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PAPER

Travelling waves with continuous profile for hyperbolic Keller-Segel equation

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Abstract
This work describes a hyperbolic model for cell-cell repulsion with population dynamics. We consider the pressure produced by a population of cells to describe their motion. We assume that cells try to avoid crowded areas and prefer locally empty spaces far away from the carrying capacity. Here, our main goal is to prove the existence of travelling waves with continuous profiles. This article complements our previous results about sharp travelling waves. We conclude the paper with numerical simulations of the PDE problem, illustrating such a result. An application to wound healing also illustrates the importance of travelling waves with a continuous and discontinuous profile.

1. Introduction

The model and its motivation: In this paper, we mainly consider the following equation:

$$\begin{cases} \partial_t u(t,x) = \underbrace{\chi\ \partial_x\ (u(t,x)\partial_x p(t,x))}_{\text{Cell-cell repulsion}} + \underbrace{\lambda\ u(t,x)\left(1-\frac{u(t,x)}{\kappa}\right)}_{\text{Vital dynamic}}, & t>0, x\in\mathbb{R}, \\ p(t,x) - \sigma^2\ \partial_{xx}p(t,x) = u(t,x), & t>0, x\in\mathbb{R}, \end{cases} \tag{1.1}$$

with the initial distribution

$$u(0,x) = u_0(x) \in \mathrm{L}^\infty(\mathbb{R}), \tag{1.2}$$

where $\lambda > 0$ is the growth rate, $\kappa > 0$ is the carrying capacity, $\chi > 0$ is the dispersion coefficient, $\sigma > 0$ is a sensing coefficient, $x \to u(t,x)$ is the density of population, and $p(t,x)$ is an external pressure.

Here, the term density of population means that

$$\int_{x_1}^{x_2} u(t,x)dx$$

is the number of individuals between x_1 and x_2 (when $x_1 < x_2$).

† P.M. was at the origin of this research and the corresponding author of this article for the original submission. Sadly, he passed away before the reviewing process was complete.

In the model the term, $\chi\,\partial_x\,(u(t,x)\partial_x p(t,x))$ describes the cell-cell repulsion, and a logistic term $\lambda u(t,x)(1-u(t,x)/\kappa)$ corresponds the cell division, cell mortality, and the quadratic term $u(t,x)^2/\kappa$ corresponds to growth limitations due to quorum sensing (for short slow down the process of cell division) and due to competition for resources.

Replacing $u(t,x)$ and $p(t,x)$ by $\widehat{u}(t,x)=u(t/\lambda,x)/\kappa$ and $\widehat{p}(t,x)=p(t/\lambda,x)/\kappa$, we obtain (dropping the hat notation)

$$\begin{cases} \partial_t u(t,x)=\chi\,\partial_x\,(u(t,x)\partial_x p(t,x))+u(t,x)(1-u(t,x)), & t>0, x\in\mathbb{R},\\ p(t,x)-\sigma^2\,\partial_{xx}p(t,x)=u(t,x), & t>0, x\in\mathbb{R}. \end{cases} \tag{1.3}$$

Therefore, through the paper we will assume that

$$\lambda=1 \text{ and } \kappa=1.$$

Our original motivation comes from the description of cell motion in a Petri dish. In a previous paper [8], we derived a two-dimensional version of (1.3) to model the cell-cell repulsion in a Petri dish. We considered that cells grow in a circular domain (the Petri dish) and generate a repulsive gradient that pushes back neighbouring cells. We built a numerical simulation framework to study the solutions of the partial differential equation and compared the results to some real experiments realized by Pasquier and collaborators [17]. When starting from an isolated disk-like islet, the solution of the PDE looks like an expanding disk whose radius seems to be growing at a constant speed. We can study the shape of an expanding islet by considering travelling waves for (1.3). Previously, we studied the well-posedness of the problem (1.3) in [9] and proved the existence of an asymptotic propagation discontinuous profile – a travelling wave – in [10], corresponding to an initial data that is equal to 0 outside of some bounded region. In other words, in [10], we considered the case of an initial population of cells with compact support: no cell exists initially outside of the islet. The travelling waves constructed in [10] are called *sharp* because the transition between the occupied space (the area where $u(t,x)>0$) and the empty space (when $u(t,x)=0$) occurs at some finite position. We also proved in [10] that sharp travelling waves are necessarily discontinuous. Our model is related to the study of Ducrot et al. [5] who introduced a complete model of in vitro cell dynamics with many different behaviours at the cellular level. Other features of closely related models have been investigated in [4, 6, 7, 12, 13].

In the previous paper, we proved the existence of sharp travelling waves for (1.3). Our goal here is to complete the description of existing travelling waves that are not sharp. Formally, our work relates to the result of de Pablo and Vazquez [18], who studied the existence of sharp and not sharp travelling waves for a porous medium equation. The porous medium equation corresponds (formally) to the case $\sigma\to 0$. The convergence of the travelling waves when $\sigma\to 0$ has been observed only numerically in [10] and proved in [11], where the authors also propose an alternate method for the construction of discontinuous fronts by vanishing viscosity among other results.

The notion of solution: Throughout this paper, we impose that $p\in L^\infty(\mathbb{R})$ so the second line of (1.3) has a unique solution for a given $u(t,\cdot)\in L^\infty(\mathbb{R})$. In order to give a sense of the solution (1.3), we first assume that $x\to p(t,x)$ is regular enough. Then, the nonlinear diffusion can be understood as

$$\chi\,\partial_x\,(u(t,x)\partial_x p(t,x))=\chi\,\partial_x u(t,x)\partial_x p(t,x)+\chi\, u(t,x)\partial_{xx}p(t,x)$$

and by using the second equation of (1.3), we obtain

$$\chi\,\partial_x\,(u(t,x)\partial_x p(t,x))=\chi\,\partial_x u(t,x)\partial_x p(t,x)+\frac{\chi}{\sigma^2}u(t,x)\left[p(t,x)-u(t,x)\right].$$

Therefore, the system (1.3) is understood for $t\geq 0$ and $x\in\mathbb{R}$ as

$$\begin{cases} \partial_t u(t,x)=\chi\,\partial_x u(t,x)\partial_x p(t,x)+u(t,x)\left(\left(1+\frac{\chi}{\sigma^2}p(t,x)\right)-\left(1+\frac{\chi}{\sigma^2}\right)u(t,x)\right),\\ p(t,x)-\sigma^2\partial_{xx}p(t,x)=u(t,x), \end{cases} \tag{1.4}$$

with the initial distribution

$$u(0,x)=u_0(x)\in L^\infty(\mathbb{R}). \tag{1.5}$$

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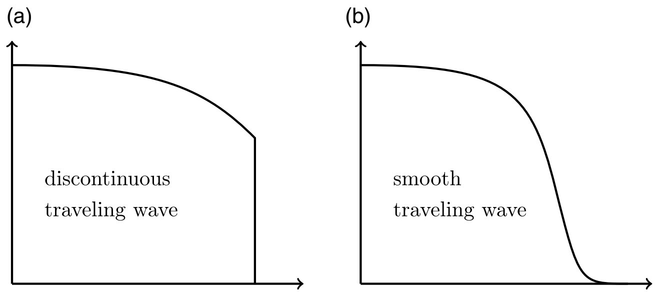

Figure 1. *An illustration of two types of travelling wave solutions.*

The existence and uniqueness of solutions of (1.4) in $L^\infty(\mathbb{R})$ have been considered as a subset of the weighted space $L^1_\eta(\mathbb{R})$ (with $\eta > 0$) with the norm

$$\|u\|_{L^1_\eta} = \int_{\mathbb{R}} e^{-\eta|x|}|u(x)|dx.$$

The existence and uniqueness of solutions for (1.4) have been studied by Fu, Griette, and Magal [9, Theorem 2.2].

1.1. Notion of travelling wave:

Definition 1.1. *A **travelling wave** is a special solution of (1.3) such that $u(t,x)$ has the specific form*

$$u(t,x) = U(x-ct), \text{ for a.e. } (t,x) \in \mathbb{R}^2,$$

*where the **profile** U has the following behaviour at $\pm\infty$:*

$$\lim_{z\to-\infty} U(z) = 1, \quad \lim_{z\to\infty} U(z) = 0.$$

*A travelling wave is **sharp** if there exists $x_0 \in \mathbb{R}$, such that*

$$U(x) = 0, \text{ for all } x > x_0.$$

*A travelling wave is **not sharp** if*

$$U(x) > 0, \text{ for all } x \in \mathbb{R}.$$

*We will say that system (1.3) has a **travelling wave with continuous profile** if we can find a bounded, continuous, and decreasing continuous function $U : \mathbb{R} \to \mathbb{R}$ that is the profile of a travelling wave.*

In Fu, Griette, and Magal [10, Proposition 2.4], we proved that the sharp travelling waves must be discontinuous. That is to say that $x \to U(x)$, the travelling wave profile of (1.3) can be either continuous or discontinuous. We illustrated both situations in Figure 1.

Estimations on the travelling speed for the discontinuous profile: Under a technical assumption on $\hat{\chi} = \frac{\chi}{\sigma^2}$, we can prove the existence of sharp travelling waves which present a jump at the vanishing point.

Assumption 1.2 (Bounds on $\hat{\chi}$). *Let $\chi > 0$ and $\sigma > 0$ be given and define $\hat{\chi} := \frac{\chi}{\sigma^2}$. We assume that $0 < \hat{\chi} < \bar{\chi}$, where $\bar{\chi}$ is the positive unique root of the function*

$$\hat{\chi} \mapsto \ln\left(\frac{2-\hat{\chi}}{\hat{\chi}}\right) + \frac{2}{2+\hat{\chi}}\left(\frac{\hat{\chi}}{2}\ln\left(\frac{\hat{\chi}}{2}\right) + 1 - \frac{\hat{\chi}}{2}\right).$$

The existence of travelling waves with discontinuous profile has been studied by Fu, Griette, and Magal [10, Theorem 2.4].

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Theorem 1.3 (Existence of a sharp discontinuous travelling wave). *Let Assumption 1.2 be satisfied. There exists a travelling wave $u(t,x) = U(x - ct)$ travelling at speed*

$$c \in \left(\frac{\sigma \hat{\chi}}{2+\hat{\chi}}, \frac{\sigma \hat{\chi}}{2} \right),$$

where

$$\hat{\chi} = \frac{\chi}{\sigma^2}.$$

Moreover, the profile U satisfies the following properties (up to a shift in space):

(i) U is sharp in the sense that $U(x) = 0$ for all $x \geq 0$; moreover, U has a discontinuity at $x = 0$ with $U(0^-) \geq \frac{2}{2+\hat{\chi}}$.

(ii) U is continuously differentiable and strictly decreasing on $(-\infty, 0]$ and satisfies

$$-c\,U' - \chi(UP')' = U(1-U) \text{ on } (-\infty, 0),$$

and

$$U = 0 \text{ on } (0, \infty),$$

and

$$P - \sigma^2 P'' = U \text{ on } \mathbb{R}.$$

In this article, we focus on the existence of travelling waves with continuous profiles. The main result of this paper is the following theorem.

