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# On certain function spaces and group structures

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We generalize a result of Walter Rudin about the structure of a compact abelian group $G$ for which $C(G) + H^{\infty}(G)$ is a closed subalgebra of $L^{\infty}(G)$ .

## 1. Introduction

Sarason ([5], [6]) showed that $C(T) + H^{\infty}(T)$ is a closed subalgebra of $L^{\infty}(T)$ , and that $C_u(R) + H^{\infty}(R)$ is a closed subalgebra of $L^{\infty}(R)$ . On the other hand, for a compact abelian group $G$ whose dual group $\hat{G}$ is ordered, Rudin showed in [4] that $C(G) + H^{\infty}(G)$ is a closed subalgebra of $L^{\infty}(G)$ if and only if $G \cong T$ . Moreover, Rudin showed the following in [4]: for a locally compact abelian group $G$ , $C_u(G) + H$ is always a closed subspace of $L^{\infty}(G)$ for each translation invariant and weak*-closed subspace $H$ of $L^{\infty}(G)$ . For a locally compact abelian group $G$ whose dual $\hat{G}$ is an algebraically ordered group, we define the space $H_P^{\infty}(G)$ . Our purpose in this paper is to investigate the relationship between the fact that $H_P^{\infty}(G) + C_u(G)$ becomes a subalgebra of $L^{\infty}(G)$ and the group structures of $G$ . Let $G$ be a locally compact abelian group with a dual group $\hat{G}$ . $L^1(G)$ denotes the space of all integrable functions on $G$ , and $L^{\infty}(G)$ is the space of all essentially bounded measurable functions on $G$ .

Let $M(G)$ be the Banach algebra of all bounded regular measures on $G$ , and $C_u(G)$ the space of all bounded uniformly continuous functions on

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$G$ .

DEFINITION 1. Let $G$ be a locally compact abelian group. $G$ is called an algebraically ordered group if there exists a subsemigroup $P$ of $G$ satisfying the (AO)-condition, namely,

(i) $P \cup (-P) = G$ and

(ii) $P \cap (-P) = \{0\}$ .

$G$ is an algebraically ordered group if and only if it is torsion-free (see, for example, [3]).

DEFINITION 2. Let $G$ be a locally compact abelian group such that $\hat{G}$ is an algebraically ordered group. Suppose $P$ is a subsemigroup of $\hat{G}$ with the (AO)-condition. We define $H^1_P(G)$, $H^\infty_P(G)$ , and $M^a_P(G)$ as follows:

$$H^1_P(G) = \{f \in L^1(G);\ \hat{f}(\gamma) = 0 \ \text{ for } \ \gamma \notin P\} \ ;$$

$$H^\infty_P(G) = \left(H^1_P(G)\right)^{\perp} = \left\{g \in L^\infty(G);\ \int_G f(x)g(x)dx = 0 \ \text{ for } \ f \in H^1_P(G)\right\} \ ;$$

$$M^a_P(G) = \{\mu \in M(G);\ \hat{\mu}(\gamma) = 0 \ \text{ for } \ \gamma \notin P\} \ .$$

REMARK 1. If $G = R$ and $P = \{x \in R;\ x \geq 0\}$ , then $H^1_P(R)$ and $H^\infty_P(R)$ are the usual $H^1(R)$ and $H^\infty(R)$ respectively.

If $G = T$ and $P = \{n \in Z;\ n \geq 0\}$ , then $H^\infty_P(T)$ is the space $H^\infty_0(T)$ . But, in this case, we note that $H^\infty_P(T) + C(T) = H^\infty(T) + C(T)$ .

DEFINITION 3. Let $\Sigma$ be a set of functions on $G$ . We call $\Sigma$ *invariant* if $\tau_a f$ belongs to $\Sigma$ for all $f \in \Sigma$ and $a \in G$ , where $\tau_a f(x) = f(x-a)$ .

PROPOSITION 1. *Suppose $G$ is a locally compact abelian group such that $\hat{G}$ is an algebraically ordered group. Let $P$ be a subsemigroup of $\hat{G}$ with the* (AO)*-condition such that it is not dense in $\hat{G}$ . Then $H^\infty_P(G)$ is a translation invariant and weak*-closed subalgebra of $L^\infty(G)$ . Hence, in particular, by* [4], Theorem 3.3, *$H^\infty_P(G) + C_u(G)$ is a closed subspace of $L^\infty(G)$ .*

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Proof. Evidently $H_P^\infty(G)$ is a weak*-closed subspace. Since $H_P^1(G)$ is translation invariant, we have

$$\int_G f_a(x)g(x)dx = \int_G f(x)g_{-a}(x)dx$$
$$= 0$$

for each $a \in G$, $f \in H_P^\infty(G)$, and $g \in H_P^1(G)$. Hence $H_P^\infty(G)$ is translation invariant. Finally, we prove that $H_P^\infty(G)$ is an algebra. For $g \in H_P^\infty(G)$ and $h \in H_P^1(G)$, $gh$ belongs to $H_P^1(G)$.

Hence, for $f, g \in H_P^\infty(G)$, we have

$$\int_G f(x)g(x)h(x)dx = 0 \quad \text{for every } h \in H_P^1(G) .$$

Therefore $fg$ belongs to $H_P^\infty(G)$. //

Our purpose in this paper is to prove the following theorem.

THEOREM. *Let $G$ be a nondiscrete locally compact abelian group such that $\hat{G}$ is an algebraically ordered group. Let $P$ be a subsemigroup of $\hat{G}$ with the* (AO)*-condition such that it is not dense in $\hat{G}$. Then the following are equivalent:*

*(i) $C_u(G) + H_P^\infty(G)$ is a closed subalgebra of $L^\infty(G)$;*

*(ii) $G$ admits one of the following structures,*

*(a) $G = R$,*

*(b) $G = R \oplus D$,*

*(c) $G = T$,*

*(d) $G = T \oplus D$,*

*where $D$ is a discrete abelian group such that $\hat{D}$ is torsion-free.*

REMARK 2. If $P$ is dense in $\hat{G}$, $H_P^1(G) = \{0\}$. Hence $C_u(G) + H_P^\infty(G)$ is always an algebra, since $H_P^\infty(G) = L^\infty(G)$.

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If $G$ is a discrete abelian group such that $\hat{G}$ is torsion-free, then $P$ is necessarily dense in $\hat{G}$. Therefore $C_u(G) + H_P^\infty(G)$ is always an algebra.

## 2. Proof of the theorem

We prove the theorem by using the structure theorem of locally compact abelian groups. Before proving the theorem, we investigate whether $C_u(G) + H_P^\infty(G)$ becomes an algebra or not, for some special groups.

LEMMA 2. *Let $G$ be a noncompact locally compact abelian group such that $\hat{G}$ is an algebraically ordered group. Let $P$ be a subsemigroup of $\hat{G}$ with the* (AO)*-condition such that it is not dense in $\hat{G}$. Then the following are established:*

*(i) for $F \in L^\infty(G)$ and $\mu \in M_P^a(G)$, $F * \mu \in H_P^\infty(G)$;*

*(ii) for $F \in H_P^\infty(G)$ and $\mu \in M(G)$, $F * \mu \in H_P^\infty(G)$.*

Proof. *(i)* For $f \in H_P^1(G)$, we put $\tilde{f}(x) = f(-x)$. Then $\tilde{f}$ belongs to $H^1_{(-P)}(G)$. Hence we have

$$\int_G F * \mu(x) f(x)dx = \int_G F * \mu(x)\tilde{f}(-x)dx$$
$$= (F*\mu) * \tilde{f}(0)$$
$$= F * (\mu*\tilde{f})(0) .$$

Since $\hat{G}$ is not discrete, $\mu * \tilde{f} = 0$. Hence $F * \mu \in H_P(G)$.

