A NOTE ON GROUP INVARIANT CONTINUA

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Let (X, T, π) be a topological transformation group, where X is a Hausdorff continuum. We will say that X is *irreducibly T-invariant* if no proper subcontinuum of X is T-invariant. Wallace, [6], has shown that if T is abelian and X is irreducibly T-invariant, then X has no cut point; he then asked if this statement remains true if "abelian" is replaced by "compact". In this paper we answer this question in the affirmative, and prove a related result when T satisfies a recursive property.

A general problem due to Wallace is as follows: Assume T leaves an endpoint of X fixed. Under what conditions on X and T does T have another fixed point? This problem has been investigated by Wallace [8], Wang, [5], Chu, [1], and Gray, [3, 4]. In Theorem 2, we show that if X is locally connected, and T is generated by a compact subgroup and a connected subgroup, then T has another fixed point.

Departing from [7] slightly, we call a subcontinuum C of X a *universal subcontinuum* (USC) if given a subcontinuum D of X, $D \cap C$ is a continuum. The intersection of arbitrarily many USC is again a USC. If $X-x = U \cup V$, where U and V are non-empty separated subsets of X (hereafter referred to as a "separation of $X-x$"), then $U \cup \{x\}$ is a *USC*. The property of being a *USC* is topological. The proofs of these statements are to be found in [7].

The terminology pertaining to transformation groups is taken from [2].

THEOREM 1. *Let (X, T, π) be a topological transformation group where X is a Hausdorff continuum and one of the following conditions is satisfied:*

(i) *T is compact,*

(ii) *X is locally connected and T is pointwise regularly almost periodic.*

If X is irreducibly T-invariant, then X contains no cut point.

PROOF. Assume that X is irreducibly T-invariant. We make the following observations:

1. If $x \in X$, no proper *USC* of X contains the orbit Tx of x.

For otherwise the intersection, D, of all *USC* which contains Tx is a proper subcontinuum of X containing Tx. We easily verify that that D is T-invariant, and hence X is not irreducibly T-invariant.

2. If $X-x = U \cup V$ is a separation of $X-x$, then U and V both contain a cut point of X.

For by 1, we cannot have $Tx \subset U \cup \{x\}$, hence $Tx \cap V \neq \emptyset$. Likewise $Tx \cap U \neq \emptyset$. Since Tx contains only cut points of X, we have the desired result.

We now prove the theorem by contradiction. Assume X contains a cut point. Let y be a fixed non-cut point of X; for each cut point $x \in X$, choose a fixed separation $X-x = U_x \cup V_x$ of $X-x$ with $y \in U_x$ and let $H_x = V_x \cup \{x\}$. Order the collection of all such H_x by inclusion and let $\mathscr{C} = \{H_\alpha;\ \alpha \in I\}$ be a maximal totally ordered subcollection. Set $H = \cap \{H_\alpha;\ \alpha \in I\} \neq \emptyset$.

If H contains a cut point x of X, let $X-x = U \cup V$ be any separation of $X-x$ with $y \in U$. Let w be a cut point in V and $X-w = W_1 \cup W_2$ be any separation of $X-w$ with $y \in W_1$. If $\alpha \in I$ such that $x \neq x_\alpha$, then $x \in V_\alpha$, hence $U_\alpha \cup \{x_\alpha\} \subset U$. Then $V \cup \{x\} \subset V_\alpha$. Also, since $w \in V$, $U \cup \{x\} \subset W_1$, hence $W_2 \cup \{w\} \subset V$. Thus if $x \neq x_\alpha$ for all $\alpha \in I$, H_x is a proper subset of each H_α. Otherwise, H_ω is a proper subset of each H_α. Thus in any case $\mathscr{C}$ is not maximal. This contradiction shows that H contains no cut point of X.

Assume T is compact and let x be any cut point of X. Since H contains no cut points, $H \cap Tx = \emptyset$. Since T is compact, $X-Tx$ is an open set containing $H = \cap \{H_\alpha;\ \alpha \in I\}$. Therefore for some $\beta \in I$, we have $H_\beta = V_\beta \cup \{x_\beta\} \subset X-Tx$. Then $Tx \subset U_\beta$, and $U_\beta \cup \{x_\beta\}$ is a proper USC of X containing Tx, which contradicts 2. This completes the proof in case (i).

Now assume (ii) holds. Let $X-x = U \cup V$ be a separation of $X-x$, and let y be a cut point of X with $y \in U$; by 1., there is a $t \in T$ such that $z = ty \in V$. Since T is regularly almost periodic at y, it easily follows that there is a syndetic invariant subgroup S of T such that $Sy \subset U$ and $Sz \subset V$. Let $E(y, z) = \{w;\ w \text{ separates } y \text{ and } z \text{ in } X\} \cup \{y, z\}$. It is clear that $Sx \subset E(y, z)$. Since X is locally connected, $E(y, z)$ is closed, thus $\overline{Sx} \subset E(y, z)$ so that $\overline{Sx}$ contains cut points alone. Let K be a compact subset of T such that $T = KS$. Then $K\overline{Sx}$ is a compact set of cut points of X. We may complete the proof as in case (i).

The definition of an end point is that found in [5] namely: e is an end-point of X if e does not separate X and given any open set U containing e, there exists $y \in U$ such that $X-y = P \cup Q$, P and Q separated, and $e \in P \subsetneqq U$.

THEOREM 2. *Let (X, T, π) be a topological transformation group, where X is a non-trivial locally connected Hausdorff continuum and T is generated by a compact subgroup C and a connected subgroup K. If T leaves an end point e of X fixed, then T has another fixed point.*

PROOF. Let z be a non-cut point of X other than e and let $X-x = U \cup V$ be a separation of $X-x$ such that $e \in U$ and $Cz \subset V$. If $p \in Cz$, then Kp is a connected set of non-cut points of X and it follows that $Kp \subset V$; thus $KCz \subset V$. Let

$$H = \{e\} \cup \{y;\, y \text{ separates } e \text{ and } KCz \text{ in } X\}.$$

If KCz is a point, K and C have a fixed point in common, and we are through. Otherwise, H contains no non-cut points of X other than e, and H is closed since X is locally connected. This means that $\overline{Kx} \subset H$; order $\overline{Kx}$ in a standard fashion as follows: e is the first element of $\overline{Kx}$ (assuming $e \in \overline{Kx}$); if $p,\, y \in \overline{Kx}$, $p \neq e$, $y \neq e$, then $p \leqq y$ iff $p = y$ or p separates e from y in X. $\leqq$ is a total order on $\overline{Kx}$. By virtue of [8], $\overline{Kx}$ has a largest element w; w is a cut point of X, and w is evidently fixed under K. Further, every point of Cw separates e from Cz. We note that Cw is compact and proceed as above to show that Cw contains a fixed point of C. But this means that w is fixed under C. Certainly $w \neq e$, so that C and K have a fixed point, other than e, in common.

References

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