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ON SOME CLASSES OF PRIMARY BANACH SPACES

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Introduction. A Banach space X is called *primary* (respectively, *prime*) if for every (bounded linear) projection P on X either PX or $(I - P)X$ (respectively, PX with dim $PX = \infty$) is isomorphic to X. It is well-known that c_0 and l_p, $1 \leqq p \leqq \infty$ [**8**; **14**] are prime. However, it is unknown whether there are other prime Banach spaces. For a discussion on prime and primary Banach spaces, we refer the reader to [**9**].

If E is a Banach sequence space and $\{X_n\}$ is a sequence of Banach spaces, we shall let $(\sum_n X_n)_E = (X_1 \oplus X_2 \oplus \ldots)_E$ be the Banach space of all sequences $\{x_n\}$ such that $x_n \in X_n$, $n = 1, 2, \ldots$ and $(\|x_1\|, \|x_2\|, \ldots) \in E$ with the norm $\|\{x_n\}\| = \|(\|x_1\|, \|x_2\|, \ldots)\|_E$. It is known that $C[0, 1]$ [**10**] and $L^p[0, 1]$, $1 < p < \infty$ [**2**] are primary. Other known classes of primary Banach spaces are the $\mathscr{L}_p$-spaces $(X_p \oplus X_p \oplus \ldots)_{l_p}$, $(l_2 \oplus l_2 \oplus \ldots)_{l_p}$ and B_p, $1 < p < \infty$ [**2**] and the spaces $C[1, \alpha]$ where α is a countable ordinal or the first uncountable ordinal [**1**; **20**]. Let X be a Banach space with symmetric basis $\{x_n\}$ and let X_n be the linear span of $\{x_1, x_2, \ldots, x_n\}$, $n = 1, 2, \ldots$. In this paper, we show that the following Banach spaces are primary:

(1) $(X \oplus X \oplus \ldots)_E$, $E = l_p$, $1 < p < \infty$ or c_0 where X is not isomorphic to l_1;

(2) $(X_1 \oplus X_2 \oplus \ldots)_E$, $E = l_p$, $1 < p < \infty$ or c_0;

(3) $(l_\infty \oplus l_\infty \oplus \ldots)_{l_p}$, $1 \leqq p < \infty$.

We shall follow the standard notation and terminology in the theory of Banach spaces [**12**]. In particular, for Banach spaces X and Y we write $X \sim Y$ if X is isomorphic to Y and $d(X, Y) = \inf \{\|T\| \cdot \|T^{-1}\|: T$ is an isomorphism from X onto $Y\}$. For a sequence of elements $\{x_n\}$ in a Banach space X, we write $[x_n]$ or $[x_1, x_2, \ldots]$ to denote the closed linear subspace in X spanned by $\{x_n\}$. For the notation on basis theory, we refer the reader to [**19**]. Throughout this paper, if X is a Banach space with symmetric basis, we shall assume that X is equipped with the associated symmetric norm (cf. [**19**]).

1. In this section, we prove that if X is a Banach space with symmetric basis which is not isomorphic to l_1 then the spaces $(X \oplus X \oplus \ldots)_E$, $E = l_p$, $1 < p < \infty$ or c_0 are primary.

PROPOSITION 1. *Let X be a Banach space with symmetric basis $\{x_n\}$ and let Y*

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be any Banach space. If P is any projection on Y, then

$$(Y \oplus Y \oplus \ldots)_X \sim (PY \oplus PY \oplus \ldots)_X \oplus ((I - P)Y \oplus (I - P)Y \oplus \ldots)_X.$$

Proof. For any element $(y_1, y_2, \ldots)$ in $(Y \oplus Y \oplus \ldots)_X$, since $\|y_n\| \leqq \|Py_n\| + \|(I - P)y_n\|$, $n = 1, 2, \ldots$, we have

$$\left\| \sum_n \|y_n\| x_n \right\| \leqq \left\| \sum_n (\|P\| \cdot \|y_n\| + \|I - P\| \cdot \|y_n\|) x_n \right\|$$
$$\leqq \left\| \sum_n \|P\| \cdot \|y_n\| x_n \right\| + \left\| \sum_n \|I - P\| \cdot \|y_n\| x_n \right\|$$
$$= (\|P\| + \|I - P\|) \left\| \sum_n \|y_n\| x_n \right\|.$$

This completes the proof of the proposition.

LEMMA 2. *Let $\{x_n, x_n^*\}$ be an unconditional basis of a Banach space X. Then no subsequence of $\{x_n\}$ spans a subspace isomorphic to l_1 if and only if* $\lim_n x_k^*(Tx_n) = 0$, $k = 1, 2, \ldots$, *for any operator T on X.*

Proof. For the necessity, see the proof of the theorem in [**5**]. Conversely, if $\{x_n\}$ is the unit vector basis of l_1, then it is easy to construct an operator T on l_1 such that $\lim_n x_k^*(Tx_n) \neq 0$ for some $k = 1, 2, \ldots$.

THEOREM 3. *Let X be a Banach space with symmetric basis $\{x_n\}$ which is not isomorphic to l_1. Then the spaces $Y = (X \oplus X \oplus \ldots)_E$, $E = c_0$ or l_p, $1 < p < \infty$ are primary.*

Proof. For $i, j = 1, 2, \ldots$, let $y_{i,j} = (0, 0, \ldots, 0, x_j, 0, 0, \ldots)$ where x_j is in the ith coordinate. Let $\{y_n\}$ be the usual Cantor ordering of $\{y_{i,j}\}$. Then it is easy to show that $\{y_n\}$ is an unconditional basis of Y.

Let P be a projection on Y and let $P(y_n) = \sum_k a_k^{(n)} y_k = \sum_{i,j} a_{i,j}^{(n)} y_{i,j}$, $n = 1, 2, \ldots$. Now for any subsequence of $\{y_n\}$, there exists a subsequence, say $\{y_{n_k}\}$ such that either $[y_{n_k}]$ is isomorphic to l_p (or c_0) or $\{y_{n_k}\}$ is equivalent to a subsequence of $\{x_n\}$ and so $[y_{n_k}]$ is isomorphic to X. In either case, $[y_{n_k}]$ is not isomorphic to l_1 and thus no subsequence of $\{y_n\}$ spans a subspace isomorphic to l_1. By Lemma 2, we conclude that $\lim_n a_k^{(n)} = 0$ for all $k = 1, 2, \ldots$.

Now there exists $\epsilon > 0$ (for example, $\epsilon = \frac{1}{2}$) such that for each $i = 1, 2, \ldots$ there exist infinitely many j with $|a_{i,j}^{(i,j)}| \geqq \epsilon$ or $|1 - a_{i,j}^{(i,j)}| \geqq \epsilon$. Hence we may assume that there exist $i_1 < i_2 < \ldots$ and $j_1 < j_2 < \ldots$ such that $|a_{i_k,j_h}^{(i_k,j_h)}| \geqq \epsilon$, $k, h = 1, 2, \ldots$. For each $k = 1, 2, \ldots$, since $\{x_n\}$ is symmetric, $[y_{i_k,j_h}]_h$ is isomorphic to X. We now follow the Cantor ordering and proceed as the proof of the theorem [**5**]; by taking subsequences of $\{i_k\}$ and $\{j_h\}$ if necessary, we conclude that $\{Py_{i_k,j_h}\}_{k,h}$ is equivalent to $\{y_{i_k,j_h}\}_{k,h}$ and the restriction of the natural projection from Y onto $[y_{i_k,j_h}]_{k,h}$ is an isomorphism from $[Py_{i_k,j_h}]_{k,h}$ onto $[y_{i_k,j_h}]_{k,h}$. Thus $[Py_{i_k,j_h}]_{k,h}$ is complemented in Y and is isomorphic to Y. The proof that PY is isomorphic to Y is completed by Proposition 1 and Pelczynski's decomposition method.

Remark. For many projections P, there exists an $\epsilon > 0$ such that both $|a_n^{(n)}| \geqq \epsilon$ and $|1 - a_m^{(m)}| \geqq \epsilon$ for infinitely many n, m. In this case, the proof of the theorem yields that both PY and $(I - P)Y$ are isomorphic to Y.

Remark. Let $Z_p = (l_2 \oplus l_2 \oplus \ldots)_{l_p}$, $1 < p < \infty$. Schechtman [**18**] recently showed that every infinite dimensional complemented subspace X with unconditional basis of Z_p is isomorphic to either l_2, l_p, $l_2 \oplus l_p$ or Z_p. The condition that X has unconditional basis was later removed by Odell [**13**]. Thus Z_p is primary. See [**2**] for another proof that Z_p is primary.