*(ii)* For $\Phi \in H_P^1(G)$, $F \in H_P^\infty(G)$, and $\mu \in M(G)$, we have

$$\int_G F * \mu(x)\Phi(x)dx = \int_G \int_G F(x-y)d\mu(y)\Phi(x)dx$$
$$= \int_G \int_G F(x-y)\Phi(x)dxd\mu(y) .$$

On the other hand, by Proposition 1, $H_P^\infty(G)$ is translation invariant. Hence $F(x-y) \in H_P^\infty(G)$ for every $y \in G$. Hence $F * \mu$ belongs to $H_P^\infty(G)$. //

**LEMMA 3.** *Let $G$ be a locally compact abelian group such that $\hat{G}$ is an algebraically ordered group. Let $P$ be a subsemigroup of $\hat{G}$ with the (AO)-condition such that it is not dense in $\hat{G}$. For $f \in H_P^\infty(G)$ and $\mu \in M_{(-P)}^a(G)$, we have $f * \mu = 0$.*

Proof. For $g \in L^1(G)$ we have

$$\begin{aligned}\int_G f * \mu(x)g(x)dx &= \int_G f * \mu(x)\tilde{g}(-x)dx \\ &= (f*\mu) * \tilde{g}(0) \\ &= f * (\mu*\tilde{g})(0) \\ &= \int_G f(x)\mu * \tilde{g}(-x)dx .\end{aligned}$$

On the other hand, $(\mu*\tilde{g})^{\sim} \in H_P^1(G)$. Indeed, $(\mu*\tilde{g})^{\sim\wedge}(\gamma) = \hat{\mu}(-\gamma)\hat{\tilde{g}}(-\gamma) = 0$ if $\gamma \notin P$. Hence $\int_G f * \mu(x)g(x)dx = 0$ for all $g \in L^1(G)$; that is, $f * \mu = 0$. //

**LEMMA 4.** *Let $G$ be a locally compact abelian group. Let $\mu$ and $\nu$ be in $M(G)$ and $g$ in $L^\infty(G)$. If $\hat{\nu} = 1$ on $\operatorname{supp}(\hat{\mu}) + \gamma_0$ for some $\gamma_0 \in \hat{G}$, then we have*

$$(\gamma_0 g*\mu) * \nu = \gamma_0 g * \mu .$$

Proof.

$$\begin{aligned}(\gamma_0 g*\mu) * \nu(x_0) &= \int_G (x_0-x, \gamma_0)g * \mu(x_0-x)d\nu(x) \\ &= \int_G \int_G (x_0-x-y, \gamma_0)g(x_0-x-y)d(\gamma_0\mu)(y)d\nu(x) \\ &= (\gamma_0 g) * ((\gamma_0\mu)*\nu)(x_0) \\ &= (\gamma_0 g) * (\gamma_0\mu)(x_0) \\ &= \int_G (x_0-x, \gamma_0)g(x_0-x)(x, \gamma_0)d\mu(x) \\ &= (x_0, \gamma_0)g * \mu(x_0) . \quad //\end{aligned}$$

The following lemma is a sufficient condition for $C_u(G) + H_P(G)$ not

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to become an algebra.

LEMMA 5. *Let* $G$ *be a noncompact locally compact abelian group such that* $\hat{G}$ *is an algebraically ordered group. Let* $P$ *be a subsemigroup of* $\hat{G}$ *with the* (AO)*-condition such that it is not dense in* $\hat{G}$ *. If there exist* $\mu \in M_P^a(G)$ *,* $\nu \in M_{(-P)}^a(G)$ *,* $g \in L^\infty(G)$ *, and* $\gamma_0 \in \hat{G}$ *such that* $g * \mu \notin C_u(G)$ *and* $\hat{\nu} = 1$ *on* $\operatorname{supp}(\hat{\mu}) + \gamma_0$ *, then* $C_u(G) + H_P^\infty(G)$ *is not an algebra.*

Proof. Suppose $C_u(G) + H_P^\infty(G)$ is an algebra. Since $\gamma_0$ is in $C_u(G)$ and $g * \mu$ in $H_P^\infty(G)$ by Lemma 2, there exist $H \in C_u(G)$ and $K \in H_P^\infty(G)$ such that

$$(x, \gamma_0)g * \mu(x) = H(x) + K(x) .$$

Hence, by Lemmas 3 and 4, we have

$$\begin{aligned} \gamma_0 g * \mu &= (\gamma_0 g * \mu) * \nu \\ &= H * \nu + K * \nu \\ &= H * \nu . \end{aligned}$$

Since $H \in C_u(G)$ , $H * \nu$ belongs to $C_u(G)$ . On the other hand, $\gamma_0 g * \mu$ does not belong to $C_u(G)$ . This is a contradiction. //

LEMMA 6. *Let* $F$ *be a compact abelian torsion-free group. Let* $P$ *be a subsemigroup of* $R^n \oplus F$ *with the* (AO)*-condition such that it is not dense in* $R^n \oplus F$ *. Then* $P$ *includes* $\mathring{P}_{R^n} \oplus F$ *, where* $P_{R^n} = P \cap R^n$ *and* $\mathring{P}_{R^n}$ *denotes its interior.*

Proof. Since $P$ is necessarily dense in $F$ , $P$ is not dense in $R^n$ . Hence there exists a unitary transformation $\tau$ on $R^n$ such that $\tau(\mathring{P}_{R^n}) = \{x = (x_1, x_2, \ldots, x_n) \in R^n ; x_1 > 0\}$ . Define an automorphism $\tilde{\tau}$ on $R^n \oplus F$ by $\tilde{\tau}(z, t) = (\tau(z), t)$ for $(z, t) \in R^n \oplus F$ . Then $\tilde{\tau}(P)$ is a subsemigroup of $R^n \oplus F$ with the (AO)-condition such that it is not dense in $R^n \oplus F$. Let '$\leq_P$' and '$\leq_{\tilde{\tau}(P)}$' denote the orders on $R^n \oplus F$ induced by $P$ and $\tilde{\tau}(P)$ respectively. Suppose there exist $y = (y_1, \ldots, y_n) \in \overset{\circ}{P}_{R^n}$ and $s \in F$ such that $y \lneq_P s$. Then $\tilde{\tau}(y) \lneq_{\tilde{\tau}(P)} \tilde{\tau}(s) = s$. Let $\tilde{\tau}(y) = (x_1, \ldots, x_n)$ $(x_1 > 0)$. Then, for $z = (z_1, \ldots, z_n)$ with $z_1 \leq x_1$, we obtain that

$$z = (0, s) + (z, -s) \in \overline{[-\tilde{\tau}(P)]} + \overline{[-\tilde{\tau}(P)]} = \overline{[-\tilde{\tau}(P)]} .$$

Since $\overline{[-\tilde{\tau}(P)]}$ is a semigroup, $R^n$ is included in $\overline{[-\tilde{\tau}(P)]}$. Therefore $-\tilde{\tau}(P)$ is dense in $R^n$. This is a contradiction. Hence $P \supset \overset{\circ}{P}_{R^n} \oplus F$.

REMARK 3. We note that Lemma 6 remains valid if we replace $R^n$ by $R \oplus Z$.

LEMMA 7. *Let $G$ be a nondiscrete locally compact abelian group such that $\hat{G}$ is an algebraically ordered group. Let $P$ be a subsemigroup of $\hat{G}$ with the* (AO)*-condition such that it is not dense in $\hat{G}$. Let $\tau$ be an automorphism on $\hat{G}$. Then the following are equivalent:*

(1) $C_u(G) + H_P^\infty(G)$ *is an algebra;*

(2) $C_u(G) + H_{\tau(P)}^\infty(G)$ *is an algebra.*

Proof. Let $\tau^*$ be the dual automorphism of $\tau$, that is, $(\tau^*(x), \gamma) = (x, \tau(\gamma))$ for $x \in G$ and $\gamma \in \hat{G}$. We only prove (1) implies (2).

Claim 1. For $f \in H^1_{\tau(P)}(G)$, $f \circ \tau^{*-1} \in H^1_P(G)$. Indeed, for $\gamma \notin P$, we have

$$\begin{aligned} \widehat{f \circ \tau^{*-1}}(\gamma) &= \int_G f(\tau^{*-1}(x))(-x, \gamma)dx \\ &= k \int_G f(x)(\tau^*(-x), \gamma)dx \\ &= k \int_G f(x)(-x, \tau(\gamma))dx \\ &= k\hat{f}(\tau(\gamma)) , \end{aligned}$$

where $k$ is a constant depending on $\tau^{*-1}$.

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Claim 2. For $g \in H_P^\infty(G)$ , $g \circ \tau^* \in H_{\tau(P)}(G)$ . Indeed, for $f \in H^1_{\tau(P)}(G)$ , since $f \circ \tau^{*-1} \in H_P^1(G)$ , we have

$$\int_G g\left(\tau^*(x)\right)f(x)dx = k \int_G g(x)f\left(\tau^{*-1}(x)\right)dx$$
$$= 0 .$$

In the same way, for $g \in H^\infty_{\tau(P)}(G)$ , we have $g \circ \tau^{*-1} \in H_P(G)$ .

