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A COMPLETE CLASSIFICATION OF FINITE HOMOGENEOUS GROUPS

CAI HENG LI

In this short note, we obtain a complete classification of finite homogeneous groups.

A group G is called a *homogeneous group* if each isomorphism between any two isomorphic subgroups of G extends to an automorphism of G. A classification of finite soluble homogeneous groups was published in [**3**]. The purpose of this note is to give a complete classification of arbitrary finite homogeneous groups. A group is called a *homocyclic group* if it is a direct product of cyclic subgroups of the same order. For two groups G and H, denote by $G \rtimes H$ a semidirect product of G by H. We use $\mathrm{Q}(64)$ to denote a Sylow 2-subgroup of $\mathrm{PSU}(3,16)$, which is a Suzuki 2-group of order 64. The main result of this paper is the following theorem.

MAIN THEOREM. *A finite group G is homogeneous if and only if $G = U \times V$ such that $\big(|U|, |V|\big) = 1$, U is Abelian with all Sylow subgroups homocyclic, and either $V = 1$, or V is one of the following:*

(1) *$W \rtimes \mathbb{Z}_{2^n}$, where W is Abelian of odd order with all Sylow subgroups homocyclic, and $\mathbb{Z}_{2^n}$ inverses all elements of W;*

(2) *Q_8 and $\mathrm{Q}(64)$;*

(3) *A_4, $\mathrm{Q}_8 \rtimes \mathbb{Z}_3$, $\mathbb{Z}_3^2 \rtimes \mathrm{Q}_8$ and $\mathrm{Q}(64) \rtimes \mathbb{Z}_3$;*

(4) *$\mathrm{L}_2(5)$, $\mathrm{L}_2(7)$, $\mathrm{SL}_2(5)$ and $\mathrm{SL}_2(7)$.*

One of the motivations for the study of homogeneous groups comes from model theory, that is, a complete first-order theory that admits quantifier elimination has a homogeneous model. In particular, a finite structure is homogeneous if and only if it admits quantifier elimination, see [**2, 3**]. This would be the principal motivation for the work of Cherlin and Felgner in [**2, 3**].

However, the motivation for the present work comes from graph theory, namely a problem about isomorphisms of Cayley graphs, see [**5, 8**] for references. In the study of this problem, we need to know the homogeneity of some groups, see [**5**]. Although it was claimed in [**3**] that a classification of finite homogeneous groups had been obtained,

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such a classification has never been published so far. Because of the demand for such a classification and the briefness of the argument, we in this short note give a complete classification of finite homogeneous groups.

Some properties of finite groups similar to homogeneity have been investigated, for example, [**4**] proved that a group of which all elements of the same order are conjugate is isomorphic to S_2 or S_3; [**10**] gave a description for finite groups G such that all elements of the same order are conjugate in $\mathrm{Aut}(G)$; Praeger and the author in [**6, 7**] obtained a description for finite groups G in which any two elements of the same order are conjugate or inverse-conjugate in $\mathrm{Aut}(G)$; Stroth [**9**] classified finite groups in which all isomorphic subgroups are conjugate.

The rest of the note is devoted to giving a short proof of the Main Theorem.

PROOF OF THE MAIN THEOREM: Assume that G is a finite homogeneous group. If G is soluble, then by [**3**], G is on the list in the theorem. Thus we assume that G is insoluble. Since G is homogeneous, all elements of the same order are conjugate in $\mathrm{Aut}(G)$. Then by [**8**, Corollary 2.4] and [**7**, Corollary 1.3 (4)], $G = U \times V$ such that $\big(|U|, |V|\big) = 1$, U is Abelian of odd order with all Sylow subgroups homocyclic, and V is one of $L_2(5)$, $L_2(7)$, $L_2(8)$, $L_2(9)$, $L_3(4)$, $SL_2(5)$, $SL_2(7)$ and $SL_2(9)$. Thus we need to prove that none of $L_2(8)$, $L_2(9)$, $L_3(4)$ and $SL_2(9)$ is homogeneous.

Assume that V is one of $L_2(8)$, $L_2(9)$, $SL_2(9)$ and $L_3(4)$. Let K be a subgroup of V. Suppose that V is homogeneous. Then by the definition, all automorphisms of K extend to automorphisms of V. Suppose further that K is elementary Abelian, and let Ω be the set of minimal generating subsets for K. Then it follows since V is homogeneous that $\mathrm{N}_{\mathrm{Aut}(V)}(K)$ is transitive on Ω and so induces a transitive permutation group P on Ω. Clearly, $P \cong \mathrm{N}_{\mathrm{Aut}(V)}(K)/\mathrm{C}_{\mathrm{Aut}(V)}(K)$.

