\right)(\Phi_1^t) + D\Phi_0^t \circ \left(\frac{d}{dt}\Phi_1^t\right)\right) \\ &= H_t^0(\Phi_0^t \circ \Phi_1^t) + e^{f_t \circ \Phi_1^t} H_t^1 \circ \Phi_1^t. \end{aligned}$$

□

Lemma 3.2. *If $\Phi^t\colon M \to M$ is a contact isotopy generated by a time-dependent contact Hamiltonian $H_t\colon M \to \mathbb{R}$, then $(\Phi^1)^{-1} \circ \Phi^{1-t}$ is a contact isotopy generated by $G_t = -e^{f \circ \Phi^1} H_{1-t} \circ \Phi^1$ where the smooth function $f\colon M \to \mathbb{R}$ is determined by $((\Phi^1)^{-1})^*\alpha = e^f\alpha$. In particular, if H_t vanishes along the image of K under Φ^t, then G_t vanishes along the image of K under $(\Phi^1)^{-1} \circ \Phi^{1-t}$.*

Proof.

$$\begin{aligned} G_t((\Phi^1)^{-1} \circ \Phi^{1-t}) &= \alpha\left(\frac{d}{dt}((\Phi^1)^{-1} \circ \Phi^{1-t})\right) = \alpha\left(D(\Phi^1)^{-1}\left(\frac{d}{dt}\Phi^{1-t}\right)\right) \\ &= -e^{f \circ \Phi^{1-t}} H_{1-t}(\Phi^{1-t}). \end{aligned}$$

□

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Lemma 3.3. *Let $\Psi \in \mathrm{Cont}(M,\xi)$ be a contactomorphism not necessarily contact isotopic to the identity. If $\Phi^t\colon M \to M$ is a contact isotopy generated by a time-dependent contact Hamiltonian $H_t\colon M \to \mathbb{R}$, then $\Psi \circ \Phi^t \circ \Psi^{-1}$ is a contact isotopy generated by $G_t = e^{f\circ\Psi^{-1}} H_t \circ \Psi^{-1}$ where $\Psi^*\alpha = e^f\alpha$. In particular, if H_t vanishes along the image of K under Φ^t, then G_t vanishes along the image of $\Psi(K)$ under $\Psi \circ \Phi^t \circ \Psi^{-1}$.*

Proof.

$$G_t(\Psi \circ \Phi^t \circ \Psi^{-1}) = \alpha\left(\frac{d}{dt}\big(\Psi \circ \Phi^t \circ \Psi^{-1}\big)\right) = \alpha\left(D\Psi\left(\frac{d}{dt}\Phi^t\right)\big(\Psi^{-1}\big)\right)$$
$$= e^{f\circ\Phi^t\circ\Psi^{-1}} H_t\big(\Phi^t \circ \Psi^{-1}\big). \qquad \square$$

3.2 Proof of Theorem B

Usher proved in [Ush15, Corollary 2.7] that the 'rigid locus' of a half-dimensional non-Lagrangian submanifold of a symplectic manifold is empty. This was later generalized to the contact setting by Rosen and Zhang in [RZ20] and independently by Nakamura [Nak20a]. This means, in particular, that the unparameterized Hofer–Chekanov–Shelukhin pseudo-norm vanishes when restricted to non-Legendrians, i.e. that two contact isotopic non-Legendrians are contact isotopic via contact Hamiltonians of arbitrarily small norm. Our strategy here is to translate the proofs in the aforementioned works to yield a more direct construction of the deformed contact isotopy generated by a small contact Hamiltonian. This leads to Theorem B, which sharpens the result from [RZ20] in the following two ways.

- The deformed contact isotopy can be assumed to be generated by a contact Hamiltonian that *vanishes* along the image of the non-Legendrian (as opposed to just being arbitrarily small there).
- The time-one map of the deformed contact isotopy can be assumed to induce the same parametrization as the original one, when restricted to the non-Legendrian.