2. In this section, we prove that if X is a Banach space with symmetric basis $\{x_n\}$ and $X_n = [x_1, \ldots, x_n]$, $n = 1, 2, \ldots$ then the spaces $(X_1 \oplus X_2 \oplus \ldots)_E$, $E = l_p$, $1 < p < \infty$ or $E = c_0$ are primary. We first prove a combinatorial lemma which is interesting in itself. We shall let N be the set of all natural numbers.

LEMMA 4. *If $M = \{m_i\}$ is a sequence of positive integers such that* $\limsup_i m_i = \infty$ *then there exist rearrangements of N and M into two sequences each, $\{n_1', n_2', \ldots; n_1'', n_2'', \ldots\}$ and $\{m_1', m_2', \ldots; m_1'', m_2'', \ldots\}$ such that $n_{2i-1}' + n_{2i}' = m_i'$ and $m_{2i-1}'' + m_{2i}'' = n_i''$ for all $i = 1, 2, \ldots$.*

Proof. We construct the rearrangements simultaneously and inductively.

Let $n_1' = 1$ and $n_2' = \min\{n \in N : n \neq n_1' \text{ and } n_1' + n \in M\}$. Let $\gamma_1 = \min\{i \in N : n_1' + n_2' = m_i \in M\}$ and $m_1' = m_{\gamma_1}$. Now, let

$$\alpha_1 = \min\{i \in N : m_i \in M\backslash\{m_1'\}\}$$

and

$$\beta_1 = \min\{i \in N : i \neq \alpha_1, m_i \in M\backslash\{m_1'\} \text{ and } m_i + m_{\alpha_1} \in N\backslash\{n_1', n_2'\}\}.$$

Define $m_1'' = m_{\alpha_1}$, $m_2'' = m_{\beta_1}$, and $n_1'' = m_1'' + m_2''$.

Assume that $n_1', n_2', \ldots, n_{2k}'; n_1'', n_2'', \ldots, n_k''$ and $m_1', m_2', \ldots, m_k'; m_1'', m_2'', \ldots, m_{2k}''$ are chosen such that $n_{2i-1}' + n_{2i}' = m_i'$ and $m_{2i-1}'' + m_{2i}'' = n_i''$, $i = 1, 2, \ldots, k$. Let

$$n_{2k+1}' = \min\{n \in N : n \neq n_i', i = 1, 2, \ldots, 2k \text{ and } n \neq n_i'',\ i = 1, 2, \ldots, k\}$$

and

$$n_{2k+2}' = \min\{n \in N : n \neq n_i', i = 1, 2, \ldots, 2k+1, n \neq n_i'',\ i = 1, 2, \ldots, k \text{ and } n_{2k+1}' + n \in M\backslash\{m_1', \ldots, m_k'; m_1'', m_2'', \ldots, m_{2k}''\}\}.$$

Since $\limsup_i m_i = \infty$, n_{2k+2}' is well-defined. Now let

$$\gamma_{k+1} = \min\{j \in N : m_j = n_{2k+1}' + n_{2k+2}', m_j \neq m_i', i = 1, 2, \ldots, k \text{ and } m_j \neq m_i'', i = 1, 2, \ldots, 2k\}.$$

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Define $m_{2k+1}' = m_{\gamma_{k+1}}$. Finally, let

$$\alpha_{k+1} = \min \{i \in N : m_i \in M \backslash \{m_1', \ldots, m_{k+1}'; m_1'', m_2'', \ldots, m_k''\}\}$$

and

$$\beta_{k+1} = \min \{i \in N : i \neq \alpha_{k+1}, m_i \in M \backslash \{m_1', \ldots, m_{k+1}'; m_1'', \ldots, m_k''\} \text{ and } m_i + m_{\alpha_{k+1}} \in N \backslash \{n_1', \ldots, n_{2k+2}'; n_1'', \ldots, n_k''\}\}.$$

Define $m_{2k+1}'' = m_{\alpha_{k+1}}$, $m_{2k+2}'' = m_{\beta_{k+1}}$ and $n_{k+1}'' = m_{2k+1}'' + m_{2k+2}''$. By induction, the proof of Lemma 4 is complete .

PROPOSITION 5. *Let $\{B_n\}$ be a sequence of finite dimensional Banach spaces and let X be a Banach space with symmetric basis. If $\{n_1', n_2', \ldots ; n_1'', n_2'', \ldots\}$ is a rearrangement of N then $(B_1 \oplus B_2 \oplus \ldots)_X$ is isomorphic to $(B_{n_1'} \oplus B_{n_2'} \oplus \ldots)_X \oplus (B_{n_1''} \oplus B_{n_2''} \oplus \ldots)_X$.*

We omit the simple proof of the proposition.

THEOREM 6. *If $\{B_n\}$ is a sequence of finite dimensional Banach spaces such that* $\sup_{n,m} d(B_n \oplus B_m, B_{n+m}) < \infty$ *and if X is a Banach space with symmetric basis then $(B_1 \oplus B_2 \oplus \ldots)_X$ is isomorphic to $(B_{m_1} \oplus B_{m_2} \oplus \ldots)_X$ for any sequence $\{m_i\}$ in N such that* $\limsup_i m_i = \infty$.

Proof. By Lemma 4, there exist rearrangements of N and $\{m_i\}$ into two sequences each, $\{n_1', n_2', \ldots ; n_1'', n_2'', \ldots\}$ and $\{m_1', m_2', \ldots ; m_1'', m_2'', \ldots\}$ such that $n_{2i-1}' + n_{2i}' = m_i'$ and $m_{2i-1}'' + m_{2i}'' = n_i''$, $i = 1, 2, \ldots$. Since X is a Banach space with symmetric basis, by Proposition 5 and the fact that $\sup_{n,m} d(B_n \oplus B_m, B_{n+m}) < \infty$, it follows that

$$\left(\sum_n B_n\right)_X \sim \left(\sum_i B_{n_i'}\right)_X \oplus \left(\sum_i B_{n_i''}\right)_X$$

$$\sim \left(\sum_i (B_{n_{2i-1}'} \oplus B_{n_{2i}'})\right)_X \oplus \left(\sum_i B_{m_{2i-1}''+m_{2i}''}\right)$$

$$\sim \left(\sum_i B_{n_{2i'-1}+n_{2i}'}\right)_X \oplus \left(\sum_i (B_{m_{2i-1}''} \oplus B_{m_{2i}''})\right)_X$$

$$\sim \left(\sum_i B_{m_i'}\right)_X \oplus \left(\sum_i B_{m_i''}\right)_X \sim \left(\sum_i B_{m_i}\right)_X$$

COROLLARY 7. *Let $\{B_n\}$ be a sequence of finite dimensional Banach spaces such that* $\sup_{n,m} d(B_n \oplus B_m, B_{n+m}) < \infty$ *and let X be a Banach space with symmetric basis. Let $Y = (\sum_n B_n)_X$. Then*

(i) *the Banach spaces Y, $Y \oplus Y$ and $(Y \oplus Y \oplus \ldots)_X$ are isomorphic; and*

(ii) *for any projection P on Y, Y is isomorphic to $Y \oplus P(Y)$.*

Proof. (i) Obvious.

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(ii) We use the same argument as the proof of Corollary 5 [**5**].

$$\begin{aligned} Y &\sim (Y \oplus Y \oplus \ldots)_X \\ &\sim (P(Y) \oplus P(Y) \oplus \ldots)_X \oplus ((I-P)Y \oplus (I-P)Y \oplus \ldots)_X \\ &\sim P(Y) \oplus (P(Y) \oplus P(Y) \oplus \ldots)_X \\ &\qquad \oplus ((I-P)Y \oplus (I-P)Y \oplus \ldots)_X \\ &\sim P(Y) \oplus (Y \oplus Y \oplus \ldots)_X \sim P(Y) \oplus Y. \end{aligned}$$

Remarks. (1) If $B_n = [e_1, e_2, \ldots, e_n]$, $n = 1, 2, \ldots$, where $\{e_n\}$ is a symmetric basis, then it is clear that $\sup_{n,m} d(B_n \oplus B_m, B_{n+m}) < \infty$. However, the converse is not true. For example, let $\{e_n\}$ be the unit vector basis of the James' quasi-reflexive Banach space J.

(2) When $X = l_p$, $1 < p < \infty$, a similar result was stated in [**7**, Lemma **5**].

The following lemma is a consequence of Ramsey's combinatorial lemma; for a proof see [**17**, p. 45].