Suppose first that $V = L_2(8)$ is homogeneous. Then by [**1**], a Sylow 2-subgroup K of V is isomorphic to $\mathbb{Z}_2^3$, and $P \cong \mathrm{N}_{\mathrm{Aut}(V)}(K)/\mathrm{C}_{\mathrm{Aut}(V)}(K) \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$. Since every 3 non-identity elements of K generate a subgroup isomorphic to $\mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$, it easily follows that $|\Omega| = \binom{7}{3} - \binom{7}{2} = 14$. Thus $|\Omega|$ does not divide $|P|$, and so P is not transitive on Ω, which is a contradiction. Therefore, $L_2(8)$ is not homogeneous.

Suppose next that $V = L_2(9)$ is homogeneous. By the Atlas [**1**], a Sylow 3-subgroup K of V is isomorphic to $\mathbb{Z}_3^2$, and $P \cong \mathrm{N}_{\mathrm{Aut}(V)}(K)/\mathrm{C}_{\mathrm{Aut}(V)}(K)$ is a group of order 16. Now K has 8 non-identity elements and any two of them generate a subgroup isomorphic to $\mathbb{Z}_3$ or $\mathbb{Z}_3^2$. It follows that $|\Omega| = \binom{8}{2} - \binom{8}{1} = 20$. Thus $|\Omega|$ does not divide $|P|$, and so P is not transitive on Ω, which is a contradiction. So $L_2(9)$ is not homogeneous. The same argument shows that $SL_2(9)$ is not homogeneous.

Suppose finally that $V = L_3(4)$ is homogeneous. By the Atlas [**1**], V has a maximal subgroup $M = K \rtimes H$ such that $K \cong \mathbb{Z}_2^4$ and $H \cong A_5$. We notice that any subset of 4 non-identity elements of K generates a subgroup isomorphic to $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$ or $\mathbb{Z}_2^4$. It is easily shown that $|\Omega| = \binom{15}{4} - \binom{15}{3} + \binom{15}{2}$ and is divisible by 29. On the other hand,

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as V is homogeneous, all minimal generating subsets of K are conjugate in $\mathrm{Aut}(V)$, which is a contradiction since $\mathrm{Aut}(V)$ is of order coprime to 29. Therefore, $\mathrm{L}_3(4)$ is not homogeneous.

Conversely, we need to prove that all groups listed in the theorem are homogeneous. By [**3**, Proposition 8], U is homogeneous, and by [**5**, Lemma 3.1], if V is homogeneous then $G = U \times V$ is homogeneous. Thus we need to verify that V is homogeneous. By [**3**, Proposition 8], if V is soluble then V is homogeneous. Hence we only need to show that the groups listed in item (3), namely $\mathrm{L}_2(5)$, $\mathrm{L}_2(7)$, $\mathrm{SL}_2(5)$ and $\mathrm{SL}_2(7)$, is homogeneous.

(1) Assume that $V = \mathrm{L}_2(p)$, where $p = 5$ or 7.

Let $K, L < V$ be such that $K \cong L$, and let σ be an isomorphism from K to L. From the information given in the Atlas [**1**], we easily conclude that all elements of V of the same order are conjugate in $\mathrm{Aut}(V)$. It follows that if K, L are cyclic then σ extends to an automorphism of V. Thus we suppose that K is not cyclic. Then $K \cong \mathbb{Z}_2^2$, D_6, D_8, D_{10}, A_4, S_4, or $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$. Again by the Atlas [**1**], it is easily shown that all isomorphic subgroups of V are conjugate in $\mathrm{Aut}(V)$. Thus we may assume that $K = L$, and so σ is an automorphism of K. By [**1**], $\mathrm{Aut}(K) \cong \mathrm{N}_{\mathrm{Aut}(V)}(K) < \mathrm{Aut}(V)$. Therefore, σ extends to an automorphism of V, and so $\mathrm{L}_2(p)$ for $p \in \{5, 7\}$ is homogeneous.

(2) Assume that $V = \mathrm{SL}_2(p)$, where $p = 5$ or 7.

Let L be a subgroup of V. Then $L \cong L_0$ or $\mathbb{Z}_2.L_0$, where L_0 is a subgroup of $\mathrm{L}_2(p)$. Since all isomorphic subgroups of $\mathrm{L}_2(p)$ are conjugate in $\mathrm{Aut}(\mathrm{L}_2(p))$, it follows that all isomorphic subgroups of V are conjugate in $\mathrm{Aut}(V)$. Thus we only need to prove that each automorphism of L extends to an automorphism of V. This is clearly true if $L \not\cong \mathbb{Z}_4, \mathbb{Z}_8$ as $\mathrm{L}_2(p)$ is homogeneous. Further, an element of V of order 4 is conjugate to its inverse. Suppose that $L \cong \mathbb{Z}_8$. Then each element of $\mathrm{Aut}(L)$ is induced by an element of $\mathrm{N}_{\mathrm{Aut}(V)}(L)$. Thus each element of $\mathrm{Aut}(L)$ extends to an automorphism of V. So V is homogeneous.

This completes the proof of the theorem. □

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Department of Mathematics
University of Western Australia
Perth, 6907 WA
Australia