The latter property can be rephrased as saying that the parameterized version of the Hofer–Chekanov–Shelukhin pseudo-norm vanishes when restricted to non-Legendrian submanifolds; see Corollary 2.6.

We first simplify the problem to the case when Φ^t is C^∞-small.

Lemma 3.4. *Fix $\epsilon > 0$. If Theorem B holds for any contact isotopy whose C^∞-norm is bounded by ϵ, then Theorem B holds for any contact isotopy.*

Proof. This follows since (a finite number of) concatenated isotopies preserve the three properties of Theorem B. Note that concatenation here is not a composition of maps at each time t, as it was in § 3.1. $\square$

By Banyaga's fragmentation result [Ban97, p. 148] (see also Rybicki [Ryb10]), the concatenation preservation of the three properties of Theorem B enables us to assume, when proving Theorem B, that Φ^t is not only C^∞-small, but also supported in a small neighborhood of some pt $\in K$. (If the small support of Φ^t does not intersect K, the proof is trivial as we set $\Psi^t := \mathrm{Id}$ in Theorem B.)

Lemma 3.5. *Consider the given non-Legendrian K, contact isotopy Φ^t, and constant ϵ from Theorem B. Further, assume that Φ^t is supported in a neighborhood $U \subset M$ that is displaced from a neighborhood $V \subset M$ of the non-Legendrian $K \subset M$ by a contact isotopy $\widetilde{\Phi}^t$. If $\operatorname{supp}\widetilde{\Phi}^t \supset U$ is contained inside an ϵ-neighborhood of K and if $\widetilde{\Phi}^t$ is generated by a Hamiltonian which vanishes on $\widetilde{\Phi}^t(K)$, then Theorem B holds for this K, Φ^t, ϵ.*

Proof. By Lemma 3.2 the contact isotopy $(\widetilde{\Phi}^1)^{-1} \circ \widetilde{\Phi}^{1-t}$ satisfies the property that its generating Hamiltonian vanishes along the image of K under the isotopy. Moreover, the time-one map of this contact isotopy displaces V from U (because $\widetilde{\Phi}^1(U) \cap V = \emptyset$ implies $(\widetilde{\Phi}^1)^{-1} \circ \widetilde{\Phi}^{1-1}(V) \cap U = \emptyset$).

The contact isotopy Ψ^t is constructed by concatenating (locally) the contact isotopy $(\widetilde{\Phi}^1)^{-1} \circ \widetilde{\Phi}^{1-t}$ that displaces V from U, with the contact isotopy

$$\Phi^1 \circ (\widetilde{\Phi}^1)^{-1} \circ \widetilde{\Phi}^t \circ \widetilde{\Phi}^1 \circ (\Phi^1)^{-1} = \left(\Phi^1 \circ (\widetilde{\Phi}^1)^{-1}\right) \circ \widetilde{\Phi}^t \circ \left(\Phi^1 \circ (\widetilde{\Phi}^1)^{-1}\right)^{-1},$$

which is the conjugation of the isotopy $\widetilde{\Phi}^t$ with a contactomorphism $\Phi^1 \circ (\widetilde{\Phi}^1)^{-1}$ that might not be equal to the identity inside $U \supset \operatorname{supp} \Phi^1$ at $t = 0$. (We concatenate these two paths as in the proof of Lemma 3.4. Technically, Ψ^t is defined for $0 \leqslant t \leqslant 2$, but to simplify notation, we omit this needed reparameterization of t.)