Therefore, since $\left(\tau^{-1}\right)^* \circ \tau^* = I$ , (1) implies (2) is proved. //

PROPOSITION 8. *Let* $n \geq 2$ . *Let* $P_1$ *be a subsemigroup of* $R^n$ *with the* (AO)*-condition such that* $P_1 \supset \left\{x = \left(x_1, x_2, \ldots, x_n\right) \in R^n;\ x_1 > 0\right\}$ . *Then* $C_u\left(R^n\right) + H^\infty_{P_1}\left(R^n\right)$ *is not an algebra.*

Proof. Define measures $\mu \in M^a_{P_1}\left(R^n\right)$ and $\nu \in M^a_{\left(-P_1\right)}\left(R^n\right)$ as follows:

$$d\mu\left(x_1, x'\right) = h\left(x_1\right)dx_1 \times d\delta_0(x') ,$$
$$d\nu\left(x_1, x'\right) = k\left(x_1\right)dx_1 \times d\delta_0(x') ,$$

where $h$ is a function in $L^1(R)$ with $\operatorname{supp}(\hat{h}) \subset (1, 2)$ , $k$ is a function in $L^1(R)$ such that $\operatorname{supp}(\hat{k}) \subset (-\infty, 0)$ and $\hat{k} = 1$ on $[-2, -1]$ , and $\delta_0$ denotes the Dirac measure at 0 in $R^{n-1}$ .

We choose functions $g_1 \in L^\infty(R)$ and $g_2 \in L^\infty\left(R^{n-1}\right)$ such that $g_1 * h \neq 0$ and $g_2 \notin C_u\left(R^{n-1}\right)$ . Define a function $g \in L^\infty\left(R^n\right)$ by $g\left(x_1, x'\right) = g_1\left(x_1\right)g_2(x')$ . Moreover, we define $\gamma_0 \in \hat{R}^n = R^n$ by $\gamma_0(x) = e^{-i3x_1}$ , where $x = \left(x_1, x_2, \ldots, x_n\right) \in R^n$ . Then $g * \mu\left(x_1, x'\right) = g_1 * h\left(x_1\right)g_2(x\ ) \notin C_u\left(R^n\right)$ and $\hat{\nu} = 1$ on $\operatorname{supp}(\hat{\mu}) + \gamma_0$ . Hence, by Lemma 5, $C_u\left(R^n\right) + H^\infty_{P_1}\left(R^n\right)$ is not an algebra. //

PROPOSITION 9. *Let* $n \geq 2$ *and* $P$ *be a subsemigroup of* $R^n$ *with the* (AO)*-condition such that it is not dense in* $R^n$ . *Then* $C_u(R^n) + H_P^\infty(R^n)$ *is not an algebra.*

Proof. There is a unitary transformation $\tau$ on $R^n$ such that $\tau(P) \supset \left\{x = (x_1, x_2, \ldots, x_n) \in R^n;\ x_1 > 0\right\}$ . Hence, by Lemma 7 and Proposition 8, $C_u(R^n) + H_P^\infty(R^n)$ is not an algebra. //

PROPOSITION 10. *Let* $G$ *be a locally compact abelian group such that* $\hat{G}$ *is an algebraically ordered group and* $\hat{G} \cong R^n \oplus F$ , *where* $n \geq 2$ , *and* $F$ *is a locally compact abelian group containing* $F_0$ *as a compact open subgroup. Let* $P$ *be a subsemigroup of* $\hat{G}$ *with the* (AO)*-condition such that it is not dense in* $\hat{G}$ . *Then* $C_u(G) + H_P^\infty(G)$ *is not an algebra.*

Proof. Case 1: $F_0 = \{0\}$ . In this case, $F$ is a discrete abelian group, and $P_{R^n}$ $(= R^n \cap P)$ is a subsemigroup of $R^n$ with the (AO)-condition.

Case 1.1: $F_0 = \{0\}$ and $P_{R^n}$ is not dense in $R^n$ . Since $n \geq 2$ , by Proposition 9, $C_u(R^n) + H_{P_{R^n}}^\infty(R^n)$ is not an algebra. Hence there exist functions $f \in C_u(R^n)$ and $g \in H_{P_{R^n}}^\infty(R^n)$ such that

$fg \notin C_u(R^n) + H_{P_{R^n}}^\infty(R^n)$ . Let $F$ and $L$ be functions on $G$ defined by $F(x, y) = f(x)$ and $L(x, y) = g(x)$ for $(x, y) \in R^n \oplus \hat{F}$ . Evidently, $F$ belongs to $C_u(G)$ .

CLAIM 1. $L$ belongs to $H_P^\infty(G)$ .

Indeed, for each $A \in H_P^1(G)$ and positive integer $n$ , there exists $B_n \in L^1(G)$ such that $\hat{B}_n$ has compact support and

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$$\|A - A * B_n\|_1 < 1/n .$$

Since $F$ is discrete, $\text{proj}_F(\text{supp}(A \hat{*} B_n)) = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ for some $\gamma_1, \ldots, \gamma_m \in F$, where $\text{proj}_F$ denotes the projection from $\hat{G}$ onto $F$. There exist functions $\varphi_k \in L^1(R^n)$ $(k = 1, \ldots, m)$ such that $A * B_n(x, y) = \sum_{k=1}^{n} \varphi_k(x)(y, \gamma_k)$. If $0 \notin \{\gamma_1, \ldots, \gamma_k\}$, then

$$\int_G LA * B_n dm_G = \int_{R^n} g(x) \sum_{k=1}^{m} \varphi_k(x) \int_{\hat{F}} (y, \gamma_k) dm_F(y) dx$$
$$= 0 .$$

If $0 \in \{\gamma_1, \ldots, \gamma_m\}$, we assume $\gamma_1 = 0$ without loss of generality. Then $\varphi_1$ must belong to $H^1_{P_{R^n}}(R^n)$. Hence we have

$$\int_G LA * B_n dm_G = \int_{R^n} g(x)\varphi_1(x) \int_{\hat{F}} (y, \gamma_1) dm_{\hat{F}}(y) dx$$
$$+ \sum_{k=2}^{m} \int_{R^n} g(x)\varphi_k(x) \int_{\hat{F}} (y, \gamma_k) dm_F(y) dx$$
$$= \int_{R^n} g(x)\varphi_1(x) dx$$
$$= 0 .$$

Hence, in each case, we have $\int_G LA * B_n dm_G = 0$. On the other hand, $\lim_{n\to\infty} \|A - A * B_n\|_1 = 0$. Hence, we have,

$$\int_G LA dm_G = \lim_{n\to\infty} \int_G LA * B_n dm_G$$
$$= 0 .$$

Thus $L$ belongs to $H^\infty_P(G)$.

Suppose that $C_u(G) + H^\infty_P(G)$ is an algebra. Then there exist functions $S \in C_u(G)$ and $T \in H^\infty_P(G)$ such that $FL = S + T$.

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We define functions $p(x) \in C_u(R^n)$ and $q(x) \in L^\infty(R^n)$ as follows:

$$p(x) = \int_{\hat{F}} S(x, y) dm_{\hat{F}}(y) ;$$

$$q(x) = \int_{\hat{F}} T(x, y) dm_{\hat{F}}(y) .$$

CLAIM 2. $q(x) \in H^1_{P_{R^n}}(R^n)$ .

Indeed, for each $\eta \in H^1_{P_{R^n}}(R^n)$ , put $Y(x, y) = \eta(x)$ for $(x, y) \in R^n \oplus \hat{F}$ . Then $Y$ belongs to $H^1_P(G)$ . Hence we have

$$\begin{aligned} 0 &= \int_G YT dm_G \\ &= \int_{R^n} \int_{\hat{F}} Y(x, y) T(x, y) dm_{\hat{F}}(y) dx \\ &= \int_{R^n} \eta(x) \int_{\hat{F}} T(x, y) dm_{\hat{F}}(y) dx \\ &= \int_{R^n} \eta(x) q(x) dx . \end{aligned}$$

Therefore $q(x)$ belongs to $H^\infty_{P_{R^n}}(R^n)$ . Evidently, $p(x) \in C_u(R^n)$ . Hence, by Claim 2, we have

$$\begin{aligned} f(x)g(x) &= \int_{\hat{F}} F(x, y) L(x, y) dm_{\hat{F}}(y) \\ &= \int_{\hat{F}} S(x, y) dm_{\hat{F}}(y) + \int_{\hat{F}} T(x, y) dm_{\hat{F}}(y) \\ &= p(x) + q(x) \in C_u(R^n) + H^\infty_{P_{R^n}}(R^n) . \end{aligned}$$

This is a contradiction. Hence, in this case, $C_u(G) + H^\infty_P(G)$ is not an algebra.

