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ON THE RELATIVE BEHAVIOUR OF MODULI OF SMOOTHNESS

S.P. ZHOU

Some general theorems concerning the relative behaviour of moduli of smoothness are established. In particular an open problem raised by Dickmeis, Nessel and van Wickeren is negated.

1. INTRODUCTION

Let $C_{2\pi}$ be the space of all continuous functions with period 2π. For $f \in C_{2\pi}$, let $\omega_r(f,t)$ denote the rth modulus of smoothness:

$$\omega_r(f,t) = \sup_{|h|\leqslant t} \max_x \left| \sum_{i=1}^{r} (-1)^{r-i} \binom{r}{i} f(x+ih) \right|,$$
$$\omega(f,t) = \omega_1(f,t).$$

One simple fact on the relative behaviour of moduli of smoothness is for $1 \leqslant s < r$,

$$\omega_r(f,t) \leqslant 2^{r-s}\omega_s(f,t).$$

Converse results are much more difficult. In 1927, Marchaud [1] showed that for $1 \leqslant s < r$,

$$\omega_s(f,t) \leqslant C(s,r)t^s \int_t^1 \frac{\omega_r(f,u)}{u^{s+1}}du + O(t^s), t > 0$$

(here and for the rest of the paper, C(x) always indicates a positive constant, which at most depends upon x). In particular,

$$(1) \qquad \omega_s(f,t) = O\Big(\omega_r\big(f,t^{s/r}\big)\Big) = O\big(t^{s-r}\omega_r(f,t)\big),$$

and

$$(2) \qquad \omega(f,t) = O\Big(\omega_2\big(f,t^{1/2}\big)\Big) = O\big(t^{-1}\omega_2(f,t)\big).$$

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It is thus natural to enquire whether the inequality (1) is sharp. Dickmeis, Nessel and van Wickeren (see [2]) studied a particular case, and proved, for each $0 < \alpha, \beta < 1$, that

$$\limsup_{t\to 0+} \frac{t^{\beta}\omega(f_{\alpha\beta}, t)}{\omega_2(f_{\alpha\beta}, t)} > 0 \tag{3}$$

for some function $f_{\alpha\beta} \in C_{2\pi}$ satisfying the Lipschitz condition

$$\omega_2(f_{\alpha\beta}, t)\begin{cases} = O(t^{\alpha}), \\ \neq o(t^{\alpha}), \end{cases} \quad t \to 0+. \tag{4}$$

This result says little about the sharpness of (2), and in [2] they raised the following problem:

PROBLEM. Given $0 < \alpha < 1$, does there exist a function $f \in C_{2\pi}$, satisfying (4), such that (3) holds true for $\beta = 1$?

The present paper is devoted to the establishment of a general theorem related to this topic to show that Marchaud's inequality (1) is only sharp in very trival cases $\omega_r(f, t) = O(t^r)$, which in particular negates the open problem above. The fact that the corresponding conclusion involving derivatives cannot hold true is proved in Section 3. Finally we give a generalisation to (3) in Section 4.

2. Sharpness on Marchaud's Inequality

THEOREM 1. *If $f \in C_{2\pi}$, then*

$$\lim_{t\to 0+} \frac{t^{r-s}\omega_s(f, t)}{\omega_r(f, t)} = 0 \tag{5}$$

if and only if

$$r > s \tag{6}$$

and

$$\lim_{t\to 0+} t^{-r}\omega_r(f, t) = \infty. \tag{7}$$

PROOF: First we show under conditions (6) and (7) that (5) holds true. We need the following basic results:

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If $f \in C_{2\pi}$, then

$$\omega_r(f, \lambda t) \leqslant (\lambda + 1)^r \omega_r(f, t), \tag{8}$$

and for convenience, we rewrite Marchaud's inequality in the following form:

$$\omega_s\Big(f, (n+1)^{-1}\Big) \leqslant C(s, r) n^{-s} \sum_{j=0}^{n} (j+1)^{s-1} \omega_r\Big(f, (j+1)^{-1}\Big). \tag{9}$$