Set Ψ and Φ^t, as used in the notation of Lemma 3.3, equal to $\Phi^1 \circ (\widetilde{\Phi}^1)^{-1}$ and $\widetilde{\Phi}^t$ as used to define the second contact isotopy in the preceding paragraph. Lemma 3.3 implies that this second contact isotopy is generated by a Hamiltonian which vanishes on the image of $\Phi^1 \circ (\widetilde{\Phi}^1)^{-1}(K) = (\widetilde{\Phi}^1)^{-1}(K)$ under this second isotopy. For this last equality, recall that $\Phi^1|_{(\widetilde{\Phi}^1)^{-1}(K)} = \mathrm{Id}|_{(\widetilde{\Phi}^1)^{-1}(K)}$ because Φ^t is supported in U which does not intersect $(\widetilde{\Phi}^1)^{-1}(V) \supset (\widetilde{\Phi}^1)^{-1}(K)$. Finally, we conclude that

$$\Psi^1|_V = \Phi^1 \circ (\widetilde{\Phi}^1)^{-1} \circ \widetilde{\Phi}^1 \circ \widetilde{\Phi}^1 \circ (\Phi^1)^{-1} \circ (\widetilde{\Phi}^1)^{-1} \circ \widetilde{\Phi}^0|_V = \Phi^1|_V$$

where in the last equality we use $(\Phi^1)^{-1}|_{(\widetilde{\Phi}^1)^{-1}(V)} = \mathrm{Id}|_{(\widetilde{\Phi}^1)^{-1}(V)}$ and $\widetilde{\Phi}^0 = \mathrm{Id}$. □

LEMMA 3.6. *Consider $\epsilon > 0$ and K from Theorem B. Fix $p \in K$. There exist a neighborhood $U \subset M$ of p, a neighborhood $V \subset M$ of K and a contact isotopy $\widetilde{\Phi}^t$ which displaces U from $V \cup U$, such that $\widetilde{\Phi}^t$ satisfies its assumptions in Lemma 3.5 (i.e.* $\operatorname{supp} \widetilde{\Phi}^t \supset U$ *is contained inside an ϵ-neighborhood of K and $\widetilde{\Phi}^t$ is generated by a Hamiltonian which vanishes on $\widetilde{\Phi}^t(K)$).*

Proof. Given any choice of neighborhood of K, the construction below can be carried out inside that neighborhood. This implies the desired property of the support of the contact isotopy that we now proceed to define.

1: the case when T_pK is not a Lagrangian subspace of ξ_p. This part of the argument is similar to [RZ20, Proposition 8.6].

The property that T_pK is not a Lagrangian subspace is equivalent to

$$(T_pK \cap \xi_p)^{d\alpha} \neq T_pK \cap \xi_p,$$

where $(T_pK \cap \xi_p)^{d\alpha} \subset \xi_p$ denotes the symplectic orthogonal (recall that $\dim T_pK \leqslant n$). Hence, we can find a non-zero vector

$$X_H \in (T_pK \cap \xi_p)^{d\alpha} \setminus T_pK \subset \xi_p \setminus T_pK$$

such that the one-form $\eta \coloneqq d\alpha(\cdot, X_H)$ on T_pM vanishes on T_pK. The one-form η extends to the exterior derivative dH of a function $H\colon M \to \mathbb{R}$ that can be taken to vanish on all of K.

Consider the contact isotopy $\widetilde{\Psi}^t$ generated by the autonomous contact Hamiltonian H. Since H vanishes on p we get $\dot{\Psi}^0(p) = X_H$. In view of Lemma 3.1, the inverse $\widetilde{\Phi}^t \coloneqq (\widetilde{\Psi}^t)^{-1}$ is generated by the non-autonomous contact Hamiltonian $G_t \coloneqq -e^{f_t \circ (\widetilde{\Phi}^t)^{-1}} H \circ (\widetilde{\Phi}^t)^{-1}$. In particular, $\widetilde{\Phi}^t$ is generated by a contact Hamiltonian that vanishes along the image of K under $\widetilde{\Phi}^t$. Finally, since the contact vector field $-X_H = X_{G_0}$ is normal to K at p, it follows that G_t generates a contact isotopy that displaces a small neighborhood $p \in U \subset M$ from $V \cup U$ for some small neighborhood V of K.

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2: the case when $T_pK \subset \xi_p$ is a Lagrangian subspace. Consider the closed subset $\mathcal{L}(K) \subset K$ of points for which $T_pK \subset \xi$ is Lagrangian (which of course is empty whenever $\dim K < n$).