CASE 1.2. $F_0 = \{0\}$ and $P_{R^n}$ is dense in $R^n$. Since $R^n$ is an open subgroup of $\hat{G}$ and $P$ is not dense in $\hat{G}$, there exists an element $\gamma_0 \in F$ with $\gamma_0 >_P 0$ such that $R^n + \gamma_0 \subset P$. We define measures $\mu \in M_P^a(G)$ and $\nu \in M_{(-P)}^a(G)$ as follows:

$$d\mu(x, y) = d\gamma_0(x) \times (y, \gamma_0) dm_{\hat{F}}(y) ,$$

$$d\nu(x, y) = d\gamma_0(x) \times (y, -\gamma_0) dm_{\hat{F}}(y) .$$

Choose a nonzero function $f \in L^\infty(R^n) \backslash C_u(R^n)$, and define a function $F$ on $G$ by $F(x, y) = f(x)(y, \gamma_0)$. Then $F * \mu(x, y) = f(x)(y, \gamma_0)$ does not belong to $C_u(G)$. Evidently, $\hat{\nu} = 1$ on $\text{supp}(\hat{\mu}) - 2\gamma_0$.

Hence, by Lemma 5, $C_u(G) + H_P^\infty(G)$ is not an algebra. Thus, in Case 1, we have proved that $C_u(G) + H_P^\infty(G)$ is not an algebra.

CASE 2. $F_0 \neq \{0\}$. Put $P_0 = P \cap R^n \oplus F_0$. We consider two cases, depending on whether $P_0$ is dense in $R^n \oplus F_0$ or not.

CASE 2.1. Suppose $P_0$ is dense in $R^n \oplus P_0$. Then there exists an element $\gamma_0 \in F$ with $\gamma_0 >_P 0$ such that $R^n \oplus F_0 + \gamma_0 \subset P$. Let $H$ be an annihilator of $F_0$ in $\hat{F}$; that is,

$$H = \{y \in \hat{F};\ (y, \gamma) = 1 \text{ for every } \gamma \in F_0\} .$$

Then $H$ is an open compact subgroup of $\hat{F}$. Define measures $\mu \in M_P^a(G)$ and $\nu \in M_{(-P)}^a(G)$ as follows:

$$d\mu(x, y) = d\delta_0(x) \times (y, \gamma_0) dm_H(y) ,$$

$$d\nu(x, y) = d\delta_0(x) \times (y, -\gamma_0) dm_H(y) ,$$

where $\delta_0$ and $m_H$ denote the Dirac measure at $0$ in $R^n$ and the normalised Haar measure on $H$, respectively. Choose a nonzero function $f(x) \in L^{\infty}(R^n) \setminus C_u(R^n)$, and define a function $F(x, y)$ on $G$ by $F(x, y) = f(x)(y, \gamma_0)$. Then $F * \mu = F \notin C_u(G)$.

Moreover, $\hat{\nu} = 1$ on $\operatorname{supp}(\hat{\mu}) - 2\gamma_0$. Hence, by Lemma 5, $C_u(G) + H_P^{\infty}(G)$ is not an algebra.

CASE 2.2. Suppose $P_0$ is not dense in $R^n \oplus F_0$. Then there exists an automorphism $\tau$ on $\hat{G}$ such that

$$\tau(P) \cap R^n \supset \left\{x = (x_1, \ldots, x_n) \in R^n;\ x_1 > 0\right\}.$$

Hence, by Lemma 7, we may assume that

$$P \cap R^n \supset \{x = (x_1, \ldots, x_n);\ x_1 > 0\}.$$

By Lemma 6, for $z \in \overset{\circ}{P}_{R^n}$ $(= P \overset{\circ}{\cap} R^n)$, $z >_P y$ for every $y \in F_0$. Let $c_1$ and $c_2$ be positive numbers such that $0 < c_1 < c_2$. Choose a nonzero function $h \in L^1(R)$ such that $\operatorname{supp}(\hat{h}) \subset (c_1, c_2)$. We define a measure $\mu \in M_P^a(G)$ by $d\mu(x_1, x', y) = h(x_1)dx_1 \times d\delta_0(x') \times dm_H(y)$, where $\delta_0(x')$ is the Dirac measure at $0$ in $R^{n-1}$ and $H$ is an annihilator of $F_0$ in $\hat{F}$. We choose nonzero functions $g_1 \in L^{\infty}(R)$, $g_2 \in L^{\infty}(R^{n-1}) \setminus C_u(R^{n-1})$, and $g_3 \in L^{\infty}(\hat{F})$ such that $g_1 * h \neq 0$ and $g_3 * m_H \neq 0$. Put $F(x_1, x', y) = g_1(x_1)g_2(x')g_3(y)$; then $F * \mu \notin C_u(G)$. Next we define a measure $\nu \in M_{(-P)}^a(G)$ as follows:

$$d\nu(x_1, x', y) = \xi(x_1)dx_1 \times d\xi_0(x') \times dm_H(y),$$

where $\xi$ is in $L^1(R)$ such that $\operatorname{supp}(\hat{\xi}) \subset (-\infty, 0)$ and $\hat{\xi} = 1$ on $[-c_2, -c_1]$. Define a character $\eta \in \hat{G}$ by $\eta(x_1, x', y) = e^{-ic_3x_1}$, where

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$c_3 = c_1 + c_2$ . Then $\hat{\nu} = 1$ on $\mathrm{supp}(\hat{\mu}) + \eta$ . Hence, by Lemma 5, $C_u(G) + H_P^\infty(G)$ is not an algebra. Therefore, in each case, $C_u(G) + H_P^\infty(G)$ is not an algebra. //

**PROPOSITION 11.** *Let $G$ be a locally compact abelian group such that $\hat{G}$ is an algebraically ordered group. Suppose that $\hat{G} \cong R \oplus F$ , where $F$ is a nontrivial discrete abelian group. Let $P$ be a subsemigroup of $\hat{G}$ with the* (AO)*-condition such that it is not dense in $\hat{G}$ . Then $C_u(G) + H_P^\infty(G)$ is not an algebra.*

PROOF. CASE 1. Suppose $P$ is dense in $R$ . Then there exists an element $\gamma_0 \in F$ with $\gamma_0 >_P 0$ such that $R + \gamma_0 \subset P$ . Hence we can prove that $C_u(G) + H_P^\infty(G)$ is not an algebra in the same way as in Case 2.2 of Proposition 10.

CASE 2. Suppose $P$ is not dense in $R$ . We may assume $P \cap R = [0, \infty)$ .

CASE 2.1. First we consider the case that $F \not\cong Z$ (integers) and $P \cap F$ induces an archimedean order on $F$ . Then $F$ is order preserving and isomorphic to some subgroup of $R_d$ which is dense in $R$ with respect to the usual topology of $R$ .

Fix an element $d_0 \in F$ such that $d_0 >_P 0$ . Then $\{d \in F;\ 0 <_P d <_P d_0\}$ is an infinite set. By Bochner's Theorem, there exist measures $\mu_2 \in M^a_{P\cap F}(\hat{F})$ and $\nu_2 \in M^a_{(-P\cap F)}(\hat{F})$ such that $\mathrm{supp}(\hat{\mu}_2) \subset \{d \in F;\ 0 \leq_P d \leq_P 2d_0\}$ , $\{d \in F;\ |\hat{\mu}_2(d)| \geq \frac{1}{2}\}$ is infinite, and $\hat{\nu}_2 = 1$ on $\mathrm{supp}(\hat{\mu}_2) - 4d_0$ . Since $\mu_2 \notin L^1(\hat{F})$ , there exists a function $g_2 \in L^\infty(\hat{F})$ such that $g_2 * \mu_2 \notin C(\hat{F})$ . Let $h$ be a nonzero function in $L^1(R)$ such that $\mathrm{supp}(\hat{h}) \subset [1, 2]$ . We choose a function $g_1 \in L^\infty(R)$ such that $g_1 * h \neq 0$ .

We define $F \in L^\infty(G)$ , $\mu \in M^a_P(G)$ , and $\nu \in M^a_{(-P)}(G)$ as follows:

$$F(x, y) = g_1(x)g_2(y) ,$$

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$$d\mu(x, y) = h(x)dx \times d\mu_2(y) ,$$

$$d\nu(x, y) = \xi(x)dx \times d\nu_2(y) ,$$

where $\xi$ is a function in $L^1(R)$ such that $\text{supp}(\hat{\xi}) \subset (-\infty, 0)$ and $\hat{\xi} = 1$ on $[-2, -1]$. Define $\gamma_0 \in \hat{G}$ by $\gamma_0(x, y) = e^{-i3x}(y, -4d_0)$. Then $F * \mu(x, y) = g_1 * h(x)g_2 * \mu_2(y) \notin C_u(G)$, and $\hat{\nu} = 1$ on $\text{supp}(\hat{\mu}) + \gamma_0$. Hence, by Lemma 5, $C_u(G) + H_P^\infty(G)$ is not an algebra.

CASE 2.2. Suppose $F \ncong Z$ and $P \cap F$ induces a nonarchimedean order on $F$.