LEMMA 8. *Let m be an arbitrary positive integer. Then every $(0, 1)$-matrix A of a sufficiently large order n contains a principal submatrix of order m of one of the following four types:*

$$(!) \quad \begin{bmatrix} * & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & * \end{bmatrix}, \begin{bmatrix} * & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 1 & & & * \end{bmatrix}, \begin{bmatrix} * & & & 1 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & * \end{bmatrix}, \begin{bmatrix} * & & & 1 \\ & \cdot & & \\ & & \cdot & \\ 1 & & & * \end{bmatrix}.$$

The asterisks on the main diagonal denote 0's and 1's, but the entries above the main diagonal and the entries below the main diagonal are all 0's or all 1's as illustrated in (!).

COROLLARY 9. *Let k and m be arbitrary positive integers. Then there exists an integer $N(k, m)$ such that for every $n \geqq N$ and for every $(0, 1)$-matrix $A = (a_{ij})$ of order n with $a_{ii} = 1$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^n \alpha_{ij} \leqq m$, $j = 1, 2, \ldots, n$, there is a principal submatrix $(a_{p_i p_j})$ or order k such that $a_{p_i p_j} = \delta_{ij}$ for all $i, j = 1, 2, \ldots, k$ where δ_{ij} is the Kronecker delta.*

THEOREM 10. *Let $\{x_n\}$ be a symmetric basis of a Banach space X and let B_n, $n = 1, 2, \ldots$ be the linear span of $x_1, x_2, \ldots, x_n$ in X. Then the spaces $Y = (\sum_n B_n)_E$, $E = c_0$ or l_p, $1 < p < \infty$ are primary.*

Proof. Let $y_i^n = (0, \ldots, 0, x_i, 0, \ldots)$, $i = 1, 2, \ldots, n$; $n = 1, 2, \ldots$ where x_i is in the nth coordinate of y_i^n. It is easy to see that $\{y_i^n\}_{i=1,2,\ldots,n;n=1,2,\ldots}$

is an unconditional basis of Y. Let P be a projection on Y and let

$$P(y_i^n) = \sum_{l=1}^{\infty} \left(\sum_{j=1}^{l} \alpha_j^{\,l}(n, i) y_j^{\,l} \right), \quad i = 1, 2, \ldots, n; n = 1, 2, \ldots .$$

Fix k. Let $\frac{1}{2} \geqq \epsilon > 0$ and let

(1) $\quad 0 < \epsilon_k < \epsilon / k^2 2^k, \quad k = 1, 2, \ldots$

be such that for any scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$,

(2) $\quad \epsilon_k k \sum_{i=1}^{k} |\lambda_i| \leqq \frac{1}{4} \left\| \sum_{i=1}^{k} \lambda_i x_i \right\| .$

Case I. X is not isomorphic to l_1.

Let $K = \max \{||P||, ||I - P||\}$. Since X is not isomorphic to l_1, there exists an integer m_k such that

(3) $\quad \left\| \sum_{i=1}^{m_k} x_i \right\| < \dfrac{m_k \epsilon_k}{K} .$

Let $N(k, m_k)$ be an integer determined by Corollary 9 and fix $n \geqq 2N(k, m_k)$. For each $i = 1, 2, \ldots, n$, either $|\alpha_i^n(n, i)| \geqq \frac{1}{2}$ or $|1 - \alpha_i^n(n, i)| \geqq \frac{1}{2}$. Since $\{x_n\}$ is symmetric, by taking a subsequence and considering $I - P$ if necessary, we may assume that $|\alpha_i^n(n, i)| \geqq \frac{1}{2}$ for $i = 1, 2, \ldots, n/2$ (or $(n - 1)/2$ if n is odd).

Define

$$\beta_{ij} = \begin{cases} 1 & \text{if } |\alpha_i^n(n, i)| \geqq \epsilon_k \\ 0 & \text{if } |\alpha_j^n(n, i)| < \epsilon_k \end{cases}, \quad 1 \leqq i, j \leqq n/2.$$

We claim that (β_{ij}) is an $(0, 1)$-matrix of order $n/2$ such that $\sum_{i=1}^{n/2} \beta_{ij} < m_k$ for all $j = 1, 2, \ldots, n/2$. Suppose for some j, $\sum_{i=1}^{n/2} \beta_{ij} \geqq m_k$. Hence $\beta_{i_l j} = 1$ for some $l = 1, 2, \ldots, m_k$. Let $\epsilon_{i_l} = \operatorname{sgn} \alpha_j^n(n, i_l)$, $l = 1, 2, \ldots, m_k$. Then

$$\left\| \sum_{l=1}^{m_k} \epsilon_{i_l} y_{i_l}^{\,n} \right\| \cdot ||P|| \geqq \left\| \sum_{l=1}^{m_k} \epsilon_{i_l} P(y_{i_l}^{\,n}) \right\| \geqq \left| \sum_{l=1}^{m_k} \epsilon_{i_l} \alpha_j^{\,n}(n, i_l) \right| \geqq m_k \epsilon_k.$$

Hence

$$\left\| \sum_{l=1}^{m_k} x_l \right\| = \left\| \sum_{l=1}^{m_k} \epsilon_{i_l} x_{i_l} \right\| = \left\| \sum_{l=1}^{m_k} \epsilon_{i_l} y_{i_l}^{\,n} \right\| \geqq \frac{m_k \epsilon_k}{||P||} \geqq \frac{m_k \epsilon_k}{K}$$

which contradicts (3).

By Corollary 9, there is a $k \times k$ submatrix $(\beta_{p_i p_j}) = (\delta_{ij})$ of (β_{ij}). Thus

(4) $\quad |\alpha_{p_j}^{\,n}(n, p_i)| < \epsilon_k, 1 \leqq i \neq j \leqq k \quad$ and

$$|\alpha_{p_i}^{\,n}(n, p_i)| \geqq \tfrac{1}{2}, i = 1, 2, \ldots, k.$$

For any scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$,

$$\|P\| \cdot \left\|\sum_{i=1}^{k} \lambda_i x_i\right\| = \|P\| \cdot \left\|\sum_{i=1}^{k} \lambda_i y_{p_i}{}^n\right\| \geqq \left\|\sum_{i=1}^{k} \lambda_i P(y_{p_i}{}^n)\right\|$$

$$\geqq \left\|\sum_{j=1}^{k} \sum_{i=1}^{k} \lambda_i \alpha_{p_j}{}^n(n, p_i) x_{p_j}\right\|$$

$$\geqq \left\|\sum_{j=1}^{k} \lambda_j \alpha_{p_j}{}^n(n, p_j) x_{p_j}\right\| - \left\|\sum_{j=1}^{k} \left(\sum_{\substack{i=1\\ i\neq j}}^{k} \lambda_i \alpha_{p_j}{}^n(n, p_i)\right) x_{p_j}\right\|$$

$$\geqq \tfrac{1}{2}\left\|\sum_{i=1}^{k} \lambda_i x_{p_i}\right\| - \sum_{j=1}^{k} \left|\sum_{\substack{i=1\\ i\neq j}}^{k} \lambda_i \alpha_{p_j}{}^n(n, p_i)\right|$$

$$> \tfrac{1}{2}\left\|\sum_{i=1}^{k} \lambda_i x_i\right\| - \epsilon_k \sum_{j=1}^{k} \sum_{\substack{i=1\\ i\neq j}}^{k} |\lambda_i|$$

$$> \tfrac{1}{2}\left\|\sum_{i=1}^{k} \lambda_i x_i\right\| - k\epsilon_k \sum_{i=1}^{k} |\lambda_i| > \tfrac{1}{2}\left\|\sum_{i=1}^{k} \lambda_i x_i\right\| - \tfrac{1}{4}\left\|\sum_{i=1}^{k} \lambda_i x_i\right\|$$

$$= \tfrac{1}{4}\left\|\sum_{i=1}^{k} \lambda_i x_i\right\|.$$

Hence we have proved that for every k there exists an integer $N(k)$ such that for all $n \geqq N(k)$, there are $1 \leqq p_1 < p_2 < \ldots < p_k \leqq n$ so that

$$(5) \qquad \tfrac{1}{4}\left\|\sum_{i=1}^{k} \lambda_i x_i\right\| \leqq \left\|\sum_{i=1}^{k} \lambda_i P(y_{p_i}{}^n)\right\| \leqq \|P\| \cdot \left\|\sum_{i=1}^{k} \lambda_i x_i\right\|$$

for any scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$. Notice that the norm of this isomorphism is independent of k.