Theorem 1.4 (Existence of a continuous travelling wave). *We assume that*

$$c \geq \sqrt{\chi\left(1 + \frac{\chi}{\sigma^2}\right)}.$$

There exists a travelling wave $u(t,x) = U(x - ct)$ with a continuous profile $x \to U(x)$ that is continuously differentiable and strictly decreasing, and

$$\lim_{x\to-\infty} U(x) = 1, \text{ and } \lim_{x\to+\infty} U(x) = 0, \tag{1.6}$$

and satisfies travelling wave problem

$$-c\,U' - \chi(UP')' = U(1-U), \text{ on } \mathbb{R}, \tag{1.7}$$

where

$$P - \sigma^2 P'' = U, \text{ on } \mathbb{R}. \tag{1.8}$$

Estimations on the travelling speed: We obtain the following condition for the existence of a travelling wave with a continuous profile for all-speed

$$c \geq c^*_{\text{cont}} := \sqrt{\chi\left(1 + \frac{\chi}{\sigma^2}\right)}.$$

For the Fisher-KPP equation [2, 15, 21], travelling waves only exist for half-line of positive travelling speeds. Moreover, there is a minimum speed $c_* > 0$ below which no travelling wave exists. Moreover, we can construct travelling waves for any values c above c_*. The existence of minimum speed is also true for porous medium equations with logistic dynamics [18]. By analogy with the porous medium equations, we expect that the minimal speed of the travelling waves corresponds to the sharp travelling wave constructed in [10]. In contrast, the continuous travelling waves constructed in the present paper correspond to higher velocities. Recall from [10, Theorem 2.4] that the sharp travelling wave is expected to travel at a speed $c_{\text{sharp}} \in \left(\frac{\chi/\sigma}{2+\chi/(\sigma^2)}, \frac{\chi}{2\sigma}\right)$, and indeed, we have that

$$c^*_{\text{cont}} = \sqrt{\chi\left(1 + \frac{\chi}{\sigma^2}\right)} \geq \frac{\chi}{\sigma} > \frac{1}{2}\frac{\chi}{\sigma} \geq c_{\text{sharp}}.$$

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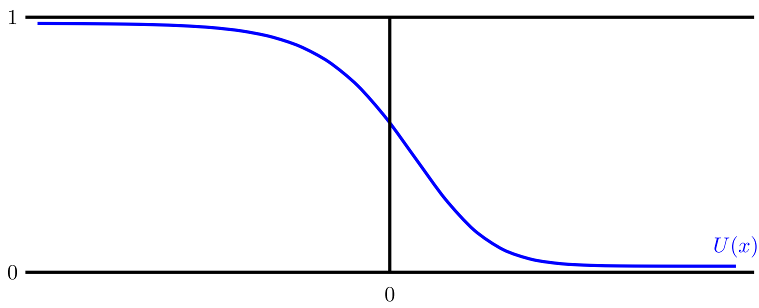

Figure 2. *In this figure, we plot the travelling wave profile $x \to U(x)$.*

Further analysis will be necessary to connect the gap between c^*_{cont} and c_{sharp} and to possibly prove the non-existence of travelling waves slower than sharp waves. Understanding the relationships between the profiles and the travelling speeds is still an open problem.

The paper is organized as follows. Section 2 is devoted to preliminary results. Section 3 presents the fixed point problem and its properties. Section 4 is devoted to the proof of Theorem 1.4. In Section 5, we present some numerical simulations. In Section 6, we present an application to wound healing.

2. Preliminary

We are interested in travelling waves of the system (1.3). In Figure 2, we illustrate the continuous travelling wave profiles.

Lemma 2.1. *Assume that system (1.4) has a travelling wave $u(t,x)=U(x-ct)$. Then, we must have*

$$p(t,x)=P(x-ct),$$

where $P:\mathbb{R}\to\mathbb{R}$ is the unique bounded continuous function satisfying the elliptic equation

$$P(x)-\sigma^2 P''(x)=U(x), \forall x\in\mathbb{R}.$$

Proof.

$$p(t,x)=\frac{1}{2\sigma}\int_{\mathbb{R}} e^{-\frac{|y|}{\sigma}}u(t,x-y)\mathrm{d}y=\frac{1}{2\sigma}\int_{\mathbb{R}} e^{-\frac{|y|}{\sigma}}U(x-y-ct)\mathrm{d}y, \tag{2.1}$$

therefore, $p(t,x)=P(x-ct)$ where

$$P(x)=\frac{1}{2\sigma}\int_{\mathbb{R}} e^{-\frac{|y|}{\sigma}}U(x-y)\mathrm{d}y \Leftrightarrow P(x)-\sigma^2P''(x)=U(x), x\in\mathbb{R}.$$

□

The following proposition was proved by Fu, Griette and Magal [10, Proposition 2.4].

Proposition 2.2. *Assume that U is a continuous profile of travelling wave. Then, $U:\mathbb{R}\to\mathbb{R}$ is continuously differentiable, and*

$$c+\chi P'(x)>0, \forall x\in\mathbb{R}\left(\Leftrightarrow 0\le -P'(x)<\frac{c}{\chi}, \forall x\in\mathbb{R}\right). \tag{2.2}$$

Transforming $U(x)$ into $\widehat{U}(x)=U(-x)$: In order to work with increasing functions rather than decreasing functions, we reverse the space variable. By Definition 1.1 and Lemma 2.1, we get the following travelling wave problem

$$-(c+\chi P'(x))\,U'(x)=U(x)\left(\left(1+\frac{\chi}{\sigma^2}P(x)\right)-\left(1+\frac{\chi}{\sigma^2}\right)U(x)\right), \forall x\in\mathbb{R}, \tag{2.3}$$

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where

$$P(x) - \sigma^2 P''(x) = U(x), \forall x \in \mathbb{R}. \tag{2.4}$$

Equation (2.3) has the following behaviour at $\pm\infty$:

$$\lim_{x\to-\infty} U(x) = 1, \quad \lim_{x\to+\infty} U(x) = 0.$$

Now, let us perform the change of variables to reverse the space direction. Setting $\widehat{U}(x) = U(-x)$, and $\widehat{P}(x) = P(-x)$, then equations (2.3) and (2.4) become

$$\left(c - \chi \widehat{P}'(x)\right) \widehat{U}'(x) = \widehat{U}(x) \left(\left(1 + \frac{\chi}{\sigma^2}\widehat{P}(x)\right) - \left(1 + \frac{\chi}{\sigma^2}\right) \widehat{U}(x)\right), \forall x \in \mathbb{R}, \tag{2.5}$$

where

$$\widehat{P}(x) - \sigma^2 \widehat{P}''(x) = \widehat{U}(x), \forall x \in \mathbb{R}.$$

Assume that $c + \chi P'(x) > 0, \forall x \in \mathbb{R}$, then we have $c - \chi \widehat{P}'(x) > 0, \forall x \in \mathbb{R}$ by using $\widehat{P}(x) = P(-x)$.

For convenience, we drop the hat notation, and system (2.5) becomes a logistic equation

$$U'(x) = \lambda(x)\, U(x) - \kappa(x)\, U^2(x), \forall x \in \mathbb{R}, \tag{2.6}$$

where

$$\lambda(x) := \frac{1 + \frac{\chi}{\sigma^2} P(x)}{c - \chi P'(x)}, \forall x \in \mathbb{R}, \tag{2.7}$$

and

$$\kappa(x) := \frac{1 + \frac{\chi}{\sigma^2}}{c - \chi P'(x)}, \forall x \in \mathbb{R}, \tag{2.8}$$

with $P(x)$ is the unique solution of the elliptic equation

$$P(x) - \sigma^2 P''(x) = U(x), \forall x \in \mathbb{R}. \tag{2.9}$$

System (2.6) has the following behaviour at $\pm\infty$

$$\lim_{x\to-\infty} U(x) = 0, \quad \lim_{x\to\infty} U(x) = 1. \tag{2.10}$$

Lemma 2.3. *Assume that $U : \mathbb{R} \to \mathbb{R}$ is an increasing C^1 function. Then, the map $x \to P(x)$ solving the elliptic equation*

$$P(x) - \sigma^2 P''(x) = U(x), \forall x \in \mathbb{R},$$

is an increasing C^3 function, and we have the following estimation of the first derivative of $P(x)$

$$\sup_{x\in\mathbb{R}} P'(x) \le \sup_{x\in\mathbb{R}} U'(x). \tag{2.11}$$

Proof. The result follows the following inequality

$$0 \le P'(x) = \frac{1}{2\sigma} \int_{\mathbb{R}} e^{-\frac{|y|}{\sigma}} U'(x-y)\mathrm{d}y \le \sup_{x\in\mathbb{R}} U'(x). \tag{2.12}$$

□

Lemma 2.4. *Assume that $c \ge \sqrt{\chi\left(1 + \frac{\chi}{\sigma^2}\right)}$,*

$$0 \le U'(x) \le C_U := \frac{c + \sqrt{c^2 - \chi\left(1 + \frac{\chi}{\sigma^2}\right)}}{2\chi}, \forall x \in \mathbb{R}, \tag{2.13}$$

and

$$0 < u_0 < \frac{\sigma^2}{2\left(\sigma^2 + \chi\right)} \left(1 - \sqrt{1 - \frac{\chi}{c^2}\left(1 + \frac{\chi}{\sigma^2}\right)}\right).$$

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Then, $U:\mathbb{R}\to\mathbb{R}$ is an increasing C^1 function satisfying (2.6), (2.10) and

$$U(0)=u_0, \tag{2.14}$$

which is given by the following formula

$$U(x)=\frac{u_0 e^{\int_0^x \lambda(s)ds}}{1+u_0\int_0^x \kappa(s)e^{\int_0^s \lambda(\tau)dl}ds}, \forall x\in\mathbb{R},$$

where $\lambda(x)$, $\kappa(x)$, and $P(x)$ are given by equations (2.7), (2.8), and (2.9) above.

Proof. Let us prove that the formula

$$\frac{u_0 e^{\int_0^x \lambda(s)ds}}{1+u_0\int_0^x \kappa(s)e^{\int_0^s \lambda(l)dl}ds} \tag{2.15}$$

is well defined for all $x\in\mathbb{R}$. So let us prove that if $0<u_0<\frac{\sigma^2}{2(\sigma^2+\chi)}\left(1-\sqrt{1-\frac{\chi}{c^2}\left(1+\frac{\chi}{\sigma^2}\right)}\right)$, then we have

$$1-u_0\int_{-\infty}^0 \kappa(s)e^{-\int_s^0 \lambda(l)dl}ds>0.$$

Indeed, since by assumption

$$0\le U'(x)\le C_U,\ \forall x\in\mathbb{R},$$

and Lemma 2.3, we deduce that

$$0\le P'(x)\le \sup_{x\in\mathbb{R}} U'(x)\le C_U,\ \forall x\in\mathbb{R}, \tag{2.16}$$

hence,

$$c-\chi P'(x)\ge c-\chi C_U=\frac{c-\sqrt{c^2-\chi\left(1+\frac{\chi}{\sigma^2}\right)}}{2}>0,\ \forall x\in\mathbb{R},$$

and since $P'(x)\ge 0$, we deduce that

$$\frac{1}{c}\le\frac{1}{c-\chi P'(x)}\le\frac{2}{c-\sqrt{c^2-\chi\left(1+\frac{\chi}{\sigma^2}\right)}},\ \forall x\in\mathbb{R}. \tag{2.17}$$

Now by combining (2.17), and $0\le P(x)\le 1$, for any $x\in\mathbb{R}$, we deduce that

$$\frac{1}{c}\le\lambda(x)\le\frac{2\left(1+\frac{\chi}{\sigma^2}\right)}{c-\sqrt{c^2-\chi\left(1+\frac{\chi}{\sigma^2}\right)}} \text{ and } 0<\frac{1}{c}\left(1+\frac{\chi}{\sigma^2}\right)\le\kappa(x)\le\frac{2\left(1+\frac{\chi}{\sigma^2}\right)}{c-\sqrt{c^2-\chi\left(1+\frac{\chi}{\sigma^2}\right)}},\ \forall x\in\mathbb{R}. \tag{2.18}$$