Then there exist two elements $d_1, d_2 \in F$ with $0 <_P d_1 <_P d_2$ such that $nd_1 <_P d_2$ for every $n \in Z$. Let $\Lambda = \{nd_1 ; n \in Z\}$, and let $H$ be an annihilator of $\Lambda$ in $\hat{F}$. Then $m_H$ (the Haar measure on $H$) is regarded as a singular measure with respect to the Haar measure on $\hat{F}$.

Hence, by [2], Theorem (35.13), there exists $g_2 \in L^\infty(\hat{F})$ such that $g_2 * (d_2 m_H) \notin C_u(\hat{F})$. Let $h_1$ and $\xi$ be functions in $L^1(R)$ such that $\text{supp}(\hat{h}_1) \subset [1, 2]$, $\text{supp}(\hat{\xi}) \subset (-\infty, 0)$, and $\hat{\xi} = 1$ on $[-2, -1]$. We choose a function $g_1 \in L^\infty(R)$ such that $g_1 * h_1 \neq 0$, and define a function $F$ on $G$ by $F(x, y) = g_1(x)g_2(y)$.

Define measures $\mu \in M_P^a(G)$ and $\nu \in M_{(-P)}^a(G)$ as follows:

$$d\mu(x, y) = h_1(x)dx \times (y, d_2)dm_H(y) ,$$

$$d\nu(x, y) = \xi(x)dx \times (y, -d_2)dm_H(y) .$$

Then $F * \mu \notin C_u(G)$. Define a character $\gamma_0 \in \hat{G}$ by $\gamma_0(x, y) = e^{-i3x}(y, -2d_2)$. Then $\hat{\nu} = 1$ on $\text{supp}(\hat{\mu}) + \gamma_0$. Hence, by Lemma 5, $C_u(G) + H_P^\infty(G)$ is not an algebra.

CASE 2.3. Suppose $F \cong Z$ (integers). In this case, we need consider only the following three cases:

(A) there exists a positive number $b$ such that

$$(P_A =)\ P = \{(n, x) \in Z \oplus R;\ x \geq n-bn \text{ if } n \leq -1\} \cup \{(n, x) \in Z \oplus R;\ x > n-bn \text{ if } n \geq 0\}\ ;$$

(B) $(P_B =)\ P = \{(n, x) \in Z \oplus R;\ n \geq 1, \text{ or } n = 0 \text{ and } x \geq 0\}$ ;

(C) $(P_C =)\ P = \{(n, x) \in Z \oplus R;\ x \geq 0 \text{ if } n \geq 0\} \cup \{(n, x) \in Z \oplus R;\ x > 0 \text{ if } n \leq -1\}$ .

But let $\tau(n, x) = (n, x+bn)$ . Then $\tau$ is an automorphism on $Z \oplus R$ , and $\tau(P_A) = P_C$ . Hence, by Lemma 7, we need investigate only Cases (B) and (C). We prove only Case (C). (In Case (B), we can proceed in the same way.)

Let $h$ be a nonzero function in $L^1(R)$ such that $\text{supp}(\hat{h}) \subset [1, 2]$ . Choose functions $g_1 \in L^\infty(R)$ and $g_2 \in L^\infty(R)$ such that $g_1 \notin C(T)$ and $g_2 * h \neq 0$ . Define measures $\mu \in M_P^a(T \oplus R)$ and $\nu \in M_{(-P)}^a(T \oplus R)$ as follows:

$$d\mu(x, y) = d\delta_0(x) \times h(y)dy\ ,$$

$$d\nu(x, y) = d\delta_0(x) \times \xi(y)dy\ ,$$

where $\xi$ is a function in $L^1(R)$ such that $\text{supp}(\hat{\xi}) \subset (-\infty, 0)$ , and $\hat{\xi} = 1$ on $[-2, -1]$ . Then $\hat{\nu} = 1$ on $\text{supp}(\hat{\mu}) + (0, -3)$ and $F * \mu \notin C_u(T \oplus R)$ , where $F(x, y) = g_1(x)g_2(y) \in L^\infty(T \oplus R)$ .

Hence, by Lemma 5, $C_u(T \oplus R) + H_P^\infty(T \oplus R)$ is not an algebra. Therefore, in each case, $C_u(G) + H_P^\infty(G)$ is not an algebra. //

LEMMA 12. *Let* $G = G_1 \oplus D$ *, where* $G_1$ *is a locally compact abelian group such that* $\hat{G}_1$ *is torsion-free, and* $D$ *is a discrete abelian group such that* $\hat{D}$ *is torsion-free. Let* $P$ *be a subsemigroup of* $\hat{G}$ *with the* (AO)*-condition such that it is not dense in* $\hat{G}$ *. Moreover, we assume that*

*(i)* $s >_P t$ *for each* $s \in (\overset{\circ}{P} \cap \hat{G}_1)$ *and* $t \in \hat{D}$ *,*

*(ii)* $P^c \subset \overline{(-\overset{\circ}{P} \cap \hat{G}_1) \times \hat{D}}$ *, and*

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*(iii)* $\left(-P \overset{\circ}{\cap} \hat{G}_1\right) \subset \overline{\left(P \cap \hat{G}_1\right)^c}$ .

*If* $C_u\left(G_1\right) + H^{\infty}_{P\cap\hat{G}_1}\left(G_1\right)$ *is not an algebra,* $C_u(G) + H^{\infty}_P(G)$ *is not an algebra.*

Proof. Suppose that $C_u(G) + H^{\infty}_P(G)$ is an algebra. For $f \in H_{P\cap\hat{G}_1}\left(G_1\right)$ , put $F(x, d) = f(x)$ for $(x, d) \in G_1 \oplus D$ .

CLAIM 1. $F$ belongs to $H^{\infty}_P(G)$ .

Indeed, for each $h \in H^1_P\left(G_1 \oplus D\right)$ , $h$ can be represented as follows:

$$h(x, d) = \sum_{n=1}^{\infty} h\left(x, d_n\right) \times d\delta d_n(d) \quad \text{for some } d_n \in D ,$$

$$\|h\|_1 = \sum_{n=1}^{\infty} \left\|h\left(\cdot, d_n\right)\right\|_{L^1\left(G_1\right)} ,$$

and

$$h\left(\cdot, d_n\right) \in H^1_{P\cap\hat{G}_1}\left(G_1\right) \quad (n = 1, 2, 3, \ldots) .$$

Hence we have

$$\begin{aligned}\int_G Fhdm_G &= \sum_{n=1}^{\infty} \int_{G_1} F\left(x, d_n\right)h\left(x, d_n\right)dm_{\hat{G}_1}(x) \\ &= \sum_{n=1}^{\infty} \int_{G_1} f(x)h\left(x, d_n\right)dm_{\hat{G}_1}(x) \\ &= 0 .\end{aligned}$$

That is, $F$ belongs to $H^{\infty}_P(G)$ .

For $g \in C_u\left(G_1\right)$ , define $K(x, y) \in C_u(G)$ by $K(x, y) = g(x)$ . Then there exist functions $H \in C_u\left(G_1 \oplus D\right)$ and $L \in H^{\infty}_P\left(G \oplus D\right)$ such that $FK = H + L$ .

Hence, in particular, $F(x, 0)K(x, 0) = H(x, 0) + L(x, 0)$ almost everywhere $x \in G_1$ . That is, $f(x)g(x) = H(x, 0) + L(x, 0)$ almost

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everywhere $x \in G_1$ . Evidently $H(\cdot, 0)$ belongs to $C_u(G_1)$ .

CLAIM 2. $K(\cdot, 0)$ belongs to $H^\infty_{P\cap\hat{G}_1}(G_1)$ .