From (8), we notice that if $n \leqslant t^{-1} < n+1$, then for any $p > 0$, there is some constant C such that

$$1 \leqslant \frac{\omega_p(f, t)}{\omega_p\Big(f, (n+1)^{-1}\Big)} \leqslant C.$$

Therefore under the conditions of Theorem 1, we need only prove

$$\lim_{n \to \infty} \frac{n^{s-r} \omega_s\Big(f, (n+1)^{-1}\Big)}{\omega_r\Big(f, (n+1)^{-1}\Big)} = 0.$$

We establish

Lemma 1. *Let (6), (7) hold true. Then*

$$\sum_{k=0}^{n} (k+1)^{s-1} \omega_r\Big(f, (k+1)^{-1}\Big) = o\Big(n^r \omega_r\Big(f, (n+1)^{-1}\Big)\Big), \quad n \to \infty. \tag{10}$$

Proof: Set

$$M_n = \min\left\{ \Big(n^r \omega_r\Big(f, (n+1)^{-1}\Big)\Big)^{\frac{1}{2s}}, n+1 \right\},$$

from (7),

$$M_n \to \infty, \quad n \to \infty.$$

Then

$$\sum_{k=0}^{n} (k+1)^{s-1} \omega_r\Big(f, (k+1)^{-1}\Big) = \sum_{k=0}^{M_n - 1} + \sum_{k=M_n}^{n} = \Sigma_1 + \Sigma_2,$$

$$\Sigma_1 \leqslant C \sum_{k=0}^{M_n - 1} (k+1)^{s-1} = O(M_n^s) = O\left(\Big(n^r \omega_r\Big(f, (n+1)^{-1}\Big)\Big)^{1/2}\right),$$

and by (8),

$$\Sigma_2 \leqslant n^r \omega_r\Big(f, (n+1)^{-1}\Big) \sum_{k=M_n}^{n} (k+1)^{s-r-1} = O\Big(M_n^{s-r} n^r \omega_r\Big(f, (n+1)^{-1}\Big)\Big),$$

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so (10) holds true. □

LEMMA 2. *The condition (7) is equivalent to the following condition:*

$$\limsup_{t\to 0+} t^{-r}\omega_r(f,t) = \infty.$$

By (8) follows that

$$2^r t^{-r}\omega_r(f,t) \geqslant n_j^r \omega_r(f, n_j^{-1}),$$

of which Lemma 2 is a simple consequence.

Lemma 1 shows the sufficiency part of Theorem 1. For the necessity part, if $r = s$, then

$$\frac{t^{r-s}\omega_s(f,t)}{\omega_r(f,t)} \equiv 1,$$

while if $r < s$, because of Marchaud's inequality,

$$\frac{\omega_r(f,t)}{t^{r-s}\omega_s(f,t)} = O(1).$$

Furthermore, if (7) does not hold true, due to Lemma 2, it follows that

$$t^{-r}\omega_r(f,t) = O(1),$$

together with $\omega_s(f,t) \geqslant C(s)t^s$, we have

$$\liminf_{t\to 0+} \frac{t^{r-s}\omega_s(f,t)}{\omega_r(f,t)} > 0.$$

Theorem 1 is thus completed. □

REMARKS.

1. Theorem 1 reveals that except for the extremely trivial case $\omega_r(f,t) \sim t^r$, Marchaud's inequality can be improved to

$$\omega_s(f,t) = o\big(t^{s-r}\omega_r(f,t)\big)$$

for $1 \leqslant s < r$.