Define

$$G:=1-u_0\int_{-\infty}^0 \kappa(s)e^{-\int_s^0 \lambda(l)dl}ds.$$

Using (2.18), we have that

$$\begin{aligned}G&\ge 1-u_0\int_{-\infty}^0 \frac{2\left(1+\frac{\chi}{\sigma^2}\right)}{c-\sqrt{c^2-\chi\left(1+\frac{\chi}{\sigma^2}\right)}}e^{-\int_s^0 \frac{1}{c}dl}ds\\ &=1-u_0\frac{2\left(1+\frac{\chi}{\sigma^2}\right)}{c-\sqrt{c^2-\chi\left(1+\frac{\chi}{\sigma^2}\right)}}\int_{-\infty}^0 e^{\frac{s}{c}}ds \qquad (2.19)\\ &=1-u_0\frac{2c\left(1+\frac{\chi}{\sigma^2}\right)}{c-\sqrt{c^2-\chi\left(1+\frac{\chi}{\sigma^2}\right)}}>0,\end{aligned}$$

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by using the assumption

$$0 < u_0 < \frac{\sigma^2}{2(\sigma^2+\chi)}\left(1-\sqrt{1-\frac{\chi}{c^2}\left(1+\frac{\chi}{\sigma^2}\right)}\right) = \frac{c-\sqrt{c^2-\chi\left(1+\frac{\chi}{\sigma^2}\right)}}{2c\left(1+\frac{\chi}{\sigma^2}\right)}.$$

□

3. The relationship between the fixed point and travelling waves

Definition 3.1. *Let $\mathcal{A}$ be the set of all admissible function $U:\mathbb{R}\to[0,1]$ satisfying*

(i) $U\in C^1(\mathbb{R})$*;*

(ii) $0\le U(x)\le 1, \forall x\in\mathbb{R}$*;*

(iii) $0\le U'(x)\le C_U = \dfrac{c+\sqrt{c^2-\chi\left(1+\frac{\chi}{\sigma^2}\right)}}{2\chi}, \forall x\in\mathbb{R}.$

Note that the upper bound in (iii) is the same as in the statement of Lemma 2.4 (2.13).

For each $U\in\mathcal{A}$, we define

$$\mathcal{T}(U)(x) := V(x), \forall x\in\mathbb{R}, \tag{3.1}$$

where

$$V(x) = \frac{u_0 e^{\int_0^x \lambda(s)ds}}{1+u_0\int_0^x \kappa(s)e^{\int_0^s \lambda(l)dl}ds}, \forall x\in\mathbb{R}, \tag{3.2}$$

with

$$0 < u_0 < \frac{\sigma^2}{2\left(\sigma^2+\chi\right)}\left(1-\sqrt{1-\frac{\chi}{c^2}\left(1+\frac{\chi}{\sigma^2}\right)}\right), \tag{3.3}$$

and

$$\lambda(x) = \frac{1+\frac{\chi}{\sigma^2}P(x)}{c-\chi P'(x)}, \forall x\in\mathbb{R}, \tag{3.4}$$

and

$$\kappa(x) = \frac{1+\frac{\chi}{\sigma^2}}{c-\chi P'(x)}, \forall x\in\mathbb{R}, \tag{3.5}$$

and $P(x)$ is the unique solution of the elliptic equation

$$P(x)-\sigma^2 P''(x) = U(x), \forall x\in\mathbb{R}. \tag{3.6}$$

Assumption 3.2. *We assume that*

$$c\ge\sqrt{\chi\left(1+\frac{\chi}{\sigma^2}\right)} \text{ and } 0<u_0<\frac{\sigma^2}{2\left(\sigma^2+\chi\right)}\left(1-\sqrt{1-\frac{\chi}{c^2}\left(1+\frac{\chi}{\sigma^2}\right)}\right).$$

Lemma 3.3 (Invariance of $\mathcal{A}$ by $\mathcal{T}$). *Let Assumption 3.2 be satisfied. Let $\mathcal{T}$ be the map defined by (3.1). Then,*

$$\mathcal{T}(\mathcal{A})\subset\mathcal{A}.$$

Proof. We divide the proof into three steps.

Step 1. We prove that $V=\mathcal{T}(U)\in C^1(\mathbb{R})$. Indeed, V is continuously differentiable and

$$V'(x) = \frac{\lambda(x)u_0 e^{\int_0^x\lambda(s)ds}\left(1+u_0\int_0^x\kappa(s)e^{\int_0^s\lambda(l)dl}ds\right) - \kappa(x)\left(u_0 e^{\int_0^x\lambda(s)ds}\right)^2}{\left(1+u_0\int_0^x\kappa(s)e^{\int_0^s\lambda(l)dl}ds\right)^2}, \tag{3.7}$$

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hence,

$$V'(x) = \lambda(x)V(x) - \kappa(x)V^2(x), \forall x \in \mathbb{R}. \tag{3.8}$$

It follows from the definitions of $\lambda(x)$ and $\kappa(x)$ (see (3.4) and (3.5)) that $\lambda(x)$ and $\kappa(x)$ are continuously differentiable. Therefore, we have

$$V \in C^1(\mathbb{R}).$$

Step 2. We prove that $0 < V(x) \leq 1, \forall x \in \mathbb{R}$. By (2.11) and $U \in \mathcal{A}$, we have that

$$0 \leq \sup_{x\in\mathbb{R}} P'(x) \leq \sup_{x\in\mathbb{R}} U'(x) \leq C_U.$$

Therefore, we have $c - \chi P'(x) > 0$. Recall that

$$V(x) = \frac{u_0 e^{\int_0^x \lambda(s)ds}}{1 + u_0 \int_0^x \kappa(s) e^{\int_0^s \lambda(l)dl} ds}, \forall x \in \mathbb{R}.$$

By using (3.3), we know that

$$1 + u_0 \int_0^x \kappa(s) e^{\int_0^s \lambda(l)dl} ds > 0, \forall x \in \mathbb{R}. \tag{3.9}$$

Therefore, by definition of $V(x)$, we have that $V(x) > 0, \forall x \in \mathbb{R}$. On the other hand, by using (3.8), we have that

$$V'(x) = \frac{1}{c - \chi P'(x)} V(x) \left(\left(1 + \frac{\chi}{\sigma^2} P(x)\right) - \left(1 + \frac{\chi}{\sigma^2}\right) V(x) \right), \forall x \in \mathbb{R}, \tag{3.10}$$

and

$$P(x) = \frac{1}{2\sigma} \int_{\mathbb{R}} e^{-\frac{|x-y|}{\sigma}} U(y)dy = \frac{1}{2\sigma} \int_{\mathbb{R}} e^{-\frac{|y|}{\sigma}} U(x-y)dy, \forall x \in \mathbb{R}.$$

Since $0 \leq U(x) \leq 1, \forall x \in \mathbb{R}$, we have

$$0 \leq P(x) \leq 1, \forall x \in \mathbb{R}. \tag{3.11}$$

We deduce that for $x \in \mathbb{R}$

$$V'(x) \leq \frac{1 + \frac{\chi}{\sigma^2}}{c - \chi P'(x)} V(x)\big(1 - V(x)\big), \forall x \in \mathbb{R}, \tag{3.12}$$

By (3.12) and the comparison principle, we have that

$$0 \leq V(x) \leq 1, \forall x \in \mathbb{R}. \tag{3.13}$$

Step 3. Let us prove that

$$0 < V'(x) \leq C_U, \forall x \in \mathbb{R}.$$

Since

$$0 \leq U'(x) \leq C_U, \forall x \in \mathbb{R},$$

then we have

$$0 \leq P'(x) = \frac{1}{2\sigma} \int_{\mathbb{R}} e^{-\frac{|y|}{\sigma}} U'(x-y)dy \leq \sup_{x\in\mathbb{R}} U'(x) \leq C_U, \forall x \in \mathbb{R}. \tag{3.14}$$

By (3.14), we have that

$$c - \chi P'(x) \geq c - \chi C_U, \forall x \in \mathbb{R}.$$

Therefore, we deduce

$$0 < \frac{1}{c - \chi P'(x)} \leq \frac{1}{c - \chi C_U}, \forall x \in \mathbb{R}. \tag{3.15}$$

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Since $0 \leq U(x) \leq 1$, and

$$P(x) = \frac{1}{2\sigma} \int_{\mathbb{R}} \mathrm{e}^{-\frac{|y|}{\sigma}} U(x-y)\mathrm{d}y, \ \forall x \in \mathbb{R},$$

we deduce that

$$0 \leq P(x) \leq 1, \ \forall x \in \mathbb{R}. \tag{3.16}$$

Therefore, by using the fact that $0 \leq V(x) \leq 1$, $\forall x \in \mathbb{R}$, (3.10), (3.15), and (3.16), we have $V(x)(1 - V(x)) \leq 1/4$ and we deduce that for $x \in \mathbb{R}$

$$V'(x) \leq \frac{1 + \frac{\chi}{\sigma^2}}{4(c - \chi C_U)}.$$

Using the definition of C_U and $c^2 \geq \chi\left(1 + \frac{\chi}{\sigma^2}\right)$, we have

$$\begin{aligned} V'(x) \leq \frac{1 + \frac{\chi}{\sigma^2}}{4(c - \chi C_U)} &= \frac{1 + \frac{\chi}{\sigma^2}}{2\left(c - \sqrt{c^2 - \chi\left(1 + \frac{\chi}{\sigma^2}\right)}\right)} \\ &= \frac{1 + \frac{\chi}{\sigma^2}}{2\left(c - \sqrt{c^2 - \chi\left(1 + \frac{\chi}{\sigma^2}\right)}\right)} \times \frac{c + \sqrt{c^2 - \chi\left(1 + \frac{\chi}{\sigma^2}\right)}}{c + \sqrt{c^2 - \chi\left(1 + \frac{\chi}{\sigma^2}\right)}} \\ &= \frac{1 + \frac{\chi}{\sigma^2}}{2\left(c^2 - c^2 + \chi\left(1 + \frac{\chi}{\sigma^2}\right)\right)}\left(c + \sqrt{c^2 - \chi\left(1 + \frac{\chi}{\sigma^2}\right)}\right) \\ &= C_U \end{aligned}$$

and we finally reach

$$V'(x) \leq C_U. \tag{3.17}$$

On the other hand, by using (3.7), we have

$$\begin{aligned} V'(x) &= \frac{\lambda(x) u_0 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s}\left(1 + u_0 \int_0^x \kappa(s)\mathrm{e}^{\int_0^s \lambda(l)\mathrm{d}l}\mathrm{d}s\right) - \kappa(x)\left(u_0 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s}\right)^2}{\left(1 + u_0 \int_0^x \kappa(s)\mathrm{e}^{\int_0^s \lambda(l)\mathrm{d}l}\mathrm{d}s\right)^2} \\ &= \frac{\lambda(x) u_0 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s} + \Lambda(x)}{\left(1 + u_0 \int_0^x \kappa(s)\mathrm{e}^{\int_0^s \lambda(l)\mathrm{d}l}\mathrm{d}s\right)^2}, \end{aligned} \tag{3.18}$$

where

$$\Lambda(x) := u_0^2 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s}\left[\lambda(x) \int_0^x \kappa(s)\mathrm{e}^{\int_0^s \lambda(l)\mathrm{d}l}\mathrm{d}s - \kappa(x)\mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s}\right].$$