Indeed, for $k \in H^1_{P\cap\hat{G}_1}(G_1)$ , define a function $N(x, d) \in L^1(G_1 \oplus D)$ as follows:

$$N(x, d) = \begin{cases} k(x) & \text{if } d = 0 , \\ 0 & \text{if } d \neq 0 . \end{cases}$$

Then $\hat{N}(\gamma_1, \gamma_2) = \hat{k}(\gamma_1)$ for $(\gamma_1, \gamma_2) \in (-P \overset{\circ}{\cap} \hat{G}_1) \times \hat{D}$ . Hence $\hat{N} = 0$ on $\overline{(-P \overset{\circ}{\cap} \hat{G}_1) \times \hat{D}} \supset P^c$ . That is, $N$ belongs to $H^1_P(G)$ . Hence we have

$$\int_{G_1} K(x, 0)k(x)dx = \int_G K(x, d)N(x, d)dm_G$$
$$= 0 .$$

Therefore $C_u(G_1) + H^\infty_{P\cap\hat{G}_1}(G_1)$ becomes an algebra. This is a contradiction. //

LEMMA 13. *Let $\Gamma$ be a locally compact abelian group and $P$ a subsemigroup of $\Gamma$ satisfying the* (AO)*-condition. Let $F$ be an open subgroup of $\Gamma$ such that $P$ is dense in it. If there exists an element $\gamma_0 \in \Gamma$ such that $-P$ is not dense in $\gamma_0 + F$ , then we have $P \supset \gamma_0 + F$ .*

Proof. Since $-P$ is not dense in $\gamma_0 + F$ , there exists an open subset $V$ of $F$ such that $(\gamma_0+V) \cap \overline{(-P)} = \emptyset$ . Hence we have $\gamma_0 + V \subset P$ . Since $P$ is dense in $F$ , we have $V + P \supset F$ . Therefore we have $\gamma_0 + F \subset \gamma_0 + V + P \subset P$ . //

PROPOSITION 14. *Let $G$ be a locally compact abelian group. Let $\hat{G}$ be torsion-free and $\hat{G} \cong R \oplus F$ , where $F$ is a locally compact abelian group which contains a compact open subgroup $F_0 \neq \{0\}$ . Let $P$ be a subsemigroup of $\hat{G}$ with the* (AO)*-condition such that it is not dense in $\hat{G}$ . If $C_u(G) + H^\infty_P(G)$ is an algebra, then $F$ is compact.*

Proof. We suppose $F$ is not compact.

CASE 1. Suppose $P$ is dense in $R \oplus F_0$ . Since $R \oplus F_0$ is an open subgroup, there exists $\gamma_0 \in F$ with $\gamma_0 >_P 0$ such that $R \oplus F_0 + \gamma_0 \subset P$ . Hence, in this case, we can prove that $C_u(G) + H_P^\infty(G)$ is not an algebra as in Case 2.1 of the proof of Proposition 10.

CASE 2. $P$ is not dense in $R \oplus F_0$ .

CASE 2.1. Suppose $P$ is not dense in $R \oplus F_0$ and $P \cap F$ is dense in $F$ . Since $P$ is necessarily dense in $F_0$ , it is not dense in $R$ . Hence we may assume $P \cap R = [0, \infty)$ . We note that $x + F \subset P$ for each $x \in R$ such that $x > 0$ . Let $h$ be a nonzero function in $L^1(R)$ such that $\operatorname{supp}(\hat{h}) \subset [1, 2]$ . Choose functions $g_1 \in L^\infty(R)$ and $g_2 \in L^\infty(\hat{F}) \backslash C_u(\hat{F})$ such that $g_1 * h \neq 0$ . Put $F(x, y) = g_1(x)g_2(y)$ . Let $\gamma_0$ be a character of $G$ defined by $\gamma_0(x, y) = e^{-i3x}$ . Define measures $\mu \in M_P^a(G)$ and $\nu \in M_{(-P)}^a(G)$ as follows:

$$d\mu(x, y) = h(x)dx \times d\delta_0(y) ,$$

$$d\nu(x, y) = \xi(x)dx \times d\delta_0(y) ,$$

where $\xi$ is a function in $L^1(R)$ such that $\operatorname{supp}(\hat{\xi}) \subset (-\infty, 0)$ , and $\hat{\xi} = 1$ on $[-2, -1]$ . Then $F * \mu \notin C_u(G)$ and $\hat{\nu} = 1$ on $\operatorname{supp}(\hat{\mu}) + \gamma_0$ . Hence, by Lemma 5, $C_u(G) + H_P^\infty(G)$ is not an algebra.

CASE 2.2. Suppose $P$ is not dense in $R \oplus F_0$ and $F$ respectively. Let $\mathsf{F} = \{\gamma+F_0;\ \gamma \in F,\ O(\gamma+F_0) < \infty\}$ , where $O(\gamma+F_0)$ denotes the order of the coset $\gamma + F_0$ in $F/F_0$ .

CLAIM 1. $\mathsf{F}$ is a finite set.

Suppose $\mathsf{F}$ is infinite. Put $F_{\mathsf{F}} = \cup\{\gamma+F_0;\ \gamma+F_0 \in \mathsf{F}\}$ . Then $F_{\mathsf{F}}$ is a noncompact open subgroup of $F$ . Evidently, $P$ is dense in $F_{\mathsf{F}}$ . We may assume $P \cap R = [0, \infty)$ . Since $P$ is not dense in $R \oplus F_{\mathsf{F}}$ ,

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$\{(x, y) \in R \oplus F_F;\ x > 0,\ y \in F_F\}$ is included in $P$. Let $h$ be a nonzero function in $L^1(R)$ such that $\mathrm{supp}(\hat{h}) \subset [1, 2]$. Let $H = F_F^-$ (the annihilator of $F_F$ in $\hat{F}$) and $H_0 = F_0^-$ (the annihilator of $F_0$ in $\hat{F}$). Then $H_0$ is a compact open subgroup of $\hat{F}$. Moreover, $H_0/H$ is infinite. Hence $m_H$ (a normalized Haar measure on $H$) is regarded as a singular measure with respect to a Haar measure on $H_0$. Hence there exists a function $g_2 \in L^\infty(\hat{F})$ such that $g_2 * m_H \notin C_u(\hat{F})$. Let $g_1$ be a function in $L^1(R)$ such that $g_1 * h \neq 0$. Put $g(x, y) = g_1(x)g_2(x)$. Let $\gamma_0$ be a character of $G$ defined by $\gamma_0(x, y) = e^{-i3x}$. Define measures $\mu \in M_P^a(G)$ and $\nu \in M_{(-P)}^a(G)$ as follows:

$$d\mu(x, y) = h(x)dx \times dm_H(y) ,$$

$$d\nu(x, y) = \xi(x)dx \times dm_H(y) ,$$

where $\xi$ is a function in $L^1(R)$ such that $\mathrm{supp}(\hat{\xi}) \subset (-\infty, 0)$, and $\hat{\xi} = 1$ on $[-2, -1]$. Then $\hat{\nu} = 1$ on $\mathrm{supp}(\hat{\mu}) + \gamma_0$, and $g * \mu$ does not belong to $C_u(G)$. Hence, by Lemma 5, $C_u(G) + H_P^\infty(G)$ is not an algebra. This is a contradiction. Hence Claim 1 is proved. Therefore $F_F$ is a compact open subgroup of $F$.

CLAIM 2. Let $\gamma$ be in $F$ such that $\gamma >_P 0$ and $\gamma + F_F \neq F_F$. Then $\gamma + F_F$ is included in $P$.

By Lemma 13, we need only prove that $\gamma + F_F \not\subset \overline{(-P)}$.

Suppose $\gamma + F_F \subset \overline{(-P)}$. Since $F/F_F$ is torsion-free, $P$ is dense in $Z \oplus F_F \cong (Z\gamma) \oplus F_F$. Hence $\{(x, y) \in R \oplus Z \oplus F_F;\ x > 0,\ y \in Z \oplus F_F\}$ is included in $P$. Let $H_0$ and $H$ be annihilators of $F_F$ and $Z \oplus F_F$ in $\hat{F}$ respectively. Then $m_H$ is a singular measure with respect to a Haar measure on $\hat{F}$. Hence there exists a function $g_2$ in $L^\infty(\hat{F})$ such that $g_2 * (\gamma m_H) \notin C_u(\hat{F})$. Let $h$ be a nonzero function in $L^1(R)$ such that $\mathrm{supp}(\hat{h}) \subset [1, 2]$. Choose a function $g_1 \in L^\infty(R)$ such that $g_1 * h \neq 0$, and define a function $g \in L^\infty(G)$ by $g(x, y) = g_1(x)g_2(y)$.

Let $\gamma_0$ be a character of $G$ defined by $\gamma_0(x, y) = e^{-i3x}(y, -2\gamma)$ . Define measures $\mu \in M_P^a(G)$ and $\nu \in M_{(-P)}^a(G)$ as follows:

$$d\mu(x, y) = h(x)dx \times (y, \gamma)dm_H(y) ,$$

$$d\nu(x, y) = \xi(x)dx \times (y, -\gamma)dm_H(y) ,$$

where $\xi$ is a function in $L^1(R)$ such that $\mathrm{supp}(\hat{\xi}) \subset (-\infty, 0)$ , and $\hat{\xi} = 1$ on $[-2, -1]$ . Then $g * \mu$ does not belong to $C_u(G)$ , and $\hat{\nu} = 1$ on $\mathrm{supp}(\hat{\mu}) + \gamma_0$ . Hence, by Lemma 5, $C_u(G) + H_P^\infty(G)$ is not an algebra. This is a contradiction. Hence, Claim 2 is proved.