Now, since $p \neq 1$, no subsequence of $\{y_i{}^n\}$ spans a subspace isomorphic to l_1, by Lemma 2, for all $j = 1, 2, \ldots, l$; $l = 1, 2, \ldots,$

$$(6) \qquad \lim_{n\to\infty} \alpha_j{}^l(n, i) = 0.$$

By (5), (6), and the standard "gliding hump" process, given $\epsilon > 0$, we can construct inductively a sequence

$$(7) \qquad Z_{p_i}{}^{n_k} = \sum_{l=q_k'}^{q_k} \sum_{j=1}^{l} \alpha_j{}^l(n_k, p_i) y_j{}^l, \quad i = 1, 2, \ldots, k; k = 1, 2, \ldots$$

where $q_1' < n_1 < q_1 < q_2' < n_2 < q_2 < \cdots < q_k' < n_k < q_k < \cdots$ such that

(i) for each $k = 1, 2, \ldots, \{P(y_{p_i}{}^{n_k})\}_{i=1,2,\ldots,k}$ satisfies (5);

(ii) $\|Z_{p_i}{}^{n_k} - P(y_{p_i}{}^{n_k})\| \leqq \epsilon/k^2 2^k$, $i = 1, 2, \ldots, k$; $k = 1, 2, \ldots,$

(Hence $\sum_k \sum_{i=1}^k \|Z_{p_i}{}^{n_k} - P(y_{p_i}{}^{n_k})\| < \epsilon$ and so $\{Z_{p_i}{}^{n_k}\}_{i=1,2,\ldots,k;k=1,2,\ldots}$ is equivalent to $\{P(y_{p_i}^{n_k})\}_{i=1,2,\ldots,k;k=1,2,\ldots}$ for sufficiently small ϵ).

$$\text{(iii)}\quad \left\|\sum_k \sum_{i=1}^k \lambda_{p_i}^{n_k} z_{p_i}^{n_k}\right\| = \begin{cases} \left(\sum_k \left\|\sum_{i=1}^k \lambda_{p_i}^{n_k} z_{p_i}^{n_k}\right\|\right)^{1/p} & (\text{when } E = l_p,\ 1 < p < \infty) \\ \sup_k \left\|\sum_{i=1}^k \lambda_{p_i}^{n_k} z_{p_i}^{n_k}\right\| & (\text{when } E = c_0) \end{cases}$$

for any scalars $\lambda_{p_i}^{n_k}$.

By (5), for each $k = 1, 2, \ldots,$ $\{P(y_{p_i}^{n_k})\}_{i=1,2,\ldots}$ is uniformly equivalent to $\{x_1, \ldots, x_k\}$. Therefore, by (ii) and (iii), we conclude that $\{z_{p_i}^{n_k}\}_{i=1,2,\ldots,k;k=1,2,\ldots}$ spans a subspace isomorphic to Y.

Case II. X is isomorphic to l_1. Then X is not isomorphic to c_0 and so there exists an integer m such that

$$(8)\qquad \left\|\sum_{i=1}^m x_i\right\| > \frac{k}{\epsilon_k}.$$

We now proceed as in Case I. Construct the (0, 1)-matrix (p_{ij}) of order $n/2$ and using (8) instead of (3) to prove that $\sum_{j=1}^{n/2} p_{ij} < m$ for all $i = 1, 2, \ldots, n/2$ (instead of $\sum_{i=1}^{n/2} p_{ij} < m$, $j = 1, 2, \ldots, n/2$). The rest of the proof is like Case I. Thus in both cases, we obtain a sequence $\{z_{p_i}^{n_k}\}_{i=1,2,\ldots,k;k=1,2,\ldots}$ satisfying conditions (i), (ii), and (iii).

By Pelczynski's decomposition method and by Corollary 7, it remains to show that, by taking a suitable subsequence if necessary, $\{z_{p_i}^{n_k}\}_{i=1,2,\ldots,k;k=1,2,\ldots}$ spans a complemented subspace in Y.

For $i = 1, 2, \ldots, k$; $k = 1, 2, \ldots,$ define

$$(9)\qquad w_{p_i}^{n_k} = \sum_{\substack{l=q_k'\\ l\neq n_k}}^{q_k} \sum_{j=1}^{l} \alpha_j^{l}(n_k, p_i) y_j^{l} + \sum_{\substack{j=1\\ j\neq p_1,\ldots,p_{i-1},p_{i+1},\ldots,p_k}}^{n_k} \alpha_j^{n_k}(n_k, p_i) y_j^{n_k}.$$

Then

$$\|z_{p_i}^{n_k} - w_{p_i}^{n_k}\| = \left\|\sum_{\substack{j=1\\ j\neq p_i}}^{k} \alpha_j^{n_k}(n_k, p_i) y_j^{n_k}\right\| \leqq \sum_{\substack{j=1\\ j\neq p_i}}^{k} |\alpha_j^{n_k}(n_k, p_i)|$$

$$< (k-1)\epsilon_k < \frac{\epsilon}{k2^k}.$$

Hence

$$\sum_{k=1}^{\infty} \sum_{i=1}^{k} \|z_{p_i}^{n_k} - w_{p_i}^{n_k}\| \leqq \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$$

and so by choosing ϵ sufficiently small, $\{z_{p_i}^{n_k}\}$ is equivalent to $\{w_{p_i}^{n_k}\}$, and $[z_{p_i}^{n_k}]$ is complemented if and only if $[w_{p_i}^{n_k}]$ is complemented in Y. Define $Q : Y \to [w_{p_i}^{n_k}]$ by

$$Q\left(\sum_{n=1}^{\infty} \sum_{i=1}^{n} \beta_i^{n} y_i^{n}\right) = \sum_{k=1}^{\infty} \sum_{i=1}^{k} \frac{\beta_{p_i}^{n_k}}{\alpha_{p_i}^{n_k}(n_k, p_i)} w_{p_i}^{n_k}.$$

Since $|\alpha_{p_i}{}^{n_k}(n_k, p_i)| \geqq \frac{1}{2}$ for all $i = 1, 2, \ldots, k$; $k = 1, 2, \ldots$ $\{y_i{}^n\}$ is an unconditional basis and by the construction $\{w_{p_i}{}^{n_k}\} \approx \{z_{p_i}{}^{n_k}\} \approx \{y_{p_i}{}^{n_k}\}$, it is easy to show that Q is a bounded projection from Y onto $[w_{p_i}{}^{n_k}]$. This completes the proof of the theorem.

By combining Theorems 3 and 10, we obtain

COROLLARY 11. *Let $\{x_n\}$ be a symmetric basis of a Banach space X and for each $n = 1, 2, \ldots$, let $B_n = X$ or the linear span of $x_1, x_2, \ldots, x_n$ in X. Then the Banach spaces $(\sum B_n)_E$, $E = c_0$ or l_p, $1 < p < \infty$, are primary.*

Remarks. (1) Since $\{y_i{}^n\}$ is an unconditional basis of Y, letting P_0 be the natural projection from Y onto $[y_{p_i}{}^{n_k}]_{i=1,2,\ldots,k;k=1,2,\ldots}$, it can be proved that the restriction of P_0 is an isomorphism from $[z_{p_i}{}^{n_k}]$ onto $[y_{p_i}{}^{n_k}]$. Hence $[z_{p_i}{}^{n_k}]$ is complemented in Y.

(2) We don't know whether the theorem is true when $p = 1$ or ∞. The first half of the proof includes the cases $p = 1$ or ∞. Namely, if T is an operator on $Y = (\sum B_n)_{l_p}$, $1 \leqq p \leqq \infty$, then for every k, there exists an integer $N(k)$ such that for any $n \geqq N$, there are $1 \leqq p_1 < p_2 < \cdots < p_k \leqq n$ such that $\{T(y_{p_i}{}^n)\}_{i=1,2,\ldots,k}$ spans a subspace isomorphic to B_k.

3. In this section, we show that if X is a Banach space with symmetric basis which is isomorphic to a complemented subspace of a Banach space E, then for any operator T on E, either TE or $(I - T)E$ contains a complemented subspace which is isomorphic to X. The technique is similar to the one used by Bessaga and Pelczynski [**4**] in generalizing some results of R. C. James. This technique also enables us to generalize some of the results in Sections 1 and 2. We first prove a stronger result when X is c_0 or l_p, $1 \leqq p < \infty$.

THEOREM 12. *Let E be a Banach space which contains a subspace X isomorphic to c_0 or l_p, $1 \leqq p < \infty$. Then for any operator $T : E \to E$, either TE or $(I - T)E$ contains a subspace isomorphic to c_0 or l_p, $1 \leqq p < \infty$.*

Proof. If X is isomorphic to l_1, then the theorem follows immediately from the beautiful result of Rosenthal [**16**] that a Banach space contains a subspace isomorphic to l_1 if and only if it contains a bounded sequence with no weak Cauchy subsequence.

Now, suppose that X is not isomorphic to l_1. Let $\{x_n\}$ be a symmetric basis of X.