2.1: the case when $p \in \operatorname{bd}\mathcal{L}(K)$. First, by a construction which is similar to the one above, for any $X \in T_pK \subset \xi$ we can construct a time-dependent contact Hamiltonian H that vanishes on the image of K under the generated isotopy Φ^t_X, and for which the corresponding contact vector field at time $t = 0$ satisfies $X_H(p) = X$. Since X is tangent to K, it is not necessarily the case that p is displaced by Φ^t_X for small $t \geqslant 0$.

If we can find some X such that $\Phi^\epsilon_X(p) \notin K$ for all small $\epsilon > 0$ then we are done. Assume not; then $\Phi^\epsilon_X(p) \in K$ for all X and for some $\epsilon \geqslant 0$. Since we are in the case $p \in \operatorname{bd}\mathcal{L}(K)$, the point p does not have a Legendrian neighborhood in K. We can thus find a direction X for which $\Phi^\epsilon_X(p)$ is contained inside $K \setminus \mathcal{L}(K)$. Note that, for this reason, K and $\Phi^\epsilon_X(K)$ are not tangent at $\Phi^\epsilon_X(p)$. Take a non-zero tangent vector $W \in T_{\Phi^\epsilon_X(p)}\Phi^\epsilon_X(K) \subset \xi_{\Phi^\epsilon_X(p)}$ that is normal to K. By the assumption above, we can find a contact isotopy of $\Phi^\epsilon_X(K)$ whose infinitesimal generator is equal to W at $\Phi^\epsilon_X(p)$, and such that the generating contact Hamiltonian vanishes along the image of $\Phi^\epsilon_X(K)$ under the isotopy.

The latter contact isotopy displaces $\Phi^\epsilon_X(p)$ from some small neighborhood V of K, and the concatenation of contact isotopies thus displaces a small neighborhood U of p from V as desired.

2.2: the case when $p \in \operatorname{int}\mathcal{L}(K)$. Finally, since K is connected, any point in the open Legendrian submanifold $\mathcal{L}(K) \setminus \operatorname{bd}\mathcal{L}(K)$ can be moved arbitrarily close to a point $p' \in \operatorname{bd}\mathcal{L}(K)$ by a contact isotopy that fixes $K \setminus (\mathcal{L}(K) \setminus \operatorname{bd}\mathcal{L}(K))$ pointwise and $\mathcal{L}(K) \setminus \operatorname{bd}\mathcal{L}(K)$ setwise. Note that the Hamiltonian of such a contact isotopy can be taken to vanish on all of K. (See [DRS20, § 1] and the proof of [Gei08, Theorem 2.6.2].) We then apply the contact isotopy from case 2.1 to p' displacing its small neighborhood U' from V. This also displaces a smaller neighborhood $U \subset U'$ of p from V as desired. $\square$

Remark 3.7. Global infinitesimal displaceability of non-Legendrians is an important ingredient in Nakamura's work [Nak20a]. The vanishing of the unparameterized Chekanov–Hofer–Shelukhin norm [RZ20] implies that any closed non-Legendrian has a displacement that can be realized by a contact Hamiltonian that is arbitrarily C^0-small. To that end we use the fact that, for any non-Legendrian K, a generic Reeb vector field is nowhere tangent to K. Hence, the Reeb flow is a contact isotopy that displaces the non-Legendrian K.

3.3 Proof of Corollary 2.6

The parameterized pseudo-metric δ_α is clearly degenerate on Legendrian submanifolds, since a Legendrian can be reparameterized by a contact Hamiltonian that vanishes along the Legendrian. See [DRS20, § 1] and the proof of [Gei08, Theorem 2.6.2]. It is non-vanishing because $\delta^{\mathrm{unp}}_\alpha$ is non-degenerate [DRS21, Theorem 1.5]. The pseudo-metric δ_α vanishes identically for any non-Legendrian because of the first and third bullet points of Theorem B.