Put $\tilde{P} = \{\gamma+F_F;\ \gamma >_P 0,\ \gamma+F_F \neq F_F\} \cup \{[0]\}$ . Then, by Claim 2, $\tilde{P}$ is a subsemigroup of $F/F_F$ with the (AO)-condition.

CASE 2.21. First we consider the case that $\tilde{P}$ induces a nonarchimedean order on $F/F_F$ .

Then there exist positive elements $[\gamma_1], [\gamma_2] \in F/F_F$ such that $n[\gamma_1] <_{\tilde{P}} [\gamma_2]$ for every $n \in Z$ . Let $[F_F, \gamma_1]$ be an open subgroup of $F$ generated by $\gamma_1$ and $F_F$ . Then $[F_F, \gamma_1] \cong F_F \oplus Z$ . Let $H_{\gamma_1} = [F_F, \gamma_1]^\perp$ (the annihilator of $[F_F, \gamma_1]$ in $F$ ). Then $m_{H_{\gamma_1}}$ (the Haar measure on $H_{\gamma_1}$ ) is a singular measure with respect to a Haar measure on $\hat{F}$ . Hence there exists a function $g_2 \in L^\infty(F)$ such that $g_2 * (\gamma_2 m_{H_{\gamma_1}}) \notin C_u(\hat{F})$ . Let $h$ be a function in $L^1(R)$ such that $\mathrm{supp}(\hat{h}) \subset [1, 2]$ . Choose a function $g_1 \in L^\infty(R)$ such that $g_1 * h \neq 0$ . Put $g(x, y) = g_1(x)g_2(y)$ for $(x, y) \in R \oplus \hat{F}$ . Define measures $\mu \in M_P^a(G)$ and $\nu \in M_{(-P)}^a(G)$ as follows:

$$d\mu(x, y) = h(x)dx \times (y, \gamma_2)dm_{H_{\gamma_1}}(y) ,$$

$$d\nu(x, y) = \xi(x)dx \times (y, -\gamma_2)dm_{H_{\gamma_1}}(y) ,$$

where $\xi$ is a function in $L^1(R)$ such that $\mathrm{supp}(\hat{\xi}) \subset (-\infty, 0)$ , and $\hat{\xi} = 1$ on $[-2, -1]$ . Let $\gamma_0$ be a character of $G$ defined by

$\gamma_0(x, y) = e^{-i3x}(y, -2\gamma_2)$ . Then $g * \mu \notin C_u(G)$ , and $\hat{\nu} = 1$ on $\mathrm{supp}(\hat{\mu}) + \gamma_0$ . Hence, by Lemma 5, $C_u(G) + H_P^\infty(G)$ is not an algebra. This is a contradiction.

CASE 2.22. Next, we consider the case that $F/F_F \not\cong Z$ and $\tilde{P}$ induces an archimedean order on $F/F_F$ .

Then $F/F_F$ is order preserving isomorphic to a subgroup of $R_d$ which is dense in $R$ with respect to the usual topology. Hence, as seen in Case 2.1 of the proof of Proposition 11, we can construct $\gamma_0 \in \hat{G}$ , $F \in L^\infty(G)$ , $\mu \in M_P^a(G)$ , and $\nu \in M_{(-P)}^a(G)$ such that $F * \mu \notin C_u(G)$ and $\hat{\nu} = 1$ on $\mathrm{supp}(\hat{\mu}) + \gamma_0$ . Hence $C_u(G) + H_P^\infty(G)$ is not an algebra by Lemma 5.

CASE 2.23. Finally, we consider the case $F/F_F \cong Z$ . Then $\hat{G} \cong R \oplus Z \oplus F_F$ . Let $G_1 = R \oplus T$ . Then $\hat{G}_1$ has the following properties:

(i) $s >_P t$ for each $s \in (\hat{G}_1 \mathring{\cap} P)$ and $t \in F_F$ ;

(ii) $P^c \subset \overline{(-P \mathring{\cap} \hat{G}_1) \times F}$ ;

(iii) $(-P \mathring{\cap} \hat{G}_1) \subset \overline{(P \cap \hat{G}_1)^c}$ .

As seen in Case 2.3 of the proof of Proposition 11, $C_u(G_1) + H^\infty_{P\cap\hat{G}_1}(G_1)$ is not an algebra. Hence, by Lemma 12, $C_u(G) + H_P^\infty(G)$ is not an algebra.

Hence, in each case, we have a contradiction. Therefore $F$ must be compact. //

The following theorems are due to Sarason (see [3], [5], and [6]).

**THEOREM 15.** *$C(T) + H^\infty(T)$ is a closed subalgebra of $L^\infty(T)$ .*

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THEOREM 16. *$C_u(R) + H^\infty(R)$ is a closed subalgebra of $L^\infty(R)$.*

The following theorem is due to Rudin ([4]).

THEOREM 17. *Let $G$ be a compact abelian group such that $\hat{G}$ is ordered. Then $C(G) + H^\infty(G)$ is a closed subalgebra of $L^\infty(G)$ if and only if $G \cong T$.*

PROPOSITION 18. *Let $G$ be a nondiscrete locally compact abelian group such that $\hat{G}$ contains a compact open subgroup $F \neq \{0\}$. Let $P$ be a subsemigroup of $\hat{G}$ with the (AO)-condition such that it is not dense in $\hat{G}$.*

*If $C_u(G) + H_P^\infty(G)$ is an algebra, then $G \cong T \oplus \hat{F}_*$ for some compact open subgroup $F_*$ of $\hat{G}$.*

Proof. Let $F_* = \cup\{\gamma+F;\ \gamma \in \hat{G},\ O(\gamma+F) < \infty\}$. Then, as seen in Case 2.2 of the proof of Proposition 14, $F_*$ is a compact open subgroup of $\hat{G}$. Put $\tilde{P} = \{\gamma+F_*;\ \gamma >_P 0,\ \gamma+F_* \neq F_*\}$.

If $\gamma >_P 0$ and $\gamma \notin F_*$, then $\gamma + F_*$ is contained in $P$.

Hence $\tilde{P}$ is a subsemigroup of $F/F_*$ with the (AO)-condition.

Let $H_0 = F_*^\perp$ (the annihilator of $F_*$). Then $H_0$ is a compact open subgroup of $G$.

CLAIM 1. For $f \in H_P^1(G)$, $f = \sum_{n=1}^\infty f_n$ with $\|f\|_1 = \sum_{n=1}^\infty \|f_n\|_1$, where $f_n$ is the restriction of $f$ to some coset $x_n + H_0$ $(n = 1, 2, \ldots)$. Put $h_n = \delta_{-x_n} * f_n$. Then $f = \sum_{n=1}^\infty \delta_{x_n} * h_n$, and $h_n$ belongs to $H_{\tilde{P}}^1(H_0)$ $(n = 1, 2, \ldots)$.

Indeed, let $\gamma$ be in $\hat{G}$ such that $[\gamma] <_{\tilde{P}} 0$. Then $\gamma + s <_P 0$ for every $s \in F_*$. Hence we have

$$0 = \hat{f}(\gamma+s)$$
$$= \sum_{n=1}^\infty \hat{h}_n([\gamma])(-x_n, \gamma)(-x_n, s) \quad \text{for every } s \in F_*.$$

Since $\{\hat{h}_n([\gamma])(-x_n, \gamma)\} \in l^1$, we obtain $\hat{h}_n([\gamma]) = 0$ $(n = 1, 2, \ldots)$. That is, $h_n \in H^1_{\tilde{P}}(H_0)$ $(n = 1, 2, \ldots)$.

Let $H^\infty_{\tilde{P},0}(H_0) = \{g \in L^\infty(H_0);\ \hat{g}(\gamma) = 0 \text{ if } \gamma \leq_{\tilde{P}} 0\}$ ; that is $H^\infty_{\tilde{P},0}(H_0) = \left(H^1_{\tilde{P}}(H_0)\right)^\perp$.

CLAIM 2. $H^\infty_{\tilde{P},0}(H_0) \subset H^\infty_P(G)$.

This is obtained from Claim 1.

We note that $C(H_0) + H^\infty_{\tilde{P}}(H_0) = C(H_0) + H^\infty_{\tilde{P},0}(H_0)$.

CLAIM 3. $C(H_0) + H^\infty_{\tilde{P}}(H_0)$ is an algebra.

For $f \in C(H_0)$ and $g \in H^\infty_{\tilde{P},0}(H_0)$, by Claim 2, $f$ and $g$ can be regarded as functions in $C_u(G)$ and $H^\infty_P(G)$, respectively.