Case I. There is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_i ||Tx_{n_i}|| = 0$ or $\lim_i ||(I - T)x_{n_i}|| = 0$.

If $\lim_i ||Tx_{n_i}|| = 0$, by choosing a subsequence if necessary, we have $\sum_i ||x_i{}^*|| \cdot ||x_{n_i} - (I - T)x_{n_i}|| = \sum_i ||x_i{}^*|| \cdot ||Tx_{n_i}|| < 1$ where $\{x_i{}^*\}$ is the coefficient functionals of $\{x_i\}$. Hence $\{(I - T)x_{n_i}\}$ is equivalent to $\{x_{n_i}\}$. That is, $(I - T)E$ contains a subspace isomorphic to X.

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Similarly, if $\lim_i ||(I - T)x_{n_i}|| = 0$ then TE contains a subspace isomorphic to X.

Case II. Both $\inf_n ||Tx_n|| > 0$ and $\inf ||(I - T)x_n|| > 0$. Since X is not isomorphic to l_1, hence $\{x_n\}$ is weakly convergent to 0 and so is $\{Tx_n\}$. In this case, we have assumed that $\inf ||Tx_n|| > 0$, hence there exists a basic subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$. Since $\{x_{n_i}\}$ dominates $\{Tx_{n_i}\}$ and every basic sequence dominates the unit vector basis of c_0, we conclude that $[Tx_{n_i}]$ is isomorphic to c_0 when X is isomorphic to c_0.

Suppose $1 < p < \infty$ and no subsequence of $\{Tx_n\}$ is equivalent to $\{x_n\}$. Then there exists a sequence $\{\alpha_i\}$ such that $\sum_i \alpha_i Tx_{n_i}$ converges and $\sum_i |\alpha_i|^p = \infty$. Choose $p_1 < p_2 < \cdots$ such that

$$1 \leqq \sum_{i=p_n+1}^{p_{n+1}} |\alpha_i|^p \leqq 2$$

and let

$$y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_{n_i}, \quad n = 1, 2, \ldots .$$

Then since $\sum \alpha_i Tx_{n_i}$ converges, we conclude that $\lim_n ||Ty_n|| = 0$. Furthermore, $\{y_n\}$ is a bounded block basic sequence of $\{x_{n_i}\}$, hence is equivalent to $\{x_n\}$. By Case I, we obtain that $(I - T)E$ contains a subspace isomorphic to l_p.

COROLLARY 13. *Let E be a Banach space with unconditional basis which is not weakly complete. Then for any operator $T : E \to E$ either TE or $(I - T)E$ is not weakly complete.*

Proof. This follows immediately from the theorem and a result of Bessaga and Pelczynski [**3**] that if X is a subspace of a Banach space with unconditional basis then X is weakly complete if and only if Y contains no subspace which is isomorphic to c_0.

We don't know whether Theorem 12 is true or not when X is an arbitrary Banach space with symmetric basis. However, we have the following:

THEOREM 14. *Let $\{x_n\}$ be a symmetric basic sequence in a Banach space E. If $\{x_n\}$ spans a complemented subspace X in E, then for any operator $T : E \to E$ either TE or $(I - T)E$ contains a subspace F which is complemented in E and is isomorphic to X.*

Proof. Let $P : E \to X$ be a projection. Then $PT|_X : X \to X$. By [**5**] when X is not isomorphic to l_1 and Rosenthal's result [**16**] when X is isomorphic to l_1, we may assume that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{PT(x_{n_i})\}$ is equivalent to $\{x_i\}$. Since $\{x_i\} > \{x_{n_i}\} > \{Tx_{n_i}\} > \{PTx_{n_i}\} \approx \{x_i\}$, we conclude that $\{Tx_{n_i}\}$ is equivalent to $\{x_i\}$ and P maps $[Tx_{n_i}]$ isomorphically onto $[PTx_{n_i}]$. Since $[PTx_{n_i}]$ is complemented in X and X is complemented in E,

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hence $[PTx_{ni}]$ is complemented in E and thus $[Tx_{ni}]$ is complemented in E and is isomorphic to X.

Remark. It is known that if E is a Banach space with unconditional basis and Y is a subspace of E which is isomorphic to l_1 then there exists a subspace F in Y which is isomorphic to l_1 and is complemented in E. However, c_0 is not complemented in l_∞ and there exist reflexive Orlicz sequence spaces which contain subspaces isomorphic to l_p, $1 < p < \infty$ but no complemented subspaces which are isomorphic to l_p, $1 < p < \infty$ [**11**].

Using the same technique and the results in Sections 1 and 2, we have:

THEOREM 15. *Let* $Y = (X \oplus X \oplus \cdots)_{c_0}$ *or* $(X \oplus X \oplus \cdots)_{l_p}$, $1 < p < \infty$ *(respectively* $(\sum B_n)_{l_p}$, $1 < p < \infty$ *or* $(\sum B_n)_{c_0}$*) where* X *is a Banach space with symmetric basis which is not isomorphic to* l_1 *(respectively,* $B_n = [x_1, \ldots, x_n]$, $n = 1, 2, \ldots$ *and* $\{x_n\}$ *is a symmetric basis of a Banach space). If* E *is a Banach space which contains a complemented subspace isomorphic to* Y *then for every operator* $T : E \to E$*, either* TE *or* $(I - T)E$ *contains a complemented subspace isomorphic to* Y.

4. In this section, we show that the spaces $(l_\infty \oplus l_\infty \oplus \ldots)_{l_p}$, $1 < p < \infty$ are primary. The proof is similar to the one used by Lindenstrauss [**8**] in proving that l_∞ is prime. Throughout this section, we shall let $Y = (l_\infty \oplus l_\infty \oplus \cdots)_{l_p}$, $1 < p < \infty$.

LEMMA 16. *Let* $y_n = (x_1^n, x_2^n, \ldots, x_i^n, \cdots)$, $n = 1, 2, \ldots$, *be elements in* Y *where* $x_i^n = (x_i^n(1), x_i^n(2), \ldots, x_i^n(k), \ldots)$. *If* $\sup_n \|\sum_{j=1}^n \epsilon_j y_j\| < \infty$ *for all* $|\epsilon_j| = 1$, $j = 1, 2, \ldots$, *then for any* $\epsilon > 0$, *there exists an integer* I *such that*

$$\sum_{n=1}^{\infty} |x_i^n(k)| \leqq \epsilon$$

for all $i \geqq I$ *and every* $k = 1, 2, \ldots$.

Proof. Suppose there exist $\epsilon_0 > 0$, $i_1 < i_2 < \cdots$ and k_j, $j = 1, 2, \ldots$ such that

$$\sum_{n=1}^{\infty} |x_{i_j}^n(k_j)| > \epsilon_0, \quad j = 1, 2, \ldots.$$

Choose m_1 such that $\sum_{n=1}^{m_1} |x_{i_1}^n(k_1)| > \epsilon_0/2$ and $\sum_{n=m_1+1}^{\infty} |x_{i_1}^n(k_1)| < \epsilon_0/8$. This can be done since for some $\{\epsilon_n\}$ with $|\epsilon_n| = 1$,

$$\sum_{n=1}^{\infty} |x_i^n(k)| = \sum_{n=1}^{\infty} \epsilon_n x_i^n(k) \leqq \sup_n \left\| \sum_{i=1}^{n} \epsilon_j y_j \right\| < \infty.$$

Note that for each $n = 1, 2, \ldots, \lim_i \|x_i^n\| = 0$. Hence for sufficiently large i, we have

$$\sum_{n=1}^{m_1} |x_i^n(k)| \leqq \sum_{n=1}^{m_1} \|x_i^n\| < \frac{\epsilon_0}{8}$$

for all $k = 1, 2, \ldots$. Thus by taking a subsequence of $\{i_j\}_{j=1,2,\ldots}$ if necessary, we may assume that

$$\sum_{n=1}^{m_1} |x_{i_2}{}^n(k_2)| < \frac{\epsilon_0}{8}.$$

Now, choose $m_2 > m_1$ such that

$$\sum_{n=1}^{m_2} |x_{i_2}{}^n(k_2)| > \frac{\epsilon_0}{2} \quad \text{and} \quad \sum_{n=m_2+1}^{\infty} |x_{i_2}{}^n(k_2)| < \frac{\epsilon_0}{8}.$$

By induction and by choosing a subsequence of $\{i_j\}_{j=1,2,\ldots}$ if necessary, there exist $0 = m_0 < m_1 < m_2 < \ldots$ such that for all $j = 1, 2, \ldots,$

$$\text{(i)} \sum_{n=1}^{m_j} |x_{i_j}{}^n(k_j)| > \frac{\epsilon_0}{2},$$

$$\text{(ii)} \sum_{n=m_j+1}^{\infty} |x_{i_j}{}^n(k_j)| < \frac{\epsilon_0}{8},$$

$$\text{(iii)} \sum_{n=1}^{m_j} |x_{i_{j+1}}{}^n(k_{j+1})| < \frac{\epsilon_0}{8}.$$