From (3.18), to prove $V'(x) > 0$ for $x \in \mathbb{R}$, we only need to prove that

$$\Lambda_1(x) := \lambda(x) u_0 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s} + \Lambda(x) > 0.$$

Indeed, we have

$$\Lambda(x) = \kappa(x)\left(u_0 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s}\right)^2\left(\frac{\lambda(x)\int_0^x \kappa(s)\mathrm{e}^{-\int_s^x \lambda(l)\mathrm{d}l}\mathrm{d}s}{\kappa(x)} - 1\right). \tag{3.19}$$

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By using the definitions $\lambda(x)$ and $\kappa(x)$ in (3.4) and (3.5), and the formula (3.19), we deduce that

$$\Lambda(x) = \kappa(x)\left(u_0 e^{\int_0^x \lambda(s)\mathrm{d}s}\right)^2 \left(\int_0^x \frac{1+\frac{\chi}{\sigma^2}P(x)}{c-\chi P'(s)} e^{-\int_s^x \frac{1+\frac{\chi}{\sigma^2}P(l)}{c-\chi P'(l)}\mathrm{d}l}\mathrm{d}s - 1\right). \tag{3.20}$$

Since by assumption U is increasing, it follows from (3.14) that P is increasing. Then, for any $s < x$, we have $P(s) \le P(x)$. We deduce that

$$\begin{aligned}\int_0^x \frac{1+\frac{\chi}{\sigma^2}P(x)}{c-\chi P'(s)} e^{-\int_s^x \frac{1+\frac{\chi}{\sigma^2}P(l)}{c-\chi P'(l)}\mathrm{d}l}\mathrm{d}s &\ge \int_0^x \frac{1+\frac{\chi}{\sigma^2}P(s)}{c-\chi P'(s)} e^{-\int_s^x \frac{1+\frac{\chi}{\sigma^2}P(l)}{c-\chi P'(l)}\mathrm{d}l}\mathrm{d}s \\ &= \int_0^x \frac{\mathrm{d}}{\mathrm{d}s}\left(e^{-\int_s^x \frac{1+\frac{\chi}{\sigma^2}P(l)}{c-\chi P'(l)}\mathrm{d}l}\right)\mathrm{d}s \\ &= 1 - e^{-\int_0^x \frac{1+\frac{\chi}{\sigma^2}P(l)}{c-\chi P'(l)}\mathrm{d}l} \\ &= 1 - e^{-\int_0^x \lambda(s)\mathrm{d}s},\end{aligned}$$

therefore by combining (3.20) and the above inequality, we obtain

$$\begin{aligned}\Lambda(x) &\ge \kappa(x)\left(u_0 e^{\int_0^x \lambda(s)\mathrm{d}s}\right)^2\left(1 - e^{-\int_0^x \lambda(s)\mathrm{d}s} - 1\right) \\ &= -\kappa(x)u_0^2 e^{\int_0^x \lambda(s)\mathrm{d}s}.\end{aligned} \tag{3.21}$$

By using (3.21) and the definitions of $\lambda(x)$ and $\kappa(x)$ for $x \in \mathbb{R}$, we have that

$$\begin{aligned}\Lambda_1(x) &= \lambda(x)u_0 e^{\int_0^x \lambda(s)\mathrm{d}s} + \Lambda(x) \\ &\ge \lambda(x)u_0 e^{\int_0^x \lambda(s)\mathrm{d}s} - \kappa(x)u_0^2 e^{\int_0^x \lambda(s)\mathrm{d}s} \\ &= u_0 e^{\int_0^x \lambda(s)\mathrm{d}s}\left[\frac{1+\frac{\chi}{\sigma^2}P(x)}{c-\chi P'(x)} - u_0\frac{1+\frac{\chi}{\sigma^2}}{c-\chi P'(x)}\right] \\ &= \frac{u_0 e^{\int_0^x \lambda(s)\mathrm{d}s}}{c-\chi P'(x)}\left[1+\frac{\chi}{\sigma^2}P(x) - u_0\left(1+\frac{\chi}{\sigma^2}\right)\right].\end{aligned} \tag{3.22}$$

To conclude, it remains to recall that by assumption we have

$$u_0 < \frac{\sigma^2}{2(\sigma^2+\chi)}\left(1-\sqrt{1-\frac{\chi}{c^2}\left(1+\frac{\chi}{\sigma^2}\right)}\right) < \frac{\sigma^2}{\sigma^2+\chi},$$

therefore since $P(x) \ge 0$ and $c - \chi P'(x) > 0$, we have that

$$\Lambda_1(x) \ge \left(u_0 e^{\int_0^x \lambda(s)\mathrm{d}s}\right)\frac{1+\frac{\chi}{\sigma^2}}{c-\chi P'(x)}\left[\frac{\sigma^2}{\sigma^2+\chi} - u_0\right] > 0, \ \forall x \in \mathbb{R},$$

which implies

$$V'(x) > 0, \ \forall x \in \mathbb{R}. \tag{3.23}$$

The conclusion of the Step 3 now follows from (3.17) and (3.23). The proof is completed. □

Let $\eta > 0$. Let BUC $(\mathbb{R})$ be the space of bounded and uniformly continuous maps from $\mathbb{R}$ to itself. Define the weighted space of continuous functions

$$BUC_\eta(\mathbb{R}) = \left\{U \in C(\mathbb{R}) : x \to e^{-\eta|x|}U(x) \in \mathrm{BUC}(\mathbb{R})\right\},$$

and the weighted space n times continuous differentiable functions

$$BUC_\eta^n(\mathbb{R}) = \left\{U \in C^n(\mathbb{R}) : x \to e^{-\eta|x|}U^{(k)}(x) \in \mathrm{BUC}(\mathbb{R}), \forall k = 0, \ldots, n\right\},$$

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which is a Banach space endowed with the norm

$$\|U\|_{n,\eta} := \sup_{x\in\mathbb{R}} \mathrm{e}^{-\eta|x|}|U(x)| + \sup_{x\in\mathbb{R}} \mathrm{e}^{-\eta|x|}|U'(x)| + \ldots + \sup_{x\in\mathbb{R}} \mathrm{e}^{-\eta|x|}|U^{(n)}(x)|. \tag{3.24}$$

Here we will use the above weighted space of $\mathrm{BUC}^1_\eta(\mathbb{R})$ maps to ensure that

$$\mathcal{A} = \left\{U \in C^1(\mathbb{R}) : 0 \le U(x) \le 1, \text{ and } 0 \le U'(x) \le C_U, \forall x \in \mathbb{R}\right\},$$

is a closed subset of $\mathrm{BUC}^1_\eta(\mathbb{R})$. As a consequence, the subset $\mathcal{A}$ equipped with the distance

$$d_\eta(U_1, U_2) = \|U_1 - U_2\|_{1,\eta}.$$

is a complete metric space.

Lemma 3.4 (Compactness of $\mathcal{T}$). *Let Assumption 3.2 be satisfied. Then, the set $\overline{\mathcal{T}(\mathcal{A})}$ is a compact subset of the metric space $\mathcal{A}$ equipped with the distance d_η.*

Proof. Let $\{U_n\}_{n\ge0} \subset \mathcal{A}$ be a sequence and define the corresponding sequence $\{P_n\}_{n\ge0}$ solution of equation (3.6) where U is replaced by U_n. Define the corresponding sequences $\{\lambda_n\}_{n\ge0}$ and $\{\kappa_n\}_{n\ge0}$ by using (3.4) and (3.5) where $P(x)$ is replaced by $P_n(x)$. Denote $V_n = \mathcal{T}(U_n)$, $\forall n \in \mathbb{N}$. By Lemma 3.3, we know that $\mathcal{T}(\mathcal{A}) \subset \mathcal{A}$. Therefore, we have

$$0 < V_n(x) \le 1 \text{ and } 0 < V_n'(x) \le C_U,\ \forall x \in \mathbb{R}. \tag{3.25}$$

Similarly to equation (3.8) in the proof of Lemma 3.3, we have that

$$V_n'(x) = \lambda_n(x)V_n(x) - \kappa_n(x)V_n^2(x),\ \forall x \in \mathbb{R}, \tag{3.26}$$

where

$$\lambda_n(x) = \frac{1 + \frac{\chi}{\sigma^2}P_n(x)}{c - \chi P_n'(x)},\ \forall x \in \mathbb{R},$$

and

$$\kappa_n(x) = \frac{1 + \frac{\chi}{\sigma^2}}{c - \chi P_n'(x)},\ \forall x \in \mathbb{R},$$

and $P_n(x)$ is the unique solution of the elliptic equation

$$P_n(x) - \sigma^2 P_n''(x) = U_n(x), \forall x \in \mathbb{R}.$$

It follows from Lemma 2.3 that the map $x \to P_n(x)$ solving the above equation is an increasing C^3 function. By (3.14), we have that $c - \chi P_n'(x) > 0$. Since $U_n(x) \in C^1(\mathbb{R})$, we have that $\lambda_n(x), \kappa_n(x) \in C^2(\mathbb{R})$ and $V_n'(x) \in C^1(\mathbb{R})$. Then, we obtain that $V_n(x) \in C^2(\mathbb{R})$. Therefore by using (3.25) and (3.26), we deduce that the families $V_n'|_{[-k,k]}$ and $V_n''|_{[-k,k]}$ are uniformly Lipschitz continuous on $[-k, k]$ for each $k \in \mathbb{N}$. Applying Ascoli-Arzelà theorem, we have that the sets $\{V_n|_{[-k,k]}\}_{n>0}$ and $\{V_n'|_{[-k,k]}\}_{n>0}$ are relatively compact on $[-k, k]$ for each $k \in \mathbb{N}$.

Using a diagonal extraction process, there exists a sub-sequence n_p and a bounded continuous function V such that $V_{n_p} \to V$ uniformly on every compact subset of $\mathbb{R}$ as $p \to \infty$. Indeed, recall that $0 < V_n(x) \le 1$ and $0 < V_n'(x) \le C_U$, $\forall x \in \mathbb{R}$. By the Ascoli-Arzelà theorem, there exists a sub-sequence $\{V_{m_p^1}\}_{p\ge0}$ of V_n and a function $V_1 \in C^1([-1, 1])$ such that

$$\lim_{p\to\infty} \|V_{m_p^1} - V_1\|_{C^1([-1,1])} = 0.$$

Now we can extract $\{V_{m_p^2}\}_{p\ge0}$ a sub-sequence of $\{V_{m_p^1}\}_{p\ge0}$, and function $V_2 \in C^1([-2, 2])$

$$\lim_{p\to\infty} \|V_{m_p^2} - V_2\|_{C^1([-2,2])} = 0.$$

By construction, we will have

$$V_2(x) = V_1(x), \forall x \in [-1, 1].$$

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Replacing eventually m_1^2 by m_1^1, we can assume that $m_p^2 = \{m_1^1, m_2^2, \cdots, m_p^2, \cdots\}$. Proceeding by induction, we can find a $\{V_{m_p^k}\}_{p\geq 0}$ a sub-sequence $\{V_{m_p^{k-1}}\}_{p\geq 0}$, such that

$$m_p^k = m_p^{k-1}, \forall p = 1, \ldots, k-1,$$

and a function $V_k \in C^1([-k, k])$ such that

$$\lim_{p\to\infty} \|V_{m_p^k} - V_k\|_{C^1([-k,k])} = 0.$$

Set

$$n_p = m_p^p,$$

and

$$V(x) = V_n(x), \text{ for any } x \in [-k, k], \forall k \geq 1.$$

Then, the sub-sequence $\{V_{n_p}\}_{p\geq 0}$ converges locally uniformly with respect to the C^1 norm, so we can define

$$V(x) = \lim_{p\to\infty} V_{n_p}(x), \forall x \in \mathbb{R},$$

and

$$V'(x) = \lim_{p\to\infty} V'_{n_p}(x), \forall x \in \mathbb{R}.$$

By construction, we will have

$$0 < V(x) \leq 1, \text{ and } 0 < V'(x) \leq C_U, \forall x \in \mathbb{R}. \tag{3.27}$$