3.4 Proof of Theorem C

Let $\rho : M \to [-1, 0]$ be a smooth compactly supported bump function such that $\rho|_U = -1$ for some 'large' (see below) neighborhood $U \supset K$. Apply Theorem B, setting K and Φ^t in Theorem B to be K in Theorem C and the flow induced by the autonomous contact Hamiltonian $\epsilon\rho$, respectively. Note that this flow is equal to the negative Reeb flow on U rescaled by $\epsilon > 0$, which is assumed to be small. Theorem B produces Ψ^t by constructing Ψ^t to be local to the image of the compact K under the negative Reeb flow. So without loss of generality, we can assume U

is sufficiently large such that $\operatorname{supp}(\Psi^t) \subset U$. We claim that $(\Phi^t)^{-1} \circ \Psi^t$ is the desired isotopy in Theorem C (which unfortunately is also called Φ^t in that theorem).

That $(\Phi^t)^{-1} \circ \Psi^t$ satisfies the first bullet point of Theorem C follows from the third bullet point of Theorem B.

Note that Φ^t is generated by a Hamiltonian which is negative on U, and thus negative on the support of Ψ^t. The second part of Lemma 3.1, setting $\Phi_1^t = \Phi^t$, implies that $(\Phi^t)^{-1}$ is generated by a Hamiltonian which is positive on U, and thus positive on the support of Ψ^t. The first part of Lemma 3.1, setting $\Phi_0^t = (\Phi^t)^{-1}$, $\Phi_1^t = \Psi^t$, combined with the first bullet point of Theorem B applied to Ψ^t, implies that $(\Phi^t)^{-1} \circ \Psi^t$ satisfies the second bullet point of Theorem C. (To see this, using the notation of Lemma 3.1, the Hamiltonian G_t which generates $\Phi_0^t \circ \Phi_1^t$, when restricted to $\Phi_0^t \circ \Phi_1^t(K)$, is a sum of the positive term $H_t^0|_{\Phi_0^t \circ \Phi_1^t(K)}$ and $e^{f_t \circ (\Phi_0^t)^{-1}} H_t^1 \circ (\Phi_0^t)^{-1}|_{\Phi_0^t \circ \Phi_1^t(K)}$. But the second term vanishes because $H_t^1|_{\Phi_1^t(K)} = 0$.)

3.5 Proof of Theorem A

First we show that the general case when K and Λ are properly embedded, but not necessarily closed, can be deduced from the statement in the case when the involved submanifolds are assumed to be closed.

Recall that the sequence Φ_k of contactomorphisms are all assumed to have support inside some fixed compact subset. Take a compact domain $U \subset M$ with smooth boundary that contains the support of all contactomorphisms Φ_k in the sequence, which in particular means that $\Lambda = K$ holds in a neighborhood of $\overline{M \setminus U}$. For a generic choice of domain U, we may further assume that the intersection

$$B \coloneqq \Lambda \cap \partial U = K \cap \partial U$$

is transverse, yielding a smooth submanifold $B \subset \Lambda$ of codimension one. After deforming the neighborhood U near B we may further assume that there is a neighborhood O of $\Lambda \cap U$ in U that is contactomorphic to $J^1(\Lambda \cap U)$, under which $j^1 0$ is identified with $\Lambda \cap U$.

Now produce an open contact manifold $\tilde{M}$ from $\operatorname{int} U$ in the following manner. Let $\tilde{\Lambda}$ denote the closed manifold obtained by gluing two disjoint copies of $\Lambda \cap U$ along its common boundary in the obvious manner. Clearly $J^1\tilde{\Lambda}$ contains $J^1(\Lambda \cap U)$ as a properly embedded submanifold with boundary. Hence, we can glue $J^1\tilde{\Lambda}$ to $\operatorname{int} U$, resulting in an open contact manifold $\tilde{M}$ in which Λ extends to a closed Legendrian $\tilde{\Lambda}$. Applying the statement to the closed Legendrian manifold $\tilde{\Lambda} \subset \tilde{M}$, we deduce that

$$\tilde{K} \coloneqq (K \cap U) \cup (\tilde{\Lambda} \setminus \operatorname{int} U)$$

is Legendrian, and hence so is the original submanifold K.