Choose $|\epsilon_n| = 1$ such that $\epsilon_n x_{i_j}{}^n(k_j) = |x_{i_j}{}^n(k_j)|, m_{j-1} < n \leqq m_j, j = 1, 2, \ldots$. Then for every $j = 1, 2, \ldots,$

$$\left\|\sum_{n=1}^{m_j} \epsilon_n y_n\right\| = \sum_{i=1}^{\infty} \left\|\sum_{n=1}^{m_j} \epsilon_n x_i{}^n\right\|^p$$

$$\geqq \sum_{h=1}^{j} \left\|\sum_{n=1}^{m_j} \epsilon_n x_{i_h}{}^n\right\|^p \geqq \sum_{h=1}^{j} \left|\sum_{n=1}^{m_j} \epsilon_n x_{i_h}{}^n(k_h)\right|^p$$

$$\geqq \sum_{h=1}^{j} \left[\left|\sum_{n=m_{h-1}+1}^{m_h} \epsilon_n x_{i_h}{}^n(k_h)\right| - \left|\sum_{n=1}^{m_{h-1}} \epsilon_n x_{i_h}{}^n(k_h)\right| - \left|\sum_{n=m_h+1}^{m_j} \epsilon_n x_{i_h}{}^n(k_h)\right|\right]^p$$

$$> \sum_{h=1}^{j} \left[\sum_{n=1}^{m_h} |x_{i_h}{}^n(k_h)| - \sum_{n=1}^{m_{h-1}} |x_{i_h}{}^n(k_h)| - \frac{\epsilon_0}{8} - \frac{\epsilon_0}{8}\right]^p$$

$$> \sum_{h=1}^{j} \left(\frac{\epsilon_0}{2} - \frac{\epsilon_0}{8} - \frac{\epsilon_0}{4}\right)^p = \left(\frac{\epsilon_0}{8}\right)^p j,$$

which is a contradiction to the hypothesis that $\sup_m ||\sum_{n=1}^{m} \epsilon_n y_n|| < \infty$ for all $|\epsilon_n| = 1, n = 1, 2, \ldots$.

LEMMA 17. *Let* $x_n = (x_n(1), \ldots, x_n(k), \ldots), n = 1, 2, \ldots$ *be elements in* l_∞. *If* $\sup ||\sum_{i=1}^{n} \epsilon_j x_j|| < \infty$ *for all* $|\epsilon_i| = 1, i = 1, 2, \ldots,$ *then for any* $\epsilon > 0$ *and* $\{k_i\}$ *there exist an integer n and a subsequence* $\{k_{ij}\}$ *of* $\{k_i\}$ *such that* $|x_n(k_{ij})| < \epsilon$ *for all* $j = 1, 2, \ldots$.

Proof. Suppose there exists $\epsilon_0 > 0$ such that for each $n = 1, 2, \ldots, |x_n(k_i)| \geqq \epsilon_0$, for all except finitely many i. Let n be an integer such that $n\epsilon_0 > \sup_n ||\sum_{j=1}^{n} \epsilon_j x_j||$. Then for each $j = 1, 2, \ldots, n$, since $|x_j(k_i)| < \epsilon_0$ for only finitely

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many i, hence there exists i_0 such that $|x_j(k_{i_0})| \geqq \epsilon_0$ for all $j = 1, 2, \ldots, n$. Let $\epsilon_j = \operatorname{sgn} x_j(k_{i_0})$, $j = 1, 2, \ldots, n$. Then

$$\left\|\sum_{j=1}^{n} \epsilon_j x_j\right\| \geqq \left|\sum_{j=1}^{n} \epsilon_j x_j(k_{i_0})\right| = \sum_{j=1}^{n} |x_j(k_{i_0})| \geqq n\epsilon_0 > \sup_n \left\|\sum_{j=1}^{n} \epsilon_j x_j\right\|,$$

which is a contradiction.

The following lemma is proved by Lindenstrauss (see the proof of **8**, Lemma 5).

LEMMA 18. *Let $\{x_n = (x_n(1), \ldots, x_n(k), \ldots)\}$ be a sequence of elements in l_∞ such that for some constant $k > 0$, $\|\sum_{i=1}^{n} \lambda_i x_i\| \leqq K \sup |\lambda_i|$ for all $\lambda_i \in R$, $i = 1, 2, \ldots, n$. If $\|x_n\| > 2$ for all $n = 1, 2, \ldots$, then for any $1/3 > \epsilon > 0$ there exists subsequences $\{n_k\}$ and $\{i_k\}$ of N such that for all $k = 1, 2, \ldots$, $|x_{n_k}(i_k)| \geqq 5/3$ and $\sum_{j\neq k} |x_{n_j}(i_k)| < \epsilon$.*

LEMMA 19. *Let $\{y_{i,j} = (x_{i,j}^1, \ldots, x_{i,j}^n, \ldots)\}_{i,j}$ be elements in Y for which there is a constant $K > 0$ such that for each $i = 1, 2, \ldots$,*

$$\left\|\sum_{j=1}^{n} \lambda_j y_{i,j}\right\| \leqq K \sup_j |\lambda_j|$$

for all $\lambda_j \in R$, $j = 1, 2, \ldots, n$. If $\|x_{i,j}^i\| > 2$ for all $i, j = 1, 2, \ldots$, then for any $1/3 > \epsilon > 0$ there exists a subsequence $\{i(l)\}_{l=1,2,\ldots}$ of N and double sequences of integers $\{j(i(l), q)\}_{l,q=1,2,\ldots}$ and $\{k(i(l), q)\}_{l,q=1,2,\ldots}$ such that for all $l, q = 1, 2, \ldots,$

(i) $|x^{i(l)}_{i(l),j(i(l),q)}(k(i(l), q))| \geqq \dfrac{5}{3}$

(ii) $\displaystyle\sum_{(h,s)\neq(l,q)} |x^{i(l)}_{i(h),j(i(h),s)}(k(i(h), q))| \leqq \frac{\epsilon}{2^l}.$

Proof. Given $1/3 > \epsilon > 0$, applying Lemma 18 to $\{x_{i,j}^i\}_{j=1,2,\ldots}$ for each fixed $i = 1, 2, \ldots$, there exist subsequences $\{j(i, q)\}_{q=1,2,\ldots}$ and $\{k(i, q)\}_{q=1,2,\ldots}$ such that

(1) $\quad |x^i_{i,j(i,q)}(k(i, q))| \geqq \dfrac{5}{3}$ for all q

and

(2) $\quad \displaystyle\sum_{s\neq q} |x^i_{i,j(i,s)}(k(i, q))| \leqq \frac{\epsilon}{2^{2i}}.$

Notice that (1) implies (i) for all $l, q = 1, 2, \ldots$. We shall choose a subsequence $\{i(l)\}$ of $\{i\}_{i=1,2,\ldots}$ which satisfies (ii).

Let $i(1) = 1$ and apply Lemma 16 to $\{y_{i(1),j}\}_{j=1,2,\ldots}$ there exists $i(2) > i(1)$ such that

(3) $\quad \displaystyle\sum_j |x^{i(2)}_{i(1),j}(k)| \leqq \frac{\epsilon}{2^2}, \quad k = 1, 2, \ldots.$

Now,

$$\sum_j |x^{i(1)}_{i(2),j}(k(1,1))| \leqq \left\| \sum_j \epsilon_j x^{i(1)}_{i(2),j} \right\| \leqq \left\| \sum_j \epsilon_j y_{i(2),j} \right\| \leqq K$$

for suitable $|\epsilon_j| = 1$. Hence there exists n such that for all $j \geqq n$

(4) $|x^{i(1)}_{i(2),j}(k(1,1))| \leqq \epsilon/2^4.$

Applying Lemma 17 to $\{x^{i(1)}_{i(2),j}\}_{j \geqq n}$, there is an integer, denoted by $j(i(2), 1)$ again, such that for some subsequence of $\{k(i(1), q)\}_{q>1}$ (which we will denote again by $\{k(i(1), q)\}_{q>1}$), we have

(5) $|x^{i(1)}_{i(2),j(i(2),1)}(k(i(1), q))| \leqq \epsilon/2^4, \quad q = 2, 3, \ldots.$