Now, we are ready to show that $\|V_{n_p} - V\|_{1,\eta} \to 0$ as $p \to +\infty$. Let $\varepsilon > 0$ be given. Let k be large enough to satisfy

$$e^{-\eta k} \leq \frac{\varepsilon}{4} \min\left\{1, \frac{1}{C_U}\right\}. \tag{3.28}$$

For all k large enough, since $0 < V_{n_p}(x) \leq 1$, $0 < V'_{n_p}(x) \leq C_U$, $\forall x \in \mathbb{R}$ and (3.27), we deduce that

$$\begin{aligned}\sup_{x\in\mathbb{R}\setminus[-k,k]} e^{-\eta|x|}|V_{n_p}(x) - V(x)| &\leq e^{-\eta k} \sup_{x\in\mathbb{R}} (|V_{n_p}(x)| + |V(x)|)\\ &\leq 2e^{-\eta k}\\ &\leq \frac{\varepsilon}{2},\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}\sup_{x\in\mathbb{R}\setminus[-k,k]} e^{-\eta|x|}|V'_{n_p}(x) - V'(x)| &\leq e^{-\eta k} \sup_{x\in\mathbb{R}} (|V'_{n_p}(x)| + |V'(x)|)\\ &\leq 2C_U e^{-\eta k}\\ &\leq \frac{\varepsilon}{2}.\end{aligned} \tag{3.30}$$

Moreover, since V_{n_p} converges locally uniformly to V and V'_{n_p} converges locally uniformly to V', for any fixed $x \in [-k, k]$, there exists an integer $p_0 > 0$ such that

$$\sup_{x\in[-k,k]} e^{-\eta|x|}|V_{n_p}(x) - V(x)| \leq \frac{\varepsilon}{2}, \forall p \geq p_0, \tag{3.31}$$

and

$$\sup_{x\in[-k,k]} e^{-\eta|x|}|V'_{n_p}(x) - V'(x)| \leq \frac{\varepsilon}{2}, \forall p \geq p_0. \tag{3.32}$$

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It follows from (3.29), (3.30), (3.31), and (3.32) that for $p \geq p_0$,

$$\|V_{n_p} - V\|_{1,\eta} = \Bigg\{ \max\left\{ \sup_{x\in\mathbb{R}\setminus[-k,k]} \mathrm{e}^{-\eta|x|}|V_{n_p}(x) - V(x)|, \sup_{x\in[-k,k]} \mathrm{e}^{-\eta|x|}|V_{n_p}(x) - V(x)| \right\}$$
$$+ \max\left\{ \sup_{x\in\mathbb{R}\setminus[-k,k]} \mathrm{e}^{-\eta|x|}|V'_{n_p}(x) - V'(x)|, \sup_{x\in[-k,k]} \mathrm{e}^{-\eta|x|}|V'_{n_p}(x) - V'(x)| \right\} \Bigg\}$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since the above inequality is true for any $\varepsilon > 0$, this completes the proof of lemma. □

The most difficult part of the proof of existence of travelling waves is the continuity of the map $\mathcal{T} : \mathcal{A} \to \mathcal{A}$. To consider this problem, we decompose the real line into several intervals $(-\infty, -K]$, $[-K, K]$, and $[K, \infty)$. Before proving the continuity of $\mathcal{T}$, we establish the continuity of its components separately.

Lemma 3.5 (Continuity of P, P', λ *and* κ). *Let Assumption 3.2 be satisfied. Assume that* $0 < \eta < \frac{1}{\sigma}$. *Let* $U_1, U_2 \in \mathcal{A}$ *and define, for* $i = 1, 2$,

$$P_i(x) = \frac{1}{2\sigma}\int_{\mathbb{R}} e^{-\frac{|x-y|}{\sigma}} U_i(y)\mathrm{d}y, \qquad P_i{}'(x) = \frac{1}{2\sigma^2}\int_{\mathbb{R}} -\mathrm{sign}(x-y) e^{-\frac{|x-y|}{\sigma}} U_i(y)\mathrm{d}y,$$
$$\lambda_i(x) = \frac{1 + \frac{\chi}{\sigma^2}P_i(x)}{c - \chi P_i{}'(x)}, \qquad \kappa_i(x) = \frac{1 + \frac{\chi}{\sigma^2}}{c - \chi P_i{}'(x)}.$$

There exist continuous functions of $x \in \mathbb{R}$, $C_P(x)$, $C_\lambda(x)$, *and* $C_\kappa(x)$ *such that, for all* $x \in \mathbb{R}$,

$$|P_1(x) - P_2(x)| \leq C_P(x)\|U_1 - U_2\|_{0,\eta}, \tag{3.33}$$
$$|P'_1(x) - P'_2(x)| \leq \frac{1}{\sigma}C_P(x)\|U_1 - U_2\|_{0,\eta}, \tag{3.34}$$
$$|\lambda_1(x) - \lambda_2(x)| \leq C_\lambda(x)\|U_1 - U_2\|_{0,\eta}, \tag{3.35}$$
$$|\kappa_1(x) - \kappa_2(x)| \leq C_\kappa(x)\|U_1 - U_2\|_{0,\eta}. \tag{3.36}$$

The functions $C_P(x)$, $C_\lambda(x)$, *and* $C_\kappa(x)$ *do not depend on the particular choice of* $U_1 \in \mathcal{A}$ *and* $U_2 \in \mathcal{A}$ *but only on* η, σ, *and* χ.

Proof. Step 1: We show (3.33). We have, for $x > 0$:

$$|P_1(x) - P_2(x)| = \frac{1}{2\sigma}\left|\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{|x-y|}{\sigma}}(U_1(y) - U_2(y))\mathrm{d}y\right|$$
$$= \frac{1}{2\sigma}\left|\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{|x-y|}{\sigma}+\eta|y|}\mathrm{e}^{-\eta|y|}(U_1(y) - U_2(y))\mathrm{d}y\right|$$
$$\leq \frac{1}{2\sigma}\int_{\mathbb{R}} \mathrm{e}^{-\frac{|x-y|}{\sigma}+\eta|y|}\mathrm{d}y\|U_1 - U_2\|_{0,\eta}$$
$$= \frac{1}{2\sigma}\|U_1 - U_2\|_{0,\eta}\left(\int_{-\infty}^{0} e^{-\frac{x-y}{\sigma}-\eta y}\mathrm{d}y + \int_0^x e^{-\frac{x-y}{\sigma}+\eta y}\mathrm{d}y + \int_x^{+\infty} \mathrm{e}^{\frac{x-y}{\sigma}+\eta y}\mathrm{d}y\right)$$
$$= \frac{1}{2\sigma}\|U_1 - U_2\|_{0,\eta}\left(\frac{1}{\frac{1}{\sigma}-\eta}e^{-\frac{x}{\sigma}} + \frac{1}{\frac{1}{\sigma}+\eta}\left(\mathrm{e}^{\eta x} - e^{-\frac{x}{\sigma}}\right) + e^{\eta x}\frac{1}{\frac{1}{\sigma}-\eta}\right), \tag{3.37}$$

and similarly for $x < 0$:

$$|P_1(x) - P_2(x)| \leq \frac{1}{2\sigma}\|U_1 - U_2\|_{0,\eta}\left(\frac{1}{\frac{1}{\sigma}-\eta}e^{-\frac{|x|}{\sigma}} + \frac{1}{\frac{1}{\sigma}+\eta}\left(\mathrm{e}^{\eta|x|} - e^{-\frac{1}{\sigma}|x|}\right) + e^{\eta|x|}\frac{1}{\frac{1}{\sigma}-\eta}\right). \tag{3.38}$$

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Rearranging the terms in (3.37) and (3.38), we have

$$|P_1(x)-P_2(x)| \leq \frac{1}{2\sigma}\left[\left(\frac{1}{\sigma}-\eta\right)^{-1}\left(\mathrm{e}^{-\frac{|x|}{\sigma}}+\mathrm{e}^{\eta|x|}\right)+\left(\frac{1}{\sigma}+\eta\right)^{-1}\left(\mathrm{e}^{\eta|x|}-\mathrm{e}^{-\frac{|x|}{\sigma}}\right)\right]\|U_1-U_2\|_{0,\eta} \tag{3.39}$$

and (3.33) is proved.

Step 2: We show (3.34). We have

$$\begin{aligned}|P_1'(x)-P_2'(x)| &= \frac{1}{2\sigma^2}\left|\int_{-\infty}^{+\infty}-\mathrm{sign}(x-y)\mathrm{e}^{-\frac{|x-y|}{\sigma}}(U_1(y)-U_2(y))\mathrm{d}y\right| \\ &\leq \frac{1}{2\sigma^2}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{|x-y|}{\sigma}+\eta|y|}\mathrm{e}^{-\eta|y|}\,|U_1(y)-U_2(y)|\,\mathrm{d}y,\end{aligned}$$

so that the exact computations leading to (3.39) can be reproduced, and we have

$$|P_1'(x)-P_2'(x)| \leq \frac{1}{\sigma}C_P(x)\|U_1-U_2\|_{0,\eta}.$$

(3.34) is proved.

Step 3: We show (3.35). It follows from the definitions of $\lambda_1(x)$ and $\lambda_2(x)$ that, for all $x\in\mathbb{R}$,

$$\begin{aligned}&|\lambda_2(x)-\lambda_1(x)| \\ &=\left|\frac{1+\frac{\chi}{\sigma^2}P_2(x)}{c-\chi P_2'(x)}-\frac{1+\frac{\chi}{\sigma^2}P_1(x)}{c-\chi P_1'(x)}\right| \\ &=\frac{\chi}{|c-\chi P_2'(x)||c-\chi P_1'(x)|}\left|\frac{c}{\sigma^2}(P_2(x)-P_1(x))+P_2'(x)-P_1'(x)+\frac{\chi}{\sigma^2}(P_1(x)P_2'(x)-P_2(x)P_1'(x))\right|.\end{aligned} \tag{3.40}$$

Since by Definition 3.1, we have $U_i'(x)\leq C_U$ $(i=1,2)$, then $P_i'(x)=\int_{\mathbb{R}}\frac{1}{2\sigma}e^{-\frac{|x-y|}{\sigma}}U'(y)\mathrm{d}y\leq C_U$ $(i=1,2)$, therefore

$$c-\chi P_i'(x)\geq c-\chi C_U=\frac{1}{2}\left(c-\sqrt{c^2-\chi\left(1+\frac{\chi}{\sigma^2}\right)}\right)>0, \qquad i=1,2. \tag{3.41}$$

It follows from (3.40) and (3.41) that

$$\begin{aligned}|\lambda_2(x)-\lambda_1(x)| &\leq \frac{c\chi}{\sigma^2(c-\chi C_U)^2}|P_2(x)-P_1(x)|+\frac{\chi}{(c-\chi C_U)^2}|P_2'(x)-P_1'(x)| \\ &\quad+\frac{\chi}{\sigma^2(c-\chi C_U)^2}|P_1(x)P_2'(x)-P_2(x)P_1'(x)| \\ &\leq \frac{c\chi}{\sigma^2(c-\chi C_U)^2}|P_2(x)-P_1(x)|+\frac{\chi}{(c-\chi C_U)^2}|P_2'(x)-P_1'(x)| \\ &\quad+\frac{\chi}{\sigma^2(c-\chi C_U)^2}\left(P_1(x)|P_2'(x)-P_1'(x)|+|P'_1(x)||P_1(x)-P_2(x)|\right).\end{aligned}$$

Using the fact that $0\leq P_1(x)\leq 1$ and $0\leq P'_1(x)\leq C_U$ for $x\in\mathbb{R}$, then (3.35) is a consequence of (3.33) and (3.34).