2. The negative answer to the problem raised in [2] is obviously a direct corollary of Theorem 1.

3. In nonperiodic spaces and L^p spaces there are corresponding results with the same proofs.

4. Considering Marchaud's result, one may naturally ask if the inequality

$$\omega_s(f,t) = o\Big(\omega_r\big(f,t^{s/r}\big)\Big), \quad t \to 0+$$

holds true under the conditions (6), (7)? The following example shows a negative answer:

$$f(x) = \sum_{n=1}^{\infty} n^{-2}\cos(3^n x).$$

In fact, let $E_n(f)$ denote the nth best approximation by trigonometric polynomials for $f \in C_{2\pi}$; it is not difficult to verify that

$$\omega_s(f,3^{-n}) \geqslant CE_{3^n}(f) \sim Cn^{-1},$$

Meanwhile for any m, $3^n < m \leqslant 3^{n+1}$,

$$\begin{aligned}\omega_r(f,m^{-1}) &= O\left(m^{-r}\left(\sum_{i=1}^{n} i^{-1}\sum_{j=3^{i-1}}^{3^i-1} j^{r-1} + n^{-1}\sum_{j=3^n}^{m} j^{r-1}\right)\right)\\ &= O\big(m^{-r}n^{-1}3^{rn} + n^{-1}\big) = O\big(\log^{-1} m\big);\end{aligned}$$

hence

$$\limsup_{n\to\infty}\frac{\omega_s(f,3^{-n})}{\omega_r\big(f,3^{-sn/r}\big)} > 0.$$

3. The Case Involving Derivatives

If considering derivatives, a natural question to ask is whether we can replace $t^m\omega_{s-m}\big(f^{(m)},t\big)$ for $\omega_s(f,t)$ in Theorem 1 for $f \in C_{2\pi}^m$ and $s > m$? The following theorem shows a negative answer.

Theorem 2. *Let $r > m$, $s \geqslant 1$; then there exists a function $f \in C_{2\pi}^s$ such that*

$$\limsup_{t\to 0+} t^{-s-1}\omega_r(f,t) = \infty,$$

and

$$\limsup_{t\to 0+}\frac{t^{r-m}\omega_m\big(f^{(s)},t\big)}{\omega_r(f,t)} = \infty.$$

Proof: For given r, m, s, take α and ε, $0 < \alpha$, $\varepsilon < 1$ such that

$$\frac{s}{r-m} > \varepsilon > \frac{\alpha}{m-\frac{1}{2}}. \tag{11}$$

Write

$$f_n(x) = n^{-s-\alpha}\cos\left(nx + \frac{s\pi}{2}\right).$$

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We select a subsequence $\{n_j\}$ from $\mathbf{N}$ by induction. Let $n_1 = 1$. Choose

$$n_{2k} > \max\{n_{2k-1}^{2(r-s-\alpha)}, 2n_{2k-1}\}, \tag{12}$$

$$n_{2k+1} = n_{2k}^{1/\varepsilon}. \tag{13}$$

Now define
$$f(x) = \sum_{k=0}^{\infty} f_{n_{2k+1}}(x).$$

Clearly, $f \in C_{2\pi}^s$,
$$\limsup_{t\to 0+} t^{-s-1}\omega_r(f,t) = \infty.$$

From the well-known Jackson theorem and (12), (13),

$$\begin{aligned}\omega_m\left(f^{(s)}, n_{2k}^{-1}\right) &\geqslant C(m)E_{n_{2k}}\left(\sum_{j=0}^{k} f_{n_{2j+1}}^{(s)}\right) - 2^m \sum_{j=k+1}^{\infty} \|f_{n_{2j+1}}^{(s)}\| \\ &= C(m)n_{2k+1}^{-\alpha} - o\left(n_{2k+1}^{-\alpha}\right).\end{aligned} \tag{14}$$