We continue our induction along the usual Cantor's ordering of $\{i, j\}_{i,j=1,2,\ldots}$. For $l = 1$, $q = 2, 3$, let $j(i(l), q) = j(i(1), 2)$ and $j(i(1), 3)$, respectively. We choose $j(i(2), 2)$ as follows. By hypothesis,

$$\sum_j |x^{i(1)}_{i(2),j}(k(i(1), q))| \leqq \left\| \sum_j \epsilon_j y_{i(2),j} \right\| < K, \quad q = 1, 2, 3.$$

Hence there exists n such that for all $j \geqq n$,

(6) $|x^{i(1)}_{k(2),j}(k(i(1), q))| \leqq \epsilon/2^5, \quad q = 1, 2, 3.$

Now, applying Lemma 19 to $\{x^{i(1)}_{i(2),j}\}_{j \geqq n}$, there exists an integer, denoted by $j(i(2), 2)$ again, such that for some subsequence of $\{k(i(1), q)\}_{q>3}$, denoted by $\{k(i(1), q)\}_{q>3}$ again, we have

(7) $|x^{i(1)}_{i(2),j(i(2),2)}(k(i(1), q))| \leqq \epsilon/2^5, \quad q = 4, 5, \ldots.$

By (6) and (7), we conclude

(8) $|x^{i(1)}_{i(2),j(i(2),2)}(k(i(1), q))| \leqq \epsilon/2^5, \quad q = 1, 2, \ldots.$

To find the next term, by applying Lemma 16 to both $\{y_{i(1),j}\}_j$ and $\{y_{i(2),j}\}_j$, there exists $i(3) > i(2)$ such that

(9) $\sum_j |x^{i(3)}_{i(l),j}(k)| \leqq \epsilon/2^{4+l}, \quad l = 1, 2; k = 1, 2, \ldots.$

By hypothesis,

$$\sum_j |x^{i(l)}_{i(3),j}(k(i(l), q))| \leqq \left\| \sum_j \epsilon_j y_{i(3),j} \right\| \leqq K$$

for $l = 1$, $q = 1, 2, 3$ and $l = 2$, $q = 1, 2$, respectively. Hence there exists n such that for all $j \geqq n$

(10) $|x^{i(l)}_{i(3),j}(k(i(l), q))| \leqq \dfrac{\epsilon}{2^{4+l}}, \quad \begin{array}{l} l = 1, q = 1, 2, 3 \text{ and} \\ l = 2, q = 1, 2, \text{ respectively} \end{array}$

Applying Lemma 17 to $\{x^{i(1)}_{i(3),j}\}_{j \geqq n}$ and $\{x^{i(2)}_{i(3),j}\}_{j \geqq n}$ simultaneously, there is an integer, denoted by $j(i(3), 1)$ again such that for some subsequences of

$\{k(i(1), q)\}_{q \geqq 4}$ and $\{k(i(2), q\}_{q \geqq 3}$ which we again denote the same way, we have

(11) $|x^{i(l)}_{i(3),j(i(3),1)}(k(i(3), q)| \leqq \frac{\epsilon}{2^{4+l}}$, $l = 1, q \geqq 4$ and $l = 2, q \geqq 3$, respectively

By (10) and (11), we conclude

(12) $|x^{i(l)}_{i(2),j(i(3),1)}(k(i(3), q))| \leqq \epsilon/2^{4+l}$, $l = 1, 2; q = 1, 2, \ldots$.

Continuing by induction, we get $\{i(l)\}_{l=1,2,\ldots}$, $\{j(i(l), q)\}_{l,q=1,2,\ldots}$, and $\{k(i(l), q)\}_{l,q=1,2,\ldots}$ such that

(13) $|x^{i(l)}_{i(h),j(i(h),s)}(k(i(l), q))| \leqq \epsilon/2^{l+h+s}$, $l, h, s, q = 1, 2, \ldots$ and $l \neq h$.

(Equation (9) yields the case $h > l$ and (12) yields the case $h < l$.) Now, for all $l, q = 1, 2, \ldots$,

$$\sum_{(h,s)\neq(l,q)} |x^{i(l)}_{i(h),j(i(h),s)}(k(i(h), q))|$$
$$= \sum_{h\neq l} \sum_{s} |x^{i(l)}_{i(h),j(i(h),s)}(k(i(h), q))| + \sum_{s\neq q} |x^{i(l)}_{i(l),j(i(l),s)}(k(i(l), q))|$$
$$\leqq \sum_{h\neq l} \sum_{s} \frac{\epsilon}{2^{l+h+s}} + \frac{\epsilon}{2^{2i(l)}} < \sum_{h\neq l} \frac{\epsilon}{2^{l+h}} + \frac{\epsilon}{2^{2l}} = \frac{\epsilon}{2^{l}}.$$

This shows that (ii) is satisfied and the proof is completed.

COROLLARY 20. *Let $\{y_{i,j}\}_{i,j}$ be elements in Y which satisfy the condition in Lemma 19. Then there exist sequences of integers $\{i(l)\}_l$ and $\{j(i(l), q\}_{l,q}$ such that for all sequences $\{\lambda_{i,j}\}_{i,j}$ with $\sum_i \sup_j |\lambda_{i,j}|)^p < \infty$, it is true that*

$$\left\{\sum_{l=1}^{\infty} \left(\sup_q |\lambda_{l,q}|\right)^p\right\}^{1/p} \leqq \left\|\sum_l \sum_q \lambda_{l,q} y_{i(l),j(i(l),q)}\right\|.$$

Proof. Choose sequences of integers $\{i(l)\}_l$, $\{j(i(l), q\}_{l,q}$, and $\{k(i(l), q)\}_{l,q}$ satisfying Lemma 19 with $\epsilon > 0$ and $\epsilon\{\sum_{l=1}^{\infty} (1/2^l)^p\}^{1/p} < 1/3$.

Let $\{\lambda_{i,j}\}$ be any sequence such that $\|\sum_{l,q} \lambda_{l,q} y_{i(l),j(i(l),q)}\| = 1$. For each $l = 1, 2, \ldots$, choose q_l such that $|\lambda_{l,q_l}| \geqq (4/5) \sup_q |\lambda_{l,q}|$. Then

$$1 = \left\|\sum_{h,s} \lambda_{h,s} y_{i(h),j(i(h),s)}\right\| = \left\{\sum_l \left\|\sum_{h,s} \lambda_{h,s} x^{i(l)}_{i(h),j(i(h),s)}\right\|^p\right\}^{1/p}.$$
$$\geqq \left\{\sum_l \left|\sum_{h,s} \lambda_{h,s} x^{i(l)}_{i(h),j(i(h),s)}(k(i(h), q_l))\right|^p\right\}^{1/p}$$
$$\geqq \left\{\sum_l |\lambda_{l,q_l} x^{i(l)}_{i(l),j(i(l),q_l)}(k(i(l), q_l))|^p\right\}^{1/p}$$
$$- \left\{\sum_l \left|\sum_{(h,s)\neq(l,q_l)} \lambda_{h,s} x^{i(h)}_{i(h),j(i(h),s)}(k(i(l), q_l))\right|^p\right\}^{1/p}$$
$$\geqq \left\{\sum_l \left|\frac{5}{3}\lambda_{l,q_l}\right|^p\right\}^{1/p} - \left\{\sum_l \left(\frac{\epsilon}{2^l}\right)^p\right\}^{1/p}$$
$$\geqq \frac{5}{3}\left\{\sum_l \left(\frac{4}{5}\sup_p |\lambda_{l,q}|\right)^p\right\}^{1/p} - \frac{1}{3}$$
$$= \frac{4}{3}\left\{\sum_l \left(\sup_q |\lambda_{l,q}|\right)^p\right\}^{1/p} - \frac{1}{3}.$$

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Hence

$$\|\sum \lambda_{l,q}y_{i(l),j(i(l),q)}\| \geqq \left\{\sum_l \left(\sup_q |\lambda_{l,q}|\right)^p\right\}^{1/p}.$$

PROPOSITION 21. *Let* $\{y_{i,j}\}_{i,j=1,2,\ldots}$ *be a sequence of elements in* Y *such that for all* $\lambda_{i,j}$ *in* R,

$$\left\|\sum_{i,j=1}^{n} \lambda_{i,j}y_{i,j}\right\| \leqq K\left(\sum_{i=1}^{n}\left(\sup_{1\leqq j\leqq n}|\lambda_{i,j}|\right)^p\right)^{1/p}$$

for some constant K *and for all* n. *Then for all* $\{\lambda_{i,j}\}_{i,j} \in Y$, $\sum_{i,j}\lambda_{i,j}y_{i,j}$ *converges in the* w^**-topology of* Y *to some element in* Y *with norm less than or equal to* $K(\sum_i(\sup_j|\lambda_{i,j}|)^p)^{1/p}$.