Since $C_u(G) + H^\infty_P(G)$ is an algebra, there exist functions $H \in C_u(G)$ and $K \in H^\infty_P(G)$ such that $gf = H + K$.

Evidently, $H|_{H_0}$ belongs to $C(H_0)$. Moreover, we can check that $K|_{H_0}$ belongs to $H^\infty_{\tilde{P}}(H_0)$. Hence we have $fg \in C(H_0) + H^\infty_{\tilde{P}}(H_0)$. Therefore $C(H_0) + H^\infty_{\tilde{P}}(H_0)$ is an algebra.

Hence, by Theorem 17, we have $F/F_* \cong Z$. Hence $G = T \oplus \hat{F}_*$. //

**THEOREM 19.** *Let $G$ be a nondiscrete locally compact abelian group such that $\hat{G}$ is an algebraically ordered group. Let $P$ be a subsemigroup of $\hat{G}$ with the* (AO)-*condition such that it is not dense in $\hat{G}$. If $C_u(G) + H^\infty_P(G)$ is a closed subalgebra of $L^\infty(G)$, then $G$ admits one of the following structures*

(a) $G \cong R$,

(b) $G \cong R \oplus D$,

(c) $G \cong T$,

(d) $G \cong T \oplus D$,

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*where $D$ is some discrete abelian group.*

Proof. By the structure theorem, $\hat{G} \cong R^n \oplus F$, where $F$ is a locally compact abelian group which contains a compact open subgroup $F_0$.

If $n \geq 2$, $C_u(G) + H_P^\infty(G)$ is not an algebra by Proposition 10. If $n = 1$, by Proposition 11 and Proposition 14, we have $G \cong R$, or $G \cong R \oplus D$ for some discrete abelian group $D$. If $n = 0$, by Theorem 17 and Proposition 18, we have $G \cong T$, or $G \cong T \oplus D$ for some discrete abelian group $D$. //

THEOREM 20. *Let $D$ be a discrete abelian group such that $\hat{D}$ is torsion-free. Let $P$ be a subsemigroup of $Z \oplus \hat{D}$ with the* (AO)*-condition such that it is not dense in $Z \oplus \hat{D}$. Then $C_u(T \oplus D) + H_P^\infty(T \oplus D)$ is a closed subalgebra of $L^\infty(T \oplus D)$.*

Proof. Evidently $C_u(T \oplus D) + H_P^\infty(T \oplus D)$ is a closed subspace of $L^\infty(T \oplus D)$. We may assume that $P \cap Z = \{n \in Z;\ n \geq 0\}$. We can easily check the following:

$$(*)\quad \begin{aligned} H_P^1(T \oplus D) &= \left\{f \in L^1(T \oplus D);\ f(\cdot, d) \in H_0^1(T) \text{ for every } d \in D\right\}, \\ H_P^\infty(T \oplus D) &= \{g \in L^\infty(T \oplus D);\ g(\cdot, d) \in H^\infty(T) \text{ for every } d \in D\}. \end{aligned}$$

For each nonnegative integer $N$, we define $Q_N$ as follows:

$$Q_N = \{f(t, d) \in L^\infty(T \oplus D);\ f(\cdot, \hat{d})(n) = 0 \text{ if } n < -N \text{ for every } d \in D\}.$$

Then $Q_N$ is a subalgebra of $L^\infty(T \oplus D)$ containing $H_P^\infty(T \oplus D)$. Moreover, $Q_N$ is contained in $C_u(T \oplus D) + H_P^\infty(T \oplus D)$.

CLAIM. $\bigcup_{N=0}^{\infty} Q_N$ is dense in $C_u(T \oplus D) + H_P^\infty(T \oplus D)$.

Indeed, for $f \in C_u(T \oplus D)$, we define a function $f_d \in C(T)$ by $f_d(x) = f(x, d)$ $(d \in D)$. Put $A_f = \{f_d;\ d \in D\}$.

Then $A_f$ is uniformly continuous and equicontinuous. Hence, by the Ascoli-Arzela Theorem, $A_f$ is relative compact in $C(T)$. That is, for

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$\varepsilon > 0$ , there exist functions $f_{d_1}, \ldots, f_{d_m} \in A_f$ such that

$$\bigcup_{k=1}^{m} \{h \in C(T);\ \|h-f_{d_k}\|_\infty < \varepsilon/3\} \supset A_f .$$

Let $\{K_n\}_{n=1}^\infty$ be Féjer's kernel. Then there exists a positive integer $N$ such that $\|K_N * f_{d_k} - f_{d_k}\|_\infty < \varepsilon/3 \quad (k = 1, \ldots, m)$ .

For $f_d \in A_f$ , there exists a positive integer $k$ $(1 \le k \le m)$ such that $\|f_d - f_{d_k}\|_\infty < \varepsilon/3$ . Hence we have

$$\|f_d * K_N - f_d\|_\infty < \|f_d * K_N - f_{d_k} * K_N\|_\infty + \|f_{d_k} * K_N - f_{d_k}\|_\infty + \|f_{d_k} - f_d\|_\infty$$
$$< \varepsilon .$$

Put $F_N(x, d) = f_d * K_N(x)$ . Since

$$F_N(x, d) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \hat{f}_d(n) e^{inx} ,$$

$F_N$ belongs to $Q_N$ ; and $\|F_N - f\|_\infty < \varepsilon$ . That is, $\bigcup_{N=0}^{\infty} Q_N$ is dense in $C_u(T \oplus D) + H_P^\infty(T \oplus D)$ . Therefore $C_u(T \oplus D) + H_P^\infty(T \oplus D)$ is a closed subalgebra of $L^\infty(T \oplus D)$ . //

THEOREM 21. *Let $D$ be a discrete abelian group with $\hat{D}$ torsion-free. $P$ is a subsemigroup of $R \oplus \hat{D}$ with the* (AO)*-condition such that it is not dense in $R \oplus \hat{D}$ . Then $C_u(R \oplus D) + H_P^\infty(R \oplus D)$ is a closed subalgebra of $L^\infty(R \oplus D)$ .*

Proof. $C_u(R \oplus D) + H_P^\infty(R \oplus D)$ is a closed subspace of $L^\infty(R \oplus D)$ .

We may assume $P \cap R = [0, \infty)$ . The following equations (*) are easily checked:

$$(*) \quad \begin{aligned} H_P^1(R \oplus D) &= \{f \in L^1(R \oplus D);\ f(\cdot, d) \in H^1(R) \text{ for every } d \in D\} , \\ H_P^\infty(R \oplus D) &= \{f \in L^\infty(R \oplus D);\ f(\cdot, d) \in H^\infty(R) \text{ for every } d \in D\} . \end{aligned}$$

For an integer $n$ , $\chi_n$ denotes a character on $R$ defined by

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$\chi_n(x) = e^{inx}$ . For each nonnegative integer $n$ , we define $A_{-n}$ as follows:

$$A_{-n} = \{g \in L^\infty(R \oplus D);\ g(\cdot, d) \in \chi_{-n}H^\infty(R) \text{ for each } d \in D\} .$$

Then $A_{-n}$ contains $H_P^\infty(R \oplus D)$ $(n = 0, 1, \ldots)$ .

CLAIM. $A_{-n} \subset C_u(R \oplus D) + H_P^\infty(R \oplus D)$ $(n = 0, 1, \ldots)$ .

For $g \in A_{-n}$ , let $h$ be a function in $L^1(R)$ such that $\hat{h} = 1$ on $[-n, 0]$ . Define a function $F$ on $R \oplus D$ by $F(x, d) = \big(g(\cdot, d)*h\big)(x)$ for $(x, d) \in R \oplus D$ . Then $F$ belongs to $C_u(R \oplus D)$ . Since $g(\cdot, d) - g(\cdot, d) * h \in H^\infty(R)$ for every $d \in D$ , by (*), we have $g - F \in H_P^\infty(R \oplus D)$ . Hence $g = F + (g-f)$ belongs to $C_u(R \oplus D) + H_P^\infty(R \oplus D)$ .

Since $H^\infty(R)$ is an algebra, we can prove that $\bigcup_{n=0}^{\infty} A_{-n}$ is a subalgebra contained in $C_u(R \oplus D) + H_P^\infty(R \oplus D)$ , from (*) and the above claim. Moreover, by using ([1], Theorem 12.11.1), we can prove that $\bigcup_{n=0}^{\infty} A_{-n}$ is dense in $C_u(R \oplus D) + H_P^\infty(R \oplus D)$ . Therefore $C_u(R \oplus D) + H_P^\infty(R \oplus D)$ $\left(= \overline{\bigcup_{n=0}^{\infty} A_{-n}}\right)$ is a closed subalgebra of $L^\infty(R \oplus D)$ .//

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