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Step 4: We show (3.36). We have:

$$\frac{1}{1+\frac{\chi}{\sigma^2}}|\kappa_1(x)-\kappa_2(x)| = \left|\frac{1}{c-\chi P_1'(x)} - \frac{1}{c-\chi P_2'(x)}\right| = \left|\frac{c-\chi P_1'(x)-c+\chi P_2'(x)}{\big(c-\chi P_2'(x)\big)\big(c-P_1'(x)\big)}\right|$$

$$= \chi\left|\frac{P_2'(x)-P_1'(x)}{\big(c-\chi P_2'(x)\big)\big(c-P_1'(x)\big)}\right| \leq \frac{\chi}{(c-\chi C_U)^2}|P_2'(x)-P_1'(x)|.$$

Thus, (3.36) is a consequence of (3.34). Lemma 3.5 is proved. □

Lemma 3.6 (Continuity of $\mathcal{T}$). *Let Assumption 3.2 be satisfied. Assume that* $0<\eta<\frac{1}{\sigma}$. *Then, the map* $\mathcal{T}:\mathcal{A}\to\mathcal{A}$ *is continuous on* $\mathcal{A}$ *endowed with distance* $d(U_1,U_2)=\|U_1-U_2\|_{1,\eta}$.

Proof. Let $U_0\in\mathcal{A}$ be fixed, and $U\in\mathcal{A}$, and define

$$V_0=\mathcal{T}(U_0) \text{ and } V=\mathcal{T}(U).$$

Part A: We prove that for each admissible profile $U_0\in\mathcal{A}$ and $\varepsilon>0$, there is a $\delta_1>0$ such that

$$\|V-V_0\|_{0,\eta}\leq\frac{\varepsilon}{2}, \tag{3.42}$$

whenever

$$\|U-U_0\|_{0,\eta}\leq\delta_1.$$

Let $K>0$ be such that

$$e^{-\eta K}\leq\frac{\varepsilon}{12}.$$

Then since $V\in\mathcal{A}$ by Lemma 3.3, we have $0\leq V(x)\leq 1$ and $0\leq V_0(x)\leq 1$ for all $x\in\mathbb{R}$, therefore

$$\begin{aligned}\|V-V_0\|_{0,\eta} &= \sup_{x\in\mathbb{R}} e^{-\eta|x|}|V(x)-V_0(x)|\\ &\leq \sup_{x\leq -K} e^{-\eta|x|}|V(x)-V_0(x)| + \sup_{|x|\leq K} e^{-\eta|x|}|V(x)-V_0(x)| + \sup_{x\geq K} e^{-\eta|x|}|V(x)-V_0(x)|\\ &\leq e^{-\eta K}\left(\sup_{x\leq -K}|V(x)-V_0(x)| + \sup_{x\geq K}|V(x)-V_0(x)|\right) + \sup_{|x|\leq K} e^{-\eta|x|}|V(x)-V_0(x)|\\ &\leq 4e^{-\eta K} + \sup_{|x|\leq K} e^{-\eta|x|}|V(x)-V_0(x)| \leq \frac{2\varepsilon}{6} + \sup_{|x|\leq K} e^{-\eta|x|}|V(x)-V_0(x)|.\end{aligned}$$

Thus, there remains only to establish that

$$\sup_{|x|\leq K} e^{-\eta|x|}|V(x)-V_0(x)|\leq\frac{\varepsilon}{6}, \tag{3.43}$$

if $\|U-U_0\|_{1,\eta}\leq\delta_1$, for $\delta_1>0$ sufficiently small. Recall that

$$V(x)=\frac{u_0\exp\left(\int_0^x \lambda(s)\mathrm{d}s\right)}{1+u_0\int_0^x \kappa(s)\exp\left(\int_0^s\lambda(l)\mathrm{d}l\right)\mathrm{d}s},$$

wherein

$$\lambda(x)=\frac{1+\frac{\chi}{\sigma^2}P(x)}{c-\chi P'(x)},$$

and

$$\kappa(x)=\frac{1+\frac{\chi}{\sigma^2}}{c-\chi P'(x)},$$

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and $P(x)$ is the unique solution of the elliptic equation

$$P(x) - \sigma^2 P''(x) = U(x), \forall x \in \mathbb{R}.$$

By using the definitions of $V_0(x)$ and $V(x)$ for $x \in \mathbb{R}$, we find that

$$\begin{aligned}
|V(x) - V_0(x)| &= \left| \frac{u_0 \exp\left(\int_0^x \lambda(s)\mathrm{d}s\right)}{1 + u_0 \int_0^x \kappa(s) \exp\left(\int_0^s \lambda(l)\mathrm{d}l\right) \mathrm{d}s} - \frac{u_0 \exp\left(\int_0^x \lambda_0(s)\mathrm{d}s\right)}{1 + u_0 \int_0^x \kappa_0(s) \exp\left(\int_0^s \lambda_0(l)\mathrm{d}l\right) \mathrm{d}s} \right| \\
&= \frac{u_0}{\left|1 + u_0 \int_0^x \kappa(s) \exp\left(\int_0^s \lambda(l)\mathrm{d}l\right) \mathrm{d}s\right| \left|1 + u_0 \int_0^x \kappa_0(s) \exp\left(\int_0^s \lambda_0(l)\mathrm{d}l\right) \mathrm{d}s\right|} \\
&\quad \times \left| \exp\left(\int_0^x \lambda(s)\mathrm{d}s\right) - \exp\left(\int_0^x \lambda_0(s)\mathrm{d}s\right) \right. \\
&\quad + u_0 \exp\left(\int_0^x \lambda(s)\mathrm{d}s\right) \int_0^x \kappa_0(s) \exp\left(\int_0^s \lambda_0(l)\mathrm{d}l\right) \mathrm{d}s \\
&\quad \left. - u_0 \exp\left(\int_0^x \lambda_0(s)\mathrm{d}s\right) \int_0^x \kappa(s) \exp\left(\int_0^s \lambda(l)\mathrm{d}l\right) \mathrm{d}s \right|. \qquad (3.44)
\end{aligned}$$

Since

$$\frac{1}{1 + u_0 \int_0^x \kappa_0(s) \exp\left(\int_0^s \lambda_0(l)\mathrm{d}l\right) \mathrm{d}s} = \frac{V_0(x)}{u_0 \exp\left(\int_0^x \lambda_0(s)\mathrm{d}s\right)},$$

and similarly

$$\frac{1}{1 + u_0 \int_0^x \kappa(s) \exp\left(\int_0^s \lambda(l)\mathrm{d}l\right) \mathrm{d}s} = \frac{V(x)}{u_0 \exp\left(\int_0^x \lambda(s)\mathrm{d}s\right)},$$

we have that

$$\begin{aligned}
&|V(x) - V_0(x)| \\
&= \left| \frac{V(x)V_0(x)}{u_0 \exp\left(\int_0^x \lambda(s)\mathrm{d}s\right) \exp\left(\int_0^x \lambda_0(s)\mathrm{d}s\right)} \right| \left| \exp\left(\int_0^x \lambda(s)\mathrm{d}s\right) - \exp\left(\int_0^x \lambda_0(s)\mathrm{d}s\right) \right. \\
&\quad + u_0 \exp\left(\int_0^x \lambda(s)\mathrm{d}s\right) \int_0^x \kappa_0(s) \exp\left(\int_0^s \lambda_0(l)\mathrm{d}l\right) \mathrm{d}s \\
&\quad \left. - u_0 \exp\left(\int_0^x \lambda_0(s)\mathrm{d}s\right) \int_0^x \kappa(s) \exp\left(\int_0^s \lambda(l)\mathrm{d}l\right) \mathrm{d}s \right|
\end{aligned}$$

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$$= \frac{|V(x)V_0(x)|}{u_0} \Bigg| \exp\left(-\int_0^x \lambda_0(s)\mathrm{d}s\right) - \exp\left(-\int_0^x \lambda(s)\mathrm{d}s\right)$$

$$+ u_0 \exp\left(-\int_0^x \lambda_0(s)\mathrm{d}s\right) \int_0^x \kappa_0(s) \exp\left(\int_0^s \lambda_0(l)\mathrm{d}l\right) \mathrm{d}s$$

$$- u_0 \exp\left(-\int_0^x \lambda(s)\mathrm{d}s\right) \int_0^x \kappa(s) \exp\left(\int_0^s \lambda(l)\mathrm{d}l\right) \mathrm{d}s \Bigg|,$$

hence,

$$|V(x) - V_0(x)| \leq \frac{1}{u_0} H(x) + I(x), \tag{3.45}$$

where

$$H(x) := \left| \exp\left(-\int_0^x \lambda_0(s)\mathrm{d}s\right) - \exp\left(-\int_0^x \lambda(s)\mathrm{d}s\right) \right|, \tag{3.46}$$

and

$$I(x) := \left| \int_0^x \kappa_0(s) \exp\left(-\int_s^x \lambda_0(l)\mathrm{d}l\right) \mathrm{d}s - \int_0^x \kappa(s) \exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right) \mathrm{d}s \right|. \tag{3.47}$$

We divide the rest of the proof of Part A into two steps, to estimate $H(x)$ and $I(x)$.