On the other hand,

$$\begin{aligned}\omega_r\left(f, n_{2k}^{-1}\right) &= O\left(\sum_{j=0}^{k-1} n_{2k}^{-r}\|f_{n_{2j+1}}^{(r)}\|\right) + \left(\sum_{j=k}^{\infty}\|f_{n_{2j+1}}\|\right) \\ &= O\left(\sum_{j=0}^{k-1} n_{2j+1}^{r-s-\alpha} n_{2k}^{-r}\right) + O\left(\sum_{j=k}^{\infty} n_{2j+1}^{-s-\alpha}\right);\end{aligned}$$

by (12), (13),

$$\omega_r\left(f, n_{2k}^{-1}\right) = O\left(n_{2k}^{-r+\frac{1}{2}}\right) + O\left(n_{2k+1}^{-s-\alpha}\right),$$

together with (11), (13) and (14) we get

$$\begin{aligned}\frac{\omega_r\left(f, n_{2k}^{-1}\right)}{n_{2k}^{-r+m}\omega_m\left(f^{(s)}, n_{2k}^{-1}\right)} &= O\left(n_{2k}^{-m+(1/2)} n_{2k+1}^{\alpha}\right) + O\left(n_{2k}^{r-m} n_{2k+1}^{-s}\right) \\ &= O\left(n_{2k}^{-m+(1/2)+(\alpha/\varepsilon)}\right) + O\left(n_{2k}^{r-m-(s/\varepsilon)}\right) = o(1), \quad k\to\infty,\end{aligned}$$

Theorem 2 is thus proved. □

4. A Generalisation to the Result of Dickmeis, Nessel and Van Wickeren

Theorem 3. *Let $1 \leqslant s < r$, $\omega(t)$ be a positive increasing function on $(0,\infty)$ with the properties $\lim_{t\to 0+}\omega(t) = 0$ and $\omega(\lambda t) \leqslant (\lambda+1)^r\omega(t)$, and $\{\rho_n\}$ be a sequence of positive numbers such that*

$$\limsup_{n\to\infty} n^{r-s}\rho_n = \infty.$$

Then there exists a function $f \in C_{2\pi}$, satisfying

$$\omega_r(f,t)\begin{cases} = O(\omega(t)), \\ \neq o(\omega(t)), \end{cases} t \to 0+, \tag{10}$$

such that $$\limsup_{t\to 0+} \frac{\rho_{[1/t]}\omega_s(f,t)}{\omega_r(f,t)} > 0.$$

Proof: If $\omega(t) = O(t^r)$, Theorem 3 obviously holds true. Suppose now $\omega(t) \neq O(t^r)$; then as Lemma 2 indicated,

$$\lim_{t\to 0+} t^{-r}\omega(t) = \infty.$$

Without loss of generality, assume also that

$$\lim_{n\to\infty} n^{r-s}\rho_n = \infty,$$

otherwise we pass to a subsequence.

We begin by selecting a subsequence of natural numbers $\{n_j\}$ as follows. Let $\varepsilon_n = n^{-r/2}\omega^{-1/2}(n^{-1})$, $n_1 = 1$, and by induction define $\{n_j\}$ with the following properties:

$$\omega\left(n_{j+1}^{-1}\right) < \frac{\omega\left(n_j^{-1}\right)}{2},$$

$$\sum_{j=1}^{k-1} n_{2j}^r\omega\left(n_{2j}^{-1}\right) \leqslant \varepsilon_{n_{2k}}^{-1},$$

$$\omega\left(n_{2k}^{-1}\right) \leqslant \min\{\varepsilon_{n_{2k-2}}\omega\left(n_{2k-2}^{-1}\right),\ n_{2k-1}^{-s}\rho_{n_{2k-1}},\ \varepsilon_{n_{2k-1}}n_{2k-1}^{-s}\},$$

$$\sum_{j=1}^{k} n_{2j}^r\omega\left(n_{2j}^{-1}\right) \leqslant n_{2k+1}^{r-s}\rho_{n_{2k+1}}.$$

Define $$f(x) = \sum_{j=1}^{\infty}\omega\left(n_{2j}^{-1}\right)\cos n_{2j}\left(x + \frac{s\pi}{2}\right).$$

It is not difficult to verify that $f(x)$ is the required function. We omit the details. □

Note added in proof. The author learned recently from Professor V. Totik that he proved a result similar to Theorem 1 in this paper as well.

References

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Department of Mathematics, Statistics
and Computing Science
Dalhousie University
Halifax NS
Canada B3H 3J5

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