Proof. Suppose for some $f \in X = (\sum l_1)_E$, $E = c_0$ or l_q, $(1/p + 1/q = 1)$ such that $\{f(\sum_{i,j=1}^{n}\lambda_{i,j}y_{i,j})\}_n$ diverges. Then $\sum_{i,j=1}^{\infty}|\lambda_{i,j}f(y_{i,j})| = \infty$. Let $\epsilon_{i,j} = 1$ such that $\epsilon_{i,j}\lambda_{i,j}f(y_{i,j}) = |\lambda_{i,j}f(y_{i,j})|$, $i, j = 1, 2, \ldots$. Then

$$\left\|\sum_{i,j=1}^{n}\epsilon_{i,j}\lambda_{i,j}y_{i,j}\right\| \leqq K\left(\sum_{i=1}^{\infty}\left(\sup_j|\lambda_{i,j}|\right)^p\right)^{1/p} < \infty.$$

But $\lim_n f(\sum_{i,j=1}^{n}\epsilon_{i,j}\lambda_{i,j}y_{i,j}) = \infty$, which is impossible.

Let $\{f_{i,j}\}_j$ be the natural basis of l_1 which is in the ith coordinate of X. Then $\{f_{i,j}\}_{i,j}$ with the usual Cantor ordering, is an unconditional basis for X. Let $\alpha_{k,l} = \lim_n f_{k,l}(\sum_{i,j=1}^{n}\lambda_{i,j}y_{i,j})$, $k, l = 1, 2, \ldots$ and let $y = (x_1, x_2, \ldots)$ where $x_i = (\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,j}, \ldots)$, $i = 1, 2, \ldots$. Since $\{f_{i,j}\}$ is a basis of X, hence the bounded sequence $\{\sum_{i,j=1}^{n}\lambda_{i,j}y_{i,j}\}_n$ converges in the w^*-topology to y. It is well-known (cf. Banach, p. 123) that

$$\|y\| \leqq \limsup_n \left\|\sum_{i,j=1}^{n}\lambda_{i,j}y_{i,j}\right\| \leqq K\left\{\sum_i\left(\sup_j|\lambda_{i,j}|\right)^p\right\}^{1/p}.$$

Remark. The proof of the proposition yields that if $\{x_n, f_n\}$ is an unconditional basis of a Banach space X, then for any sequence $\{y_n\}$ in $[f_n]$ such that for some constant $k > 0$, $\|\sum_{i=1}^{n}\lambda_n y_n\| \leqq K\|\sum_{i=1}^{n}\lambda_i f_i\|$ for all scalars $\{\lambda_i\}$, then for any $\sum_n \lambda_n f_n \in [f_n]$, $\sum_{i=1}^{n}\lambda_i y_i$ converges in the w^*-topology to some element in X^* with norm less than or equal to $K\|\sum_{n=1}^{\infty}\lambda_n f_n\|$.

THEOREM 22. *For any operator* T *on* Y *either* TY *or* $(I - T)Y$ *contains a subspace isomorphic to* Y *which is complemented in* Y.

Proof. Let $\{e_{i,j}\}_j$ be the natural basis of c_0 in its n natural embedding of the ith coordinate of Y. Let $y_{i,j} = Te_{i,j} = (x_{i,j}^{(1)}, \ldots, x_{i,j}^{(n)}, \ldots)$, $i, j = 1, 2, \ldots$. By Theorem 12, and by taking a subsequence if necessary, we may assume that $\|x_{i,j}^{(i)}\| \geqq \frac{1}{2}$, $i, j = 1, 2, \ldots$. Since

$$\left\|\sum_{i,j=1}^{n}\lambda_{i,j}y_{i,j}\right\| \leqq \|T\|\left(\sum_{i=1}^{n}\left(\sup_{1\leqq j\leqq n}|\lambda_{i,j}|\right)^p\right)^{1/p}$$

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for all $\lambda_{i,j}$ in R, let $K = 4||T||$ and apply Corollary 20 to $\{4y_{i,j}\}_{i,j}$. We obtain sequences $\{i(l)\}_l$ and $\{j(i(l), q)\}_{l,q}$ such that

$$\left\|\sum_l \sum_q \lambda_{l,q} y_{i(l),j(i(l),q)}\right\| \geqq \left(\sum_l \left(\sup_q |\lambda_{l,q}|\right)^p\right)^{1/p}$$

for all $\{\lambda_{l,q}\}$ such that $\sum_{l,q} \lambda_{l,q} e_{l,q} \in Y$. Hence, by Proposition 21, the subspace of Y which consists of all w^*-limits of $\sum_{l,q} \lambda_{l,q} y_{i(l),j(i(l),q)}$ where $\sum \lambda_{l,q} e_{l,q} \in Y$ is isomorphic to Y. We now mimic the proof of the theorem in [**8**] to obtain a subspace in TY which is isomorphic to Y. Let $\{N_\gamma\}_{\gamma \in \Gamma}$ be an uncountable collection of infinite subsets of N such that $N_\alpha \wedge N_\beta$ is finite for all $\alpha \neq \beta$. For each $\gamma \in \Gamma$, let X_γ be all w^*-limits of $\sum_l \sum_{q \in N\gamma} \lambda_{l,q} y_{i(l),j(i(l),q)}$ where $\sum \lambda_{l,q} e_{l,q} \in Y$. Then X_γ is isomorphic to Y for all $\gamma \in \Gamma$. Suppose for each $\gamma \in \Gamma$ there exists $||x_\gamma|| = 1$ in $X_\gamma \backslash TY$. Let $x_\gamma = \sum_l \sum_{q \in N\gamma} \lambda_{l,q}^{(\gamma)} y_{i(l),j(i(l),q)}$. By the same reasoning as in [**8**], we conclude that for each $l = 1, 2, \ldots,$

$$\left\|\sum_{k=1}^n \epsilon_k (I - T) \sum_{q \in N\gamma_k} \lambda_{l,q}^{(\gamma_k)} y_{i(l),j(i(l),q)}\right\| \leqq ||I - T|| \cdot ||T||$$

for all $|\epsilon_k| = 1$ and all finite $\gamma_1, \ldots, \gamma_n$. Since Y has a countable total subset $\{f_k\}$ in Y^*, $||f_k|| = 1$, $k = 1, 2, \ldots,$ hence there exists a $\gamma \in \Gamma$ such that

$$f_k\left[(I - T) \sum_{q \in N\gamma} \lambda_{l,q}^{(\gamma)} y_{i(l),j(i(l),q)}\right] = 0, \quad l, k = 1, 2, \ldots.$$

Now

$$\sum_l \left(\sup_{q \in N\gamma} |\lambda_{l,q}^{(\gamma)}|\right)^p < \infty,$$

and given $\epsilon > 0$, there exists an n such that

$$\left(\sum_{l=n+1}^\infty \sup_{q \in N\gamma} |\lambda_{l,q}^{(\gamma)}|^p\right)^{1/p} < \epsilon.$$

Hence

$$\begin{aligned} |f_k(I - T)x_\gamma| &= \left|f_k(I - T)\left(x_\gamma - \sum_{l=1}^n \sum_{q \in N\gamma} \lambda_{l,q}^{(\gamma)} y_{i(l),j(i(l),q)}\right)\right| \\ &= \left|f_k(I - T) \sum_{l=n+1}^\infty \sum_{q \in N\gamma} \lambda_{l,q}^{(\gamma)} y_{i(l),j(i(l),q)}\right| \\ &\leqq ||f_k|| \cdot ||I - T|| \cdot ||T|| \cdot \left(\sum_{l=n+1}^\infty \left(\sup_{q \in N\gamma} |\lambda_{l,q}^{(\gamma)}|\right)^p\right)^{1/p} \\ &< ||f_k|| \cdot ||I - T|| \,.\, ||T|| \cdot \epsilon. \end{aligned}$$

Thus $f_k(I - T)x_\gamma = 0$ for all $k = 1, 2, \ldots,$ which is a contradiction since $\{f_k\}$ is total and $x_\gamma \neq Tx_\gamma$. Thus we have proved that TY contains a subspace

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X_γ which is isomorphic to Y. Since $Y \sim Y \oplus Y$, to show that $TY \sim Y$, it remains to observe that X_γ is complemented in Y. This follows immediately since the restriction of the natural projection P from Y to the subspace $E = [e_{i(l),j(i(l),q)}]_{l,q \in N\gamma}$ is an isomorphism from X_γ onto E.

COROLLARY 24. *The Banach spaces* $(l_\infty \oplus l_\infty \oplus \ldots)_{l_p}$, $1 \leqq p < \infty$ *are primary.*

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