Step 1: We show that

$$H(x) \leq C_H(x) \|U - U_0\|_{0,\eta}, \ \forall x \in \mathbb{R}, \tag{3.48}$$

for some continuous function $C_H(x)$ independent of ε, U, U_0.
By Taylor's theorem, we have that

$$|\mathrm{e}^A - \mathrm{e}^B| \leq |A - B| \mathrm{e}^{\max\{A,B\}}, \quad \forall A, B \in \mathbb{R}. \tag{3.49}$$

Now we use (3.49) to estimate $H(x)$ defined in (3.46).

$$\begin{aligned} H(x) &\leq \left| \int_0^x \lambda(s) - \lambda_0(s)\mathrm{d}s \right| \exp\left(\max\left\{ -\int_0^x \lambda(s)\mathrm{d}s, -\int_0^x \lambda_0(s)\mathrm{d}s \right\} \right) \\ &\leq \exp\left(\max\left\{ -\int_0^x \lambda(s)\mathrm{d}s, -\int_0^x \lambda_0(s)\mathrm{d}s \right\} \right) \int_0^x |\lambda(s) - \lambda_0(s)| \,\mathrm{d}s. \end{aligned} \tag{3.50}$$

Recall from (3.35) in Lemma 3.5 that there is a continuous function $C_\lambda(x)$ such that

$$|\lambda(x) - \lambda_0(x)| \leq C_\lambda(x) \|U - U_0\|_{0,\eta}, \ \forall x \in \mathbb{R}.$$

Thus, we can rewrite (3.50) as

$$H(x) \leq \exp\left(-\int_0^x \lambda(s)\mathrm{d}s, -\int_0^x \lambda_0(s)\mathrm{d}s \right) \int_0^x C_\lambda(s)\mathrm{d}s \|U - U_0\|_{0,\eta}. \tag{3.51}$$

Next recall the definition of $\lambda(x)$:

$$\lambda(x) = \frac{1 + \frac{\chi}{\sigma^2} P(x)}{c - \chi P'(x)}, \ \forall x \in \mathbb{R}.$$

Since by Definition 3.1, we have $U'(x) \leq C_U$, then $P'(x) = \int_{\mathbb{R}} \frac{1}{2\sigma} e^{-\frac{|x-y|}{\sigma}} U'(y)\mathrm{d}y \leq C_U$; therefore,

$$c - \chi P'(x) \geq c - \chi C_U = \frac{1}{2} \left(c - \sqrt{c^2 - \chi \left(1 + \frac{\chi}{\sigma^2}\right)} \right) > 0, \ \forall x \in \mathbb{R},$$

and therefore,

$$\lambda(x) = \frac{1 + \frac{\chi}{\sigma^2} P(x)}{c - \chi P'(x)} \leq \frac{1 + \frac{\chi}{\sigma^2}}{c - \chi C_U}, \ \forall x \in \mathbb{R}.$$

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Clearly, we have the same upper bound for $\lambda_0(x)$ and $\lambda(x)$, and (3.51) becomes

$$H(x) \leq \exp\left(\frac{1+\frac{\chi}{\sigma^2}}{c-\chi C_U}|x|\right)\int_{[0,x]} C_\lambda(s)\mathrm{d}s\|U-U_0\|_{0,\eta} = C_H(x)\|U-U_0\|_{0,\eta},\ \forall x\in\mathbb{R},$$

where $C_H(x)$ is a continuous function. Therefore, (3.48) is proved.

Step 2: We show that

$$I(x) \leq C_I(x)\|U-U_0\|_{0,\eta},\ \forall x\in\mathbb{R}, \tag{3.52}$$

for some continuous function $C_I(x)$ independent from ε, U, U_0.
Indeed, we have

$$\begin{aligned} I(x) \leq & \left|\int_0^x \kappa_0(s)\exp\left(-\int_s^x \lambda_0(l)\mathrm{d}l\right)\mathrm{d}s - \int_0^x \kappa_0(s)\exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right)\mathrm{d}s\right| \\ & + \left|\int_0^x \kappa_0(s)\exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right)\mathrm{d}s - \int_0^x \kappa(s)\exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right)\mathrm{d}s\right| \\ = & \left|\int_0^x \kappa_0(s)\left(\exp\left(-\int_s^x \lambda_0(l)\mathrm{d}l\right) - \exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right)\right)\mathrm{d}s\right| \\ & + \left|\int_0^x (\kappa_0(s)-\kappa(s))\exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right)\mathrm{d}s\right| \\ =: & I_1(x)+I_2(x), \end{aligned} \tag{3.53}$$

where

$$I_1(x) := \left|\int_0^x \kappa_0(s)\left(\exp\left(-\int_s^x \lambda_0(l)\mathrm{d}l\right) - \exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right)\right)\mathrm{d}s\right|, \tag{3.54}$$

and

$$I_2(x) := \left|\int_0^x (\kappa_0(s)-\kappa(s))\exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right)\mathrm{d}s\right|. \tag{3.55}$$

Using (3.49), we rewrite (3.54) as

$$I_1(x) \leq \int_0^x |\kappa_0(s)|\exp\left(-\max\left\{-\int_s^x \lambda(l)\mathrm{d}l, -\int_s^x \lambda_0(l)\mathrm{d}l\right\}\right)\int_s^x |\lambda(l)-\lambda_0(l)|\mathrm{d}l\mathrm{d}s. \tag{3.56}$$

Since by Definition 3.1, we have $U'(x) \leq C_U$, then $P'(x) = \int_{\mathbb{R}} \frac{1}{2\sigma} e^{-\frac{|x-y|}{\sigma}} U'(y)\mathrm{d}y \leq C_U$; therefore,

$$c-\chi P'(x) \geq c-\chi C_U = \frac{1}{2}\left(c - \sqrt{c^2-\chi\left(1+\frac{\chi}{\sigma^2}\right)}\right) > 0,\ \forall x\in\mathbb{R},$$

and finally

$$|\lambda(x)| \leq \frac{1+\frac{\chi}{\sigma^2}}{c-\chi C_U} \text{ and } |\kappa(x)| \leq \frac{1+\frac{\chi}{\sigma^2}}{c-\chi C_U}. \tag{3.57}$$

By using (3.56), (3.57), and (3.35) in Lemma 3.5, we rewrite as

$$I_1(x) \leq \frac{1+\frac{\chi}{\sigma^2}}{c-\chi C_U}\int_{[0,x]} \exp\left(\frac{1+\frac{\chi}{\sigma^2}}{c-\chi C_U}|x-s|\right)\int_{[s,x]} C_\lambda(l)\mathrm{d}l\mathrm{d}s\|U-U_0\|_{0,\eta}.$$

Thus, there exists a continuous function $C_{I_1}(x)$ such that

$$I_1(x) \leq C_{I_1}(x)\|U-U_0\|_{0,\eta}. \tag{3.58}$$

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Next we estimate $I_2(x)$ in (3.55). By using (3.35) and (3.57), we have

$$I_2(x) \le \int_{[0,x]} |\kappa_0(s) - \kappa(s)| \exp\left(\frac{1 + \frac{\chi}{\sigma^2}}{c - \chi C_U} |x - s| \right) ds$$
$$\le \int_{[0,x]} C_\kappa(s) \exp\left(\frac{1 + \frac{\chi}{\sigma^2}}{c - \chi C_U} |x - s| \right) ds \|U - U_0\|_{0,\eta},$$

thus there exists a continuous function $C_{I_2}(x)$ such that

$$I_2(x) \le C_{I_2}(x) \|U - U_0\|_{0,\eta}. \tag{3.59}$$

Combining (3.53), (3.58) and (3.59), there exists a continuous function $C_I(x) := C_{I_1}(x) + C_{I_2}(x)$ such that (3.52) holds. Step 2 is completed.

Conclusion of Part A: By choosing δ_1 such that

$$\delta_1 := \frac{\varepsilon}{6} \left(\frac{1}{\sup\limits_{x \in [-K,K]} \frac{1}{u_0} C_H(x) + C_I(x)} \right), \tag{3.60}$$

we conclude from (3.38), (3.48), and (3.52) that indeed

$$\sup_{x \in [-K,K]} e^{-\eta|x|} |V(x) - V_0(x)| \le \sup_{x \in [-K,K]} |V(x) - V_0(x)| \le \sup_{x \in [-K,K]} \frac{1}{u_0} H(x) + I(x)$$
$$\le \sup_{x \in [-K,K]} \left(\frac{1}{u_0} C_H(x) + C_I(x) \right) \|U - U_0\|_{0,\eta}$$
$$\le \frac{\varepsilon}{6},$$

whenever $\|U - U_0\|_{0,\eta} \le \delta_1$. Thus, (3.43) holds, and this concludes Part A.

Part B: We prove that for each admissible profile $U_0 \in \mathcal{A}$ and $\varepsilon > 0$, there is $\delta > 0$ such that whenever

$$\|U - U_0\|_{0,\eta} \le \delta,$$

we have

$$\|V - V_0\|_{1,\eta} \le \varepsilon.$$

By Lemma 3.3, we know that $V = \mathcal{T}(U) \in \mathcal{A}$ and $V_0 = \mathcal{T}(U_0) \in \mathcal{A}$. Therefore,

$$|V'(x)| \le C_U \text{ and } |V_0{}'(x)| \le C_U, \ \forall x \in \mathbb{R}.$$

Let $K > 0$ be such that

$$C_U e^{-\eta K} \le \frac{\varepsilon}{12}.$$

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We have

$$\begin{aligned}\sup_{x\in\mathbb{R}} e^{-\eta|x|}|V'(x)-V_0{}'(x)| &\le \sup_{x\le -K} e^{-\eta|x|}|V'(x)-V_0{}'(x)| + \sup_{|x|\le K} e^{-\eta|x|}|V'(x)-V_0{}'(x)|\\ &\quad + \sup_{x\ge K} e^{-\eta|x|}|V'(x)-V_0{}'(x)|\\ &\le e^{-\eta K}\left(\sup_{x\le -K}|V'(x)-V_0{}'(x)| + \sup_{x\ge K}|V'(x)-V_0{}'(x)|\right)\\ &\quad + \sup_{|x|\le K} e^{-\eta|x|}|V'(x)-V_0{}'(x)|\\ &\le \frac{2\varepsilon}{6} + \sup_{|x|\le K} e^{-\eta|x|}|V'(x)-V_0{}'(x)|. \end{aligned} \tag{3.61}$$

Thus, there remains only to establish that

$$\sup_{|x|\le K} e^{-\eta|x|}|V'(x)-V_0{}'(x)| \le \frac{\varepsilon}{6}. \tag{3.62}$$

We note that V and V_0 satisfy (3.2); therefore,

$$V'(x)=\lambda(x)V(x)-\kappa(x)V^2(x), \text{ and } V_0{}'(x)=\lambda_0(x)-\kappa_0(x), \text{ for all } x\in\mathbb{R}. \tag{3.63}$$

Then, we have that

$$\begin{aligned}|V'(x)-V_0{}'(x)| &= |\lambda(x)V(x)-\kappa(x)V^2(x)-\lambda_0(x)V_0(x)+\kappa_0(x)V_0^2(x)|\\ &= |(\lambda(x)-\lambda_0(x))V(x)+\lambda_0(x)(V(x)-V_0(x))\\ &\quad + (\kappa_0(x)-\kappa(x))V_0^2(x)+\kappa(x)(V_0^2(x)-V^2(x))|\\ &\le |\lambda(x)-\lambda_0(x)||V(x)|+\lambda_0(x)|V(x)-V_0(x)|\\ &\quad + |\kappa_0(x)-\kappa(x)|V_0^2(x)+\kappa(x)|V_0^2(x)-V^2(x)|\\ &= |\lambda(x)-\lambda_0(x)||V(x)|+\lambda_0(x)|V(x)-V_0(x)|\\ &\quad + |\kappa_0(x)-\kappa(x)|V_0^2(x)+\kappa(x)|V_0(x)-V(x)||V_0(x)+V(x)|.\end{aligned}$$

By using the fact that $0\le V(x)\le 1$ and $0\le V_0(x)\le 1$ for $x\in\mathbb{R}$, we have that

$$|V'(x)-V_0{}'(x)| \le |\lambda(x)-\lambda_0(x)|+\lambda_0(x)|V(x)-V_0(x)|+|\kappa_0(x)-\kappa(x)|+2\kappa(x)|V_0(x)-V(x)|. \tag{3.64}$$

It follows from (3.45), (3.57), (3.52), (3.48), (3.35), and (3.36) that

$$|V'(x)-V_0{}'(x)| \le \left(C_\lambda(x)+\frac{1+\frac{\chi}{\sigma^2}}{u_0(c-\chi C_U)}(C_H(x)+u_0C_I(x))+C_\kappa(x)+2\frac{1+\frac{\chi}{\sigma^2}}{c-\chi C_U}\right)\|U-U_0\|_{0,\eta}. \tag{3.65}$$

Let

$$\delta_2 := \frac{\varepsilon}{3}\left[\sup_{x\in[-K,K]}\left(C_\lambda(x)+\frac{1+\frac{\chi}{\sigma^2}}{u_0(c-\chi C_U)}(C_H(x)+u_0C_I(x))+C_\kappa(x)+2\frac{1+\frac{\chi}{\sigma^2}}{c-\chi C_U}\right)e^{-\eta|x|}\right]^{-1}, \tag{3.66}$$

then whenever $\|U-U_0\|_{0,\eta}\le\delta_2$, we have

$$\sup_{x\in[-K,K]} e^{-\eta|x|}|V'(x)-V_0{}'(x)| \le \frac{\varepsilon}{3},$$

therefore, recalling (3.61),

$$\sup_{x\in\mathbb{R}} e^{-\eta|x|}|V'(x)-V_0{}'(x)| \le \frac{\varepsilon}{2}. \tag{3.67}$$

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Conclusion of Part B: Let $\delta := \min(\delta_1, \delta_2)$ where δ_1 is defined in (3.60) and δ_2 is defined in (3.66). Then if $\|U - U_0\|_{0,\eta} \le \delta$, we know from Part A (3.42) that

$$\sup_{x\in\mathbb{R}} e^{-\eta|x|}|V(x) - V_0(x)| \le \frac{\varepsilon}{2},$$

and from (3.67) that

$$\sup_{x\in\mathbb{R}} e^{-\eta|x|}|V'(x) - V_0{}'(x)| \le \frac{\varepsilon}{2}.$$

So finally,

$$\|V - V_0\|_{1,\eta} = \sup_{x\in\mathbb{R}} e^{-\eta|x|}|V(x) - V_0(x)| + \sup_{x\in\mathbb{R}} e^{-\eta|x|}|V'(x) - V_0{}'(x)| \le \varepsilon.$$

Part B is proved. Since we always have $\|U - U_0\|_{0,\eta} \le \|U - U_0\|_{1,\eta}$, the continuity holds for the norm $\|\cdot\|_{1,\eta}$. Lemma 3.5 is proved. □

4. Proof of Theorem 1.4

From the definition of admissible functions $\mathcal{A}$, it is a nonempty, closed, convex, bounded subset of the Banach space $BUC^1_\eta(\mathbb{R})$. By Lemmas 3.4 and 3.6, we obtain that $\mathcal{T}$ is a continuous compact operator on $\mathcal{A}$. Therefore, by the Schauder fixed point theorem, there exists U in $\mathcal{A}$ such that

$$\mathcal{T}(U) = U.$$

Applying Lemma 3.3, we have that $U \in C^1(\mathbb{R})$ and $0 \le U'(x) \le C_U$ for any $x \in \mathbb{R}$. Therefore, we have that

$$U(x) = \frac{u_0 e^{\int_0^x \lambda(s)\mathrm{d}s}}{1 + u_0 \int_0^x \kappa(s) e^{\int_0^s \lambda(l)\mathrm{d}l}\mathrm{d}s},$$

wherein

$$\lambda(x) = \frac{1 + \frac{\chi}{\sigma^2}P(x)}{c - \chi P'(x)},$$

and

$$\kappa(x) = \frac{1 + \frac{\chi}{\sigma^2}}{c - \chi P'(x)},$$

and $P(x)$ is the unique solution of the elliptic equation

$$P(x) - \sigma^2 P''(x) = U(x), \forall x \in \mathbb{R}. \tag{4.1}$$

Namely, we have that

$$U'(x) = \frac{1}{c - \chi P'(x)} U(x) \left(\left(1 + \frac{\chi}{\sigma^2} P(x)\right) - \left(1 + \frac{\chi}{\sigma^2}\right) U(x) \right), \forall x \in \mathbb{R}. \tag{4.2}$$

Therefore, we have that

$$cU'(x) - \chi P'(x)U'(x) - \frac{\chi}{\sigma^2} U(x)(P(x) - U(x)) = U(x)(1 - U(x)), \forall x \in \mathbb{R}.$$

By using (4.1), we have that

$$cU'(x) - \chi(P'(x)U(x))' = U(x)(1 - U(x)), \forall x \in \mathbb{R}. \tag{4.3}$$

We prove that

$$U(\infty) := \lim_{x\to+\infty} U(x) = 1 \text{ and } U(-\infty) := \lim_{x\to-\infty} U(x) = 0.$$

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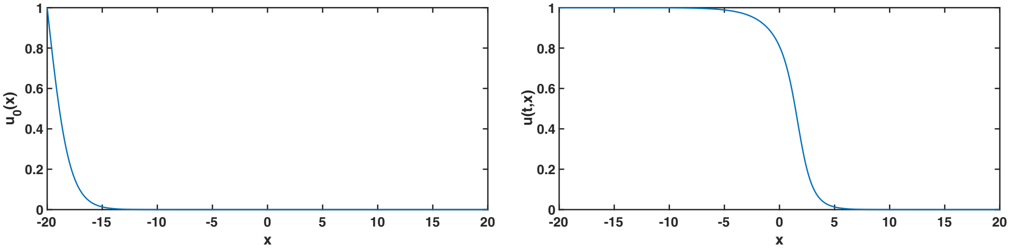

Figure 3. *On the left-hand side, we plot $x \to u_0(x)$ the initial distribution of system (A.1), obtained by using formula (5.1) with $\beta = 1$ and $K = 20$. On the right-hand side, we plot the travelling wave profile which coincides with $x \to u(t, x)$ the solution of system (A.1) at $t = 20$ days.*

Indeed, since $U'(x) \geq 0$ and $0 < U(x) \leq 1$ for any $x \in \mathbb{R}$, then $U(\infty)$ exists. By using P equation (1.8), the function $x \to P(x)$ is increasing and bounded, and by Lebesgue's dominated convergence theorem, we have

$$P(\pm\infty) = U(\pm\infty).$$

Therefore,

$$\lim_{x \to \pm\infty} U'(x) = 0, \text{ and } \lim_{x \to \pm\infty} P'(x) = 0. \tag{4.4}$$

It follows from (4.3) and (4.4), we have that

$$\lim_{x \to \pm\infty} U(x)(1 - U(x)) = 0,$$

and since $x \to U(x)$ increasing and $U(0) = u_0 > 0$, this implies that

$$U(-\infty) = 0, \text{ and } U(+\infty) = 1.$$

This completes the proof of the Theorem 1.4.

5. Numerical simulations

We choose a bounded interval $[-K, K]$ and an initial distribution $u_0 \in C([-K, K])$ as follows:

$$u_0(x) = \frac{2\,\mathrm{e}^{-\beta(x+K)}}{1 + \mathrm{e}^{-\beta(x+K)}}. \tag{5.1}$$

In the following numerical simulations, we solve the PDE numerically using the upwind scheme, and we refer to Leveque [16] and Toro [19] for more results on this subject. The numerical method used for the simulations is presented in Section A of the Appendix.

In this section, we set the parameters of the system (A.1) all equal to one. That is,

$$\sigma = \chi = \lambda = \kappa = 1.$$

In Figure 3, we plot $x \to u_0(x)$ with the parameter values $\beta = 1$, and $K = 20$, and the corresponding travelling wave profile which coincides with $x \to u(20, x)$ the solution of system (A.1) at $t = 20$ days.

In Figure 4, we run a simulation from $t = 0$ until $t = 20$ of the model (A.1). We observe that the travelling wave appears almost immediately after the starting time $t = 0$.

Next, we use the following initial value

$$u_0(x) = \max\left(1 - \beta\,(x + K)\,, 0\right). \tag{5.2}$$

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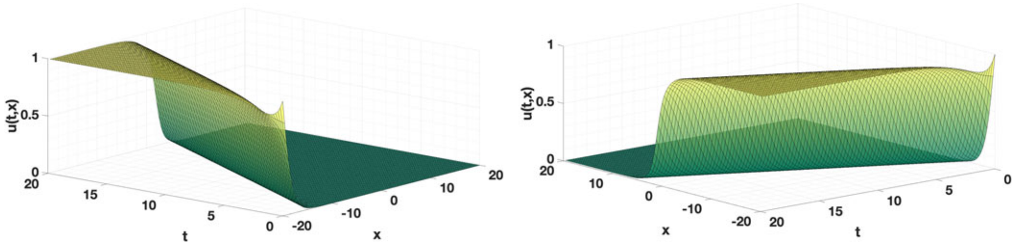

Figure 4. *In this figure, we plot the solution of the model (A.1) starting from the initial distribution (5.1) (with* $\beta = 1$ *and* $K = 20$*).*

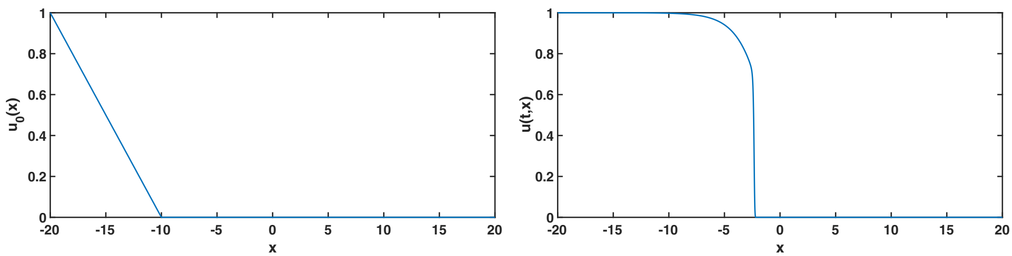

Figure 5. *On the left-hand side, we plot* $x \to u_0(x)$ *the initial distribution of system (A.1), obtained by using formula (5.2) with* $\beta = 0.1$ *and* $K = 20$*. On the right-hand side, we plot the travelling wave profile which coincide with* $x \to u(t, x)$ *the solution of system (A.1) at* $t = 20$ *days.*

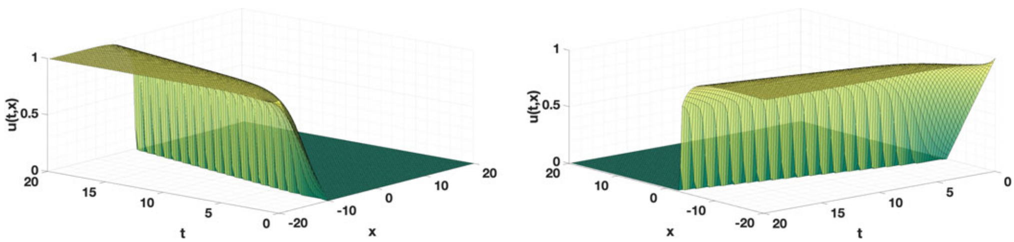

Figure 6. *In this figure, we plot the solution of the model (A.1) starting from the initial distribution (5.2) (with* $\beta = 0.1$ *and* $K = 20$*).*

In Figure 5, we plot $x \to u_0(x)$ the initial distribution of system (A.1) (on the left-hand side) and the corresponding travelling wave profile which coincides with $x \to u(20, x)$ the solution of system (A.1) at $t = 20$ days.

In Figure 6, we run a simulation from $t = 0$ until $t = 20$ of the model (A.1). We observe that the travelling wave appears almost immediately after the starting time $t = 0$.

On the one hand, our numerical simulations show that continuous travelling waves can be observed from an initial distribution decaying exponentially (slowly enough). On the other hand, sharp travelling waves can also be observed when starting the PDE with initial distributions equal to zero on the half-plane. So in practice, both types of travelling waves can be observed numerically.

Now concerning the travelling speed, we observe numerically that sharp travelling waves are slower than continuous travelling waves. In this aspect, the situation is somehow similar to what is observed with reaction-diffusion equations (like the Fisher-KPP equation), in that the ‘slowest’ wave is caught

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