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THE ASYMPTOTIC RATIO SET AND DIRECT INTEGRAL DECOMPOSITIONS OF A VON NEUMANN ALGEBRA

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The fact that any von Neumann algebra on a separable Hilbert space has an essentially unique direct integral decomposition into factors means that there is a global as well as a local aspect to any partial classification of von Neumann algebras. More precisely, suppose that J is a statement about von Neumann algebras which is either true or false for any given von Neumann algebra. Then a von Neumann algebra is said to satisfy J *globally* if it satisfies J, and to satsify J *locally* if almost all the factors appearing in some (and hence in any) central decomposition of it satisfy J. In a recent paper [**3**], H. Araki and E. J. Woods introduced the notion of the asymptotic ratio set of a factor, and by means of this they made remarkable progress in the classification of factors. The asymptotic ratio set of an arbitrary von Neumann algebra can be defined in an obvious way, and gives a partial classification of von Neumann algebras. The purpose of the present paper is to investigate the local properties of this classification.

Suppose that $\mathfrak{A}$ is a von Neumann algebra on a separable Hilbert space, and let $\zeta \rightarrow \mathfrak{A}(\zeta)$ be a Borel field of factors on some standard Borel space Z whose direct integral with respect to some Borel measure μ on Z is spatially isomorphic to $\mathfrak{A}$. The local aspect of the classification of von Neumann algebras into those types I, II, and III is, of course, well known (see, for instance, [**12**, Corollary III.1.3 and Theorem III.1.6]). Its essential features are the following: (i) Z is the disjoint union of the μ-measurable sets

$$Z_x = \{\zeta \in Z : \mathfrak{A}(\zeta) \text{ is of type } x\},$$

$x = I, II$, and III; (ii) $\mathfrak{A}$ is of type I (respectively, type II, type III) if and only if $\mathfrak{A}(\zeta)$ is of type I (respectively, type II, type III) for μ − a.a. $\zeta \in Z$ (i.e., the local and global definitions of type coincide); and (iii) the sets Z_x can be used in a natural way to obtain the unique decomposition of $\mathfrak{A}$ into a direct sum of algebras of types I, II, and III. The situation is only slightly more complicated when the $\mathfrak{A}(\zeta)$ are classified according to their asymptotic ratio sets. The exact analogue of (i) remains true (Corollary 5). The asymptotic ratio set of a von Neumann algebra $\mathfrak{A}$ is always defined (i.e., defined globally). However, the asymptotic ratio set of $\mathfrak{A}$ is not in general defined locally; in fact, it is defined locally if and only if the $\mathfrak{A}(\zeta)$ have the same asymptotic ratio set for μ − a.a. $\zeta \in Z$. (Here "global" and "local"

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are used in the sense of the previous paragraph.) Thus the statement corresponding to (ii) must fail. If, however, an algebra has a locally defined asymptotic ratio set, then the global and local definitions agree, and the algebra is said to be of pure asymptotic ratio type. Finally, (iii), the most important of the three, does carry over. The analogue must, however, be phrased in terms of a direct integral whose components are of pure asymptotic ratio type (Theorem 6).