It remains to prove Theorem A when Λ and K are closed, which we do by contradiction. Suppose $K \subset M$ is a closed non-Legendrian submanifold that is the C^0-limit of the closed Legendrian submanifolds $\Phi_n(\Lambda)$, where Φ_n are contactomorphisms that C^0-converge to a homeomorphism: $\Phi_n \xrightarrow{C^0} \Phi_\infty$. Let Φ^t be the contact isotopy from Theorem C generated by H_t such that $H_t|_{\Phi^t(K)} > 0$. Take a sufficiently small neighborhood $U \supset K$ contained inside V provided by the theorem, so that $H_t|_{\Phi^t(U)} > 0$ as well as $\Phi^1|_U = \mathrm{Id}$ are satisfied. Since $\Phi_k(\Lambda) \subset U$ holds for all $k \gg 0$, we have produced a positive loop $\Phi^t|_{\Phi_k(\Lambda)}$ of a closed Legendrian submanifold. This contradicts Theorem 2.5. Hence, K is Legendrian.

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3.6 Proof of Theorem 2.1

(i) Since Φ_∞ is a homeomorphism, $\Phi_\infty(U)$ is a neighborhood of $\Phi_\infty(\Lambda)$. (Or see Remark 2.2.) The C^0-convergence implies that $\Phi_k(U)$ is a neighborhood of $\Phi_\infty(\Lambda)$ for $k \gg 0$. The C^0-convergence ensures that the fiberwise rescaling inside $\Phi_k(U)$ projects $\Phi_\infty(\Lambda)$ onto $\Phi_k(\Lambda)$ with degree 1, as required in this squeezing [DRS20, § 1.2]. Since $\Phi_k(\Lambda)$ also may be assumed to be contained inside a one-jet neighborhood of $\Phi_\infty(\Lambda)$, there is also a squeezing of $\Phi_k(\Lambda)$ into this one-jet neighborhood in the same sense (i.e. the fiberwise projection is of degree one).

(ii) (a) Recall that two maps from the same domain that are sufficiently C^0-close are homotopic; $\Phi_k|_\Lambda$ is thus homotopic to $\Phi_\infty|_\Lambda$ inside $\Phi_k(U)$ for $k \gg 0$. Diffeomorphic and homotopic implies smoothly isotopic in high dimensions [Hae63].

(b) To show that $\Phi_\infty(\Lambda)$ is not loose in the one-jet neighborhood $\Phi_k(\Lambda)$, we claim that the zero-section in $J^1\Lambda$ cannot be squeezed into the one-jet neighborhood of a loose Legendrian, while (1) provides such a squeezing. To see that the zero-section cannot be squeezed into the one-jet neighborhood of a loose Legendrian we argue as follows. After stabilizing the ambient contact manifold (M, α) to $(M \times T^*S^1, \alpha + p\,dq)$, the Legendrian Λ to $\Lambda \times \mathbf{0}_{T^*S^1}$, and the contactomorphisms Φ_k to $\Phi_k \times \mathrm{Id}_{T^*S^1}$, we may consider the case when $U \cong J^1(\Lambda \times S^1)$ since we can stabilize the squeezing. The result follows from Lemma 3.8.

(iii) This follows directly from [DRS20, Theorem 1.7]. (In [DRS20], the term 'stabilized' Legendrian is used in a completely different sense than $\Lambda \times \mathbf{0}_{T^*X}$ as above. When $\dim(\Lambda) = 1$, in a local front projection, a neighborhood of a point of Λ is replaced by a zig-zag. We use this zig-zag construction in the proof of Lemma 3.8 below. When $\dim(\Lambda) > 1$, stabilization is defined by a more general construction which Murphy proves equivalent to the existence of a loose chart [Mur12].)

Lemma 3.8. *The zero-section in $J^1(\Lambda \times S^1)$ cannot be squeezed into the one-jet neighborhood of a loose Legendrian submanifold.*

Proof. Let $\Pi : J^1\Lambda \to T^*\Lambda$ be the projection along the standard Reeb flow. A Legendrian $\Lambda' \subset J^1\Lambda$ is horizontally displaceable if there exists a contact isotopy that disjoins Λ' from its image under this Reeb flow, $\Pi^{-1}(\Pi(\Lambda'))$. This is an open condition in the sense that if Λ' is horizontally displaceable, then so too is any Legendrian that is sufficiently C^0-close to Λ'. The Rabinowitz Floer complex (e.g. see [DRS21, § 4]) of the zero-section $j^1(0)$ of $J^1\Lambda$ is not acyclic and its homology is invariant under contact isotopy. Since the complex is generated by Reeb chords between $j^1(0)$ and its image under the contact isotopy, this implies $j^1(0)$ is not horizontally displaceable.

Let $Z_{\mathrm{st}} \subset J^1S^1$ denote the stabilized zero-section in J^1S^1 whose front is given by a single zig-zag. This Legendrian is horizontally displaced when after the contact isotopy, the minimum magnitude of the slope of the zig-zag is greater than its initial zig-zag slope's maximum magnitude. It follows that the stabilized zero-section $j^1(0) \times Z_{\mathrm{st}} \subset J^1(\Lambda \times S^1)$ also admits a horizontal displacement.

We claim that by Murphy's h-principle [Mur12], any compact loose Legendrian $\Lambda_0 \subset J^1(\Lambda \times S^1)$ can be placed inside a one-jet neighborhood of $j^1(0) \times Z_{\mathrm{st}}$ by a contact isotopy. To see the isotopy, we construct a formal Legendrian isotopy between the loose Legendrian Λ_0 and a *formal* Legendrian Λ_1 contained in the neighborhood of $j^1(0) \times Z_{\mathrm{st}}$. The formal Legendrian isotopy is constructed by, first, fiber-scaling Λ_0 towards the zero-section. Then, since $Z_{\mathrm{st}} \subset J^1S^1$ is smoothly isotopic to j^10, it is now easy to see that there is a smooth isotopy $f_t : L \to J^1(\Lambda \times S^1)$

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such that $f_i(L) = \Lambda_i$ for $i = 0, 1$, where the smooth (not necessarily Legendrian) submanifold Λ_1 is contained inside the one-jet neighborhood of $j^1 0 \times Z_{\mathrm{st}}$. The formal isotopy $(g, G) : [0,1]_t \times [0,1]_s \times L \to \big(J^1(\Lambda \times S^1), T(J^1(\Lambda \times S^1))\big)$ is defined as follows. Set $g_{t,s} = f_t$, $G_{t,0} = df_t$, $G_{0,s} = df_0$. Then use homotopy lifting to extend $G_{t,s}$ as a full-rank bundle map whose image is a Lagrangian in the contact planes for all $t \in [0,1]$ and $s = 1$. The 0-parametric version of Murphy's h-principle produces an actual loose Legendrian Λ_1' in an arbitrarily small neighborhood of Λ_1. Since Λ_1' is formally Legendrian isotopic to Λ_0 by construction, the 1-parametric version of Murphy's h-principle produces the Legendrian isotopy that takes Λ_0 into the neighborhood, as desired.

Hence the loose Legendrians are all horizontally displaceable as well. If the zero-section of $J^1(\Lambda \times S^1)$ can be squeezed into the one-jet neighborhood of some loose Legendrian, then by fiber-scaling it can be squeezed into an arbitrarily small one-jet neighborhood of the loose Legendrian. So the zero-section is horizontally displaceable, contradicting its Rabinowitz Floer calculation. □

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Conflicts of interest

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