E. G. Effros' formulation of the direct integral theory of von Neumann algebras [**6**; **7**] will be used throughout. It is not only more elegant than the customary formulation (as found in, say, [**5**; **12**]), but also technically advantageous because it facilitates the application of Borel spaces to direct integral theory. The standard results about Borel spaces (as contained in [**4**, pp. 2–13; **9**, § 1–6]) will be used frequently, and the reader is assumed to be familiar with them. A brief review of Effros' direct integral theory is included here to acquaint the reader with the notation and terminology used below. All Hilbert spaces considered are by assumption complex, separable and positive dimensional. For each $n = \infty, 1, 2, \ldots$, let $\mathscr{H}_n$ be a fixed n-dimensional Hilbert space, and let $\mathscr{L}_n$ and $\mathscr{A}_n$ be the sets of all bounded linear operators and all von Neumann algebras on $\mathscr{H}_n$, respectively. Give to $\mathscr{H}_n$ and $\mathscr{L}_n$ the weak Borel structure and to $\mathscr{A}_n$ the standard Borel structure defined in [**6**]. This means that $\mathscr{H}_n$ (respectively, $\mathscr{L}_n$) is given the Borel structure generated by the weak topology on $\mathscr{H}_n$ (respectively, the weak operator topology on $\mathscr{L}_n$), or equivalently, the strong topology $\mathscr{H}_n$ (respectively, the strong operator topology on $\mathscr{L}_n$). The important facts about the Borel structure on $\mathscr{A}_n$ are that (i) $\mathfrak{A} \to \mathfrak{A}'$ is a Borel automorphism of $\mathscr{A}_n$ and that (ii) a map $\zeta \to \mathfrak{A}(\zeta)$ from a Borel space Z into $\mathscr{A}_n$ is Borel if and only if there are Borel maps $\zeta \to A_k(\zeta)$, $k = 1, 2, \ldots$, from Z into the unit ball of $\mathscr{L}_n$ such that the $A_k(\zeta)$ are weakly dense in the unit ball of $\mathfrak{A}(\zeta)$ for each ζ in Z. Also, let $\mathscr{H}_u$ be the Borel space union (or sum) of the $\mathscr{H}_n$ and define $\mathscr{L}_u$ and $\mathscr{A}_u$ similarly. Suppose that Z is a standard Borel space, that μ is a finite positive Borel measure on Z, and that for each $\zeta \in Z$, $\mathfrak{A}(\zeta)$ is a von Neumann algebra on a Hilbert space $\mathscr{H}(\zeta)$. Put $Z_n = \{\zeta \in Z: \dim \mathscr{H}(\zeta) = n\}$ for each n. Then $\zeta \to \mathfrak{A}(\zeta)$ is said to be a Borel field of von Neumann algebras on Z if for each n and each $\zeta \in Z_n$, there is a linear isometry $\gamma(\zeta)$ of $\mathscr{H}(\zeta)$ onto $\mathscr{H}_n$ such that $\zeta \to \gamma(\zeta)\mathfrak{A}(\zeta)\gamma(\zeta)^{-1}$ is a Borel map from Z into $\mathscr{A}_u$. If this is the case, then a von Neumann algebra $\mathfrak{A}$, called the direct integral of the field $\zeta \to \mathfrak{A}(\zeta)$ with respect to μ and denoted by

$$\int_Z^{\oplus} \mathfrak{A}(\zeta) d\mu(\zeta),$$

can be defined as follows. (The notation makes no reference to the $\gamma(\zeta)$ since it turns out that, to within spatial isomorphism, the direct integral is independent of the choice of the $\gamma(\zeta)$ [**7**, Lemma 4.1].) The Hilbert space on which $\mathfrak{A}$ acts is the direct integral of the field $\zeta \to \mathscr{H}(\zeta)$ with respect to μ, which is denoted by

$$\int_Z^{\oplus} \mathscr{H}(\zeta)d\mu(\zeta).$$

It consists of all those vector fields $\zeta \to x(\zeta)$ (i.e., maps $x : Z \to \bigcup_{\zeta\in Z}\mathscr{H}(\zeta)$ with $x(\zeta) \in \mathscr{H}(\zeta)$ for each $\zeta \in Z$) which satisfy (i) $\zeta \to \gamma(\zeta)x(\zeta)$ is a Borel map from Z into $\mathscr{H}_u$ and (ii)

$$\int_Z ||x(\zeta)||^2 d\mu(\zeta) < \infty,$$

where, as usual, two such vector fields $\zeta \to x(\zeta)$ and $\zeta \to y(\zeta)$ are identified whenever

$$\int_Z ||x(\zeta) - y(\zeta)||^2 d\mu(\zeta) = 0.$$

The linear operations are the pointwise ones and the inner product is

$$(x, y) = \int_Z (x(\zeta), y(\zeta))d\mu(\zeta).$$

If, for each $\zeta \in Z$, $A(\zeta)$ is an operator in $\mathfrak{A}(\zeta)$ and if $\zeta \to \gamma(\zeta)A(\zeta)\gamma(\zeta)^{-1}$ is a uniformly bounded Borel map from Z into $\mathscr{L}_u$, then the map which assigns to a vector field $\zeta \to x(\zeta)$ in $\mathscr{H}$ the vector field $\zeta \to A(\zeta)x(\zeta)$, again in $\mathscr{H}$, is a bounded linear operator on $\mathscr{H}$. The set of all such operators constitutes a von Neumann algebra and is the desired algebra $\mathfrak{A}$. It should be pointed out that if $\zeta \to \mathfrak{B}(\zeta)$ is a second Borel field of von Neumann algebras on Z and if $\mathfrak{A}(\zeta)$ is *-isomorphic (respectively, spatially isomorphic) to $\mathfrak{B}(\zeta)$ for μ – a.a. $\zeta \in Z$, then $\mathfrak{A}$ is *-isomorphic (respectively, spatially isomorphic) to

$$\int_Z^{\oplus} \mathfrak{B}(\zeta)d\mu(\zeta)$$

(this is stated without proof as [**7**, Lemma 4.1]). In fact, the proof in the case of spatial isomorphisms is just a straightforward application of the Mackey-von Neumann cross-section theorem, and in the case of *-isomorphisms can be reduced to the first case by the device used in the proof of Proposition 1. In what follows, $\mathfrak{A} \cong \mathfrak{B}$ will mean that $\mathfrak{A}$ and $\mathfrak{B}$ are *-isomorphic.

Following [**7**, § 5], $(Z, \mu, \zeta \to \mathfrak{A}(\zeta))$ is called a *central decomposition* of a von Neumann algebra $\mathfrak{A}$ if each of the following statements is true: (i) μ is a finite positive Borel measure on a standard Borel space Z; (ii) $\zeta \to \mathfrak{A}(\zeta)$ is a Borel field of factors on Z; and (iii) $\mathfrak{A}$ and

$$\int_Z^{\oplus} \mathfrak{A}(\zeta)d\mu(\zeta)$$

are spatially isomorphic. There is a refinement (or transitivity) theorem for central decompositions. Roughly speaking, it says that the central decompositions of the components of a direct integral decomposition of a von Neumann algebra $\mathfrak{A}$ can be put together to give a central decomposition of $\mathfrak{A}$. Although this theorem can be made to follow from a result of Guichardet [**8**, § 5, Proposition 2], a simpler proof of it is included here. Let μ be a finite positive Borel measure on a standard Borel space Z, let $\zeta \rightarrow \mathfrak{A}(\zeta)$ be a Borel field of von Neumann algebras on Z, and let

$$\mathfrak{A} = \int_Z^{\oplus} \mathfrak{A}(\zeta) d\mu(\zeta).$$

Suppose that $\mathfrak{A}(\zeta)$ acts on a Hilbert space $\mathscr{H}(\zeta)$ so that $\mathfrak{A}$ acts on

$$\mathscr{H} = \int_Z^{\oplus} \mathscr{H}(\zeta) d\mu(\zeta).$$

The algebra $\mathfrak{A}_0$ of diagonal operators on $\mathscr{H}$ is *-isomorphic to $L^\infty(Z, \mu)$ and is a subalgebra of the center $\mathfrak{A} \cap \mathfrak{A}'$ of $\mathfrak{A}$. Let ν be a finite Borel measure on a standard Borel space Y such that $\mathfrak{A} \cap \mathfrak{A}'$ and $L^\infty(Y, \nu)$ are *-isomorphic. The inclusion map of $\mathfrak{A}_0$ into $\mathfrak{A} \cap \mathfrak{A}'$ is induced by a Borel mapping f of Y into Z which carries ν into a measure $\tilde{\mu}$ equivalent to μ, i.e., $\tilde{\mu} = f_*(\nu)$ and f^* is the inclusion map in the notation of [**4**] (see, e.g., [**8**, § 1, Proposition 1]). Define an equivalence relation R on Y by specifying that $\xi_1 R \xi_2$ if and only if $f(\xi_1) = f(\xi_2)$. Let Y/R be the quotient Borel space, $p : Y \rightarrow Y/R$ the quotient map, and $\tilde{\nu}$ the quotient measure. Let g be the unique mapping from Y/R into Z which satisfies $g \circ p = f$. Since g is one-one and Borel, Y/R must be countably separated and hence analytic. Therefore, by deleting a ν-null R-saturated Borel set from Y and changing notation, Y/R can be assumed to be standard. Then, however, the range of g (which is also the range f) is a Borel subset of Z and g is a Borel isomorphism of Y/R with $f(Y)$, the latter having the relative Borel structure. Notice that μ and $g_*(\tilde{\nu})$ are equivalent measures on Z and that $\mu(Z - f(Y)) = 0$.

Since the centre of $\mathfrak{A}$ is *-isomorphic to $L^\infty(Y, \nu)$, there is a Borel field $\xi \rightarrow \mathfrak{B}(\xi)$ of factors on Y such that $\mathfrak{A}$ and

$$\int_Y^{\oplus} \mathfrak{B}(\xi) d\nu(\xi)$$

are spatially isomorphic. Suppose that $\mathfrak{B}(\xi)$ acts on a Hilbert space $\mathscr{K}(\xi)$. Choose a family $(\nu_\eta)_{\eta \in Y/R}$ of finite positive Borel measures on Y in accordance with the measure disintegration theorem [**7**, Lemma 4.4]. It can be assumed that the ν_η, $\eta \in Y/R$, are concentrated on pairwise disjoint Borel subsets of Y. By [**7**, Lemma 4.5 and its proof],

$$\eta \rightarrow \int_Y^{\oplus} \mathfrak{B}(\xi) d\nu_\eta(\xi)$$

is a Borel field of von Neumann algebras on Y/R and there is a unitary operator from $\mathscr{H}$ to

$$\int_{Y/R}^{\oplus} \int_{Y}^{\oplus} \mathscr{K}(\xi) d\nu_\eta(\xi) d\tilde{\nu}(\eta)$$

which carries $\mathfrak{A}_0$ and $\mathfrak{A}$ onto the diagonal algebra and

$$\int_{Y/R}^{\oplus} \int_{Y}^{\oplus} \mathfrak{B}(\xi) d\nu_\eta(\xi) d\tilde{\nu}(\eta),$$

respectively. Now by [**5**, p. 173, Proposition 1(ii) and p. 212, Théorème 4 and its proof], there is a μ-null Borel subset N of $f(Y)$ such that

$$\int_{Y}^{\oplus} \mathfrak{B}(\xi) d\nu_\eta(\xi)$$

and $\mathfrak{A}(g(\eta))$ are spatially isomorphic for all $\eta \in Y/R - g^{-1}(N)$. The desired refinement theorem for central decompositions has now been proved. It is worth pointing out that the most general refinement theorem for direct integral decompositions of a von Neumann algebra can be obtained by a trivial modification of the above proof.

Let $\mathscr{R}_x$, $0 \leqslant x \leqslant 1$, be the family of factors defined in [**3**, Definition 3.10]. Recall that $\mathscr{R}_0$ is the I_∞ factor, that $\mathscr{R}_1$ is the hyperfinite II_1 factor and that the $\mathscr{R}_x$, $0 < x < 1$, are just the mutually non-isomorphic type III factors studied in [**11**, § 4]. It is known that the $\mathscr{R}_x$ can be realized on $\mathscr{H}_\infty$ in such a way that $x \to \mathscr{R}_x$ is a Borel map from [0, 1] into $\mathscr{A}_\infty$ (see, e.g., [**10**]). The *asymptotic ratio set* of a von Neumann algebra $\mathfrak{A}$, denoted by $r_\infty(\mathfrak{A})$, is that subset of $[0, \infty)$ defined as follows: an $x \in [0, 1]$ is in $r_\infty(\mathfrak{A})$ if and only if $\mathfrak{A} \cong \mathfrak{A} \otimes \mathscr{R}_x$, and an $x > 1$ is in $r_\infty(\mathfrak{A})$ if and only if $x^{-1} \in r_\infty(\mathfrak{A})$. Araki [**1**] has shown that the only possibilities for the asymptotic ratio set of a factor are:

$$S_\emptyset = \emptyset, S_0 = \{0\}, S_1 = \{1\}, S_{01} = \{0, 1\}, S_\infty = [0, \infty),$$

and

$$S_x = \{0\} \cup \{x^n : n = 0, \pm 1, \pm 2, \ldots\}, 0 < x < 1.$$

In [**3**] it is shown that $r_\infty(\mathscr{R}_x) = S_x$, $0 < x < 1$.

PROPOSITION 1. *Let μ be a finite positive Borel measure on a standard Borel space Z, let $\zeta \to \mathfrak{A}(\zeta)$ be a Borel field of von Neumann algebras on Z, and set*

$$\mathfrak{A} = \int_{Z}^{\oplus} \mathfrak{A}(\zeta) d\mu(\zeta).$$

Then for any non-negative number x, $x \in r_\infty(\mathfrak{A})$ if and only if $x \in r_\infty(\mathfrak{A}(\zeta))$ for μ — a.a. *$\zeta \in Z$.*

Proof. Say $x \in [0, 1]$. If $x \in r_\infty(\mathfrak{A}(\zeta))$ for μ — a.a. $\zeta \in Z$, then $x \in r_\infty(\mathfrak{A})$. This is an immediate consequence of [**7**, Lemma 4.1] and the fact that $\zeta \to \mathfrak{A}(\zeta) \otimes \mathscr{R}_x$ is a Borel field whose direct integral with respect to μ is spatially isomorphic to $\mathfrak{A} \otimes \mathscr{R}_x$ [**7**, p. 443].

Conversely, suppose that $x \in r_\infty(\mathfrak{A})$. Then $\mathfrak{A} \otimes \mathbf{C}(\mathscr{H}_\infty)$ and $\mathfrak{A} \otimes \mathscr{R}_x \otimes \mathbf{C}(\mathscr{H}_\infty)$ are spatially isomorphic. The notation that was introduced during the proof of the refinement theorem for central decompositions is used again here. $\xi \to \mathfrak{B}(\xi) \otimes \mathbf{C}(\mathscr{H}_\infty)$ and $\xi \to \mathfrak{B}(\xi) \otimes \mathscr{R}_x \otimes \mathbf{C}(\mathscr{H}_\infty)$ are two Borel fields of factors on Y whose direct integrals with respect to ν are spatially isomorphic. Since any direct integral of factors is the central decomposition, there are ν-null Borel subsets M and M_1 of Y and a Borel isomorphism h of $Y - M$ onto $Y - M_1$ such that $\mathfrak{B}(\xi)$ and $\mathfrak{B}(h(\xi)) \otimes \mathscr{R}_x$ are *-isomorphic for all $\xi \in Y - M$ [**5**, p. 212, Théorème 4 and p. 173, Proposition 1(ii)]. Then for any such ξ,

$$\begin{aligned}\mathfrak{B}(\xi) \otimes \mathscr{R}_x &\cong \mathfrak{B}(h(\xi)) \otimes \mathscr{R}_x \otimes \mathscr{R}_x \\ &\cong \mathfrak{B}(h(\xi)) \otimes \mathscr{R}_x \\ &\cong \mathfrak{B}(\xi),\end{aligned}$$

i.e., $x \in r_\infty(\mathfrak{B}(\xi))$. Now since

$$0 = \nu(M) = \int_{Y/R} \nu_\eta(M) d\tilde{\nu}(\eta),$$

there is a μ-null Borel subset P of $f(Y)$ such that $\nu_\eta(M) = 0$ whenever $\eta \in Y/R - g^{-1}(P)$. This means that $\mathfrak{A}(\zeta)$ and

$$\int_Y^{\oplus} \mathfrak{B}(\xi) d\nu_\eta(\xi)$$

are spatially isomorphic and that $x \in r_\infty(\mathfrak{B}(\xi))$ for ν_η — a.a. $\xi \in Y$ whenever ζ is in $g(Y/R) - (N \cup P)$ and is the image of $\eta \in Y/R$ by g. Therefore (by the first part of the proof) x lies in $r_\infty(\mathfrak{A}(\zeta))$ for all ζ in $f(Y) - (N \cup P)$. Since $f(Y) - (N \cup P)$ is the complement of a μ-null Borel set, this completes the proof of the Proposition.

COROLLARY 2. *The asymptotic ratio set of a von Neumann algebra must be one of the sets* $S_\emptyset$, S_{01}, S_∞, *or* S_x, $0 \leqslant x \leqslant 1$.

Proof. Let $\mathfrak{A}$ be a given von Neumann algebra. From the properties of the $\mathscr{R}_x$ it follows immediately that $r_\infty(\mathfrak{A}) - \{0\}$ is either empty or else it is a multiplicative subgroup of the positive reals (cf. the proof of [**3**, Lemma 6.5]). So to show that $r_\infty(\mathfrak{A})$ is of the desired form it is only necessary to show that it is closed. Suppose that (x_n) is a sequence in $r_\infty(\mathfrak{A})$ which converges to a non-negative number x and choose a central decomposition $(Z, \mu, \zeta \to \mathfrak{A}(\zeta))$ of $\mathfrak{A}$. By the Proposition there exist μ-null Borel subsets $W_1, W_2, \ldots$ of Z such that $x_n \in r_\infty(\mathfrak{A}(\zeta))$ for all $\zeta \in Z - W_n$ and all n. Then $x_n \in r_\infty(\mathfrak{A}(\zeta))$ for all n and all $\zeta \in Z - \bigcup_{n=1}^{\infty} W_n$. The fact that the asymptotic ratio set of any factor is closed then implies that $x \in r_\infty(\mathfrak{A}(\zeta))$ for all such ζ. Another application of Proposition 1 now completes the proof of the Corollary.

Araki has recently given a completely different proof of Corollary 2 [**2**].

When factors are classified by their asymptotic ratio set rather than by their type, (ii) of the second paragraph must be replaced by Proposition 1. The analogy, however, is far from perfect, because

$$r_\infty(\mathscr{R}_0 \oplus \mathscr{R}_x) = S_0 = r_\infty(\mathscr{R}_0) \neq r_\infty(\mathscr{R}_x)$$

whenever $0 < x < 1$. In obtaining the replacements for (i) and (iii), it will be convenient to begin with a general theorem (cf. [**5**, p. 180, Proposition 5]).

THEOREM 3. *If $\zeta \to \mathfrak{A}(\zeta)$ and $\zeta \to \mathfrak{B}(\zeta)$ are two Borel fields of von Neumann algebras on a standard Borel space Z, then $\{\zeta \in Z: \mathfrak{A}(\zeta) \cong \mathfrak{B}(\zeta)\}$ is an analytic subset of Z.*

Proof. The device used in the proof of Proposition 1 reduces the theorem to showing that

$$W_0 = \{\zeta \in Z: \mathfrak{A}(\zeta) \text{ is spatially isomorphic to } \mathfrak{B}(\zeta)\}$$

is an analytic set. If $\mathscr{H}(\zeta)$ and $\mathscr{K}(\zeta)$ denote the Hilbert spaces on which $\mathfrak{A}(\zeta)$ and $\mathfrak{B}(\zeta)$ act, respectively, then

$$\{\zeta \in Z: \dim \mathscr{H}(\zeta) = \dim \mathscr{K}(\zeta) = n\},\ 1 \leqq n \leqq \infty,$$

is a sequence of Borel sets whose union contains W_0. This makes it possible to assume that $\mathscr{H}(\zeta) = \mathscr{K}(\zeta) = \mathscr{H}_n$ for all $\zeta \in Z$ and some n. The set $\mathscr{L}_{n,u}$ of unitary operators in $\mathscr{L}_n$ is standard in the relative Borel structure. Put

$$W = \{(\zeta, T) \in Z \times \mathscr{L}_{n,u}: T\mathfrak{A}(\zeta)T^* = \mathfrak{B}(\zeta)\}.$$

Since W_0 is the projection of W onto its first component, it is enough to show that W is a Borel set.

By [**6**, Theorem 3 and the Corollary to Theorem 2], there exist Borel maps $\zeta \to S_i(\zeta)$, $\zeta \to T_i(\zeta)$, $\zeta \to S_i'(\zeta)$ and $\zeta \to T_i'(\zeta)$ $(i = 1, 2, \ldots)$ of Z into the unit ball of $\mathscr{L}_n$ such that the $S_i(\zeta)$ (respectively, the $T_i(\zeta)$, $S_i'(\zeta)$, $T_i'(\zeta)$) are weakly dense in the unit ball of $\mathfrak{A}(\zeta)$ (respectively, $\mathfrak{B}(\zeta)$, $\mathfrak{A}(\zeta)'$, $\mathfrak{B}(\zeta)'$) for each $\zeta \in Z$. A point (ζ, T) in $Z \times \mathscr{L}_{n,u}$ is in W if and only if both

$$T\mathfrak{A}(\zeta)T^* \subset \mathfrak{B}(\zeta)''$$

and

$$T\mathfrak{A}(\zeta)'T^* \subset \mathfrak{B}(\zeta)',$$

and this in turn is the case if and only if

$$TS_i(\zeta)T^*T_j'(\zeta) = T_j'(\zeta)TS_i(\zeta)T^*$$

and

$$TS_i'(\zeta)T^*T_j(\zeta) = T_j(\zeta)TS_i'(\zeta)T^*,$$

for all i and j. Each side of the first equation is a Borel function in its dependence on (ζ, T) and hence this equation holds on a Borel subset of $Z \times \mathscr{L}_{n,u}$.

Similar considerations apply to the second equation. This means that W has been expressed as the intersection of a countable number of Borel sets and therefore is itself a Borel set.

COROLLARY 4. *Let $\zeta \rightarrow \mathfrak{A}(\zeta)$ be a Borel field of von Neumann algebras on a standard Borel space Z and let $\mathscr{R}$ be any von Neumann algebra. Then*

$$\{\zeta \in Z: \mathfrak{A}(\zeta) \otimes \mathscr{R} \cong \mathfrak{A}(\zeta)\}$$

is an analytic subset of Z.

COROLLARY 5. *Suppose that $\zeta \rightarrow \mathfrak{A}(\zeta)$ is a Borel field of von Neumann algebras on a standard Borel space Z. Let $Z_\emptyset = \{\zeta \in Z: r_\infty(\mathfrak{A}(\zeta)) = S_\emptyset\}$ and define Z_{01}, Z_∞ and $Z_x (0 \leqslant x \leqslant 1)$ similarly. Then each of these sets is universally measurable, i.e., measurable with respect to any finite Borel measure on Z.*

Proof. Notice first that for any von Neumann algebra $\mathfrak{A}$,

$$\begin{aligned} r_\infty(\mathfrak{A}) &= S_\emptyset \Leftrightarrow \mathfrak{A} \otimes \mathscr{R}_0 \ncong \mathfrak{A} \text{ and } \mathfrak{A} \otimes \mathscr{R}_1 \ncong \mathfrak{A} \\ r_\infty(\mathfrak{A}) &= S_0 \Leftrightarrow \mathfrak{A} \otimes \mathscr{R}_0 \cong \mathfrak{A} \text{ and } \mathfrak{A} \otimes \mathscr{R}_1 \ncong \mathfrak{A} \\ r_\infty(\mathfrak{A}) &= S_1 \Leftrightarrow \mathfrak{A} \otimes \mathscr{R}_0 \ncong \mathfrak{A} \text{ and } \mathfrak{A} \otimes \mathscr{R}_1 \cong \mathfrak{A} \\ r_\infty(\mathfrak{A}) &= S_\infty \Leftrightarrow \mathfrak{A} \otimes \mathscr{R}_{\frac{1}{2}} \cong \mathfrak{A} \text{ and } \mathfrak{A} \otimes \mathscr{R}_{\frac{1}{3}} \cong \mathfrak{A}, \end{aligned}$$

and that for each $x \in (0, 1)$,

$$r_\infty(\mathfrak{A}) = S_x \Leftrightarrow \mathfrak{A} \otimes \mathscr{R}_x \cong \mathfrak{A} \text{ and } \mathfrak{A} \otimes \mathscr{R}_y \ncong \mathfrak{A},$$

for $y = x^{\frac{1}{2}}, x^{\frac{1}{3}}, \ldots$. With the exception of Z_{01}, the universal measurability of the sets in question is now a consequence of Corollary 4.

Since $(\zeta, x) \rightarrow \mathfrak{A}(\zeta) \otimes \mathscr{R}_x$ and $(\zeta, x) \rightarrow \mathfrak{A}(\zeta)$ are Borel fields of von Neumann algebras on $Z \times (0, 1)$ [7, p. 443], Theorem 3 implies that

$$W = \{(\zeta, x) \in Z \times (0, 1): \mathfrak{A}(\zeta) \otimes \mathscr{R}_x \cong \mathfrak{A}(\zeta)\}$$

is an analytic subset of $Z \times (0, 1)$. The image W_0 of W under the projection of $Z \times (0, 1)$ onto its first component is then universally measurable. On the other hand, W_0 is just the union of Z_∞ and the Z_x, $0 < x < 1$. This means that Z_{01} is the complement in Z of $W_0 \cup Z_\emptyset \cup Z_0 \cup Z_1$ and therefore Z_{01}, too, is universally measurable.

Of course, Corollary 6 implies the analogue to (i) of the second paragraph. The result corresponding to (iii) will now be derived.

Let μ be a finite positive Borel measure on a standard Borel space Z, let $\zeta \rightarrow \mathfrak{A}(\zeta)$ be a Borel field of von Neumann algebras on Z and set

$$\mathfrak{A} = \int_Z^\oplus \mathfrak{A}(\zeta) d\mu(\zeta).$$

Suppose that there is a map $y: Z \rightarrow (0, 1)$ such that $r_\infty(\mathfrak{A}(\zeta)) = S_{y(\zeta)}$ for all $\zeta \in Z$. As in the proof of Corollary 5, the image W_a of

$$\{(\zeta, x) \in Z \times (a, 1): \mathfrak{A}(\zeta) \otimes \mathscr{R}_x \cong \mathfrak{A}(\zeta)\}$$

under the projection of $Z \times (0, 1)$ on Z is μ-measurable for each $a \in (0, 1)$. It is easy to see that $y^{-1}((a, 1)) = W_a$ for any such a, which in turn shows that y is μ-measurable. So by deleting a μ-null Borel set form Z and changing notation, y can be assumed to be Borel. Define an equivalence relation R on Z by specifying that $\zeta_1 R \zeta_2$ if and only if $y(\zeta_1) = y(\zeta_2)$. As in the proof of the refinement theorem, it can be assumed that $y(Z)$ is a Borel set and is in a natural way Borel isomorphic to Z/R. Moreover, this isomorphism carries the quotient measure on Z/R onto ν, the restriction to $y(Z)$ of the measure $y_*(\mu)$ on $(0, 1)$ defined by μ and y. By [**7**, Lemmas 4.4 and 4.5], there is a family $(\mu_x)_{x \in y(Z)}$ of finite positive Borel measures on Z such that each μ_x is concentrated on $y^{-1}(x)$ and such that $\mathfrak{A}$ and

$$\int_{y(Z)}^{\oplus} \int_{Z}^{\oplus} \mathfrak{A}(\zeta) d\mu_x(\zeta) d\nu(x)$$

are spatially isomorphic.

Now suppose that $(Z, \mu, \zeta \to \mathfrak{A}(\zeta))$ is a central decomposition of a von Neumann algebra $\mathfrak{A}$. $\mathfrak{A}$ is said to be of *pure asymptotic ratio type* $S_\emptyset$ if $r_\infty(\mathfrak{A}(\zeta)) = S_\emptyset$ for μ – a.a. $\zeta \in Z$, and similarly for the other possibilities for the asymptotic ratio set (cf. [**12**, p. 234, Definition 4]). Restricting Corollary 5 and the above discussion to the case of a Borel field of factors gives the promised analogy to (iii).

THEOREM 6. *Let $\mathfrak{A}$ be a von Neumann algebra. There exist pairwise orthogonal projections E, F and G in the centre of $\mathfrak{A}$ such that* (i) *if E (respectively, F, G) is non-zero, then $\mathfrak{A}_E$ (respectively, $\mathfrak{A}_F$, $\mathfrak{A}_G$) is of pure asymptotic ratio type $S_\emptyset$ (respectively, S_{01}, S_∞), and* (ii) *if $I \neq E + F + G$, then there is a finite positive Borel measure μ on* $[0, 1]$ *and a field $x \to \mathfrak{A}_x$ of von Neumann algebras on* $[0, 1]$ *such that $\mathfrak{A}_{I-(E+F+G)}$ and*

$$\int_{[0,1]}^{\oplus} \mathfrak{A}_x d\mu(x)$$

are spatially isomorphic and such that $\mathfrak{A}_x$ is of pure asymptotic ratio type S_x for each $x \in [0, 1]$. Moreover, the above conditions determine E, F, G uniquely and (if $I \neq E + F + G$) determine μ to within equivalence and the $\mathfrak{A}_x$ to within spatial isomorphism for μ – a.a. $x \in [0, 1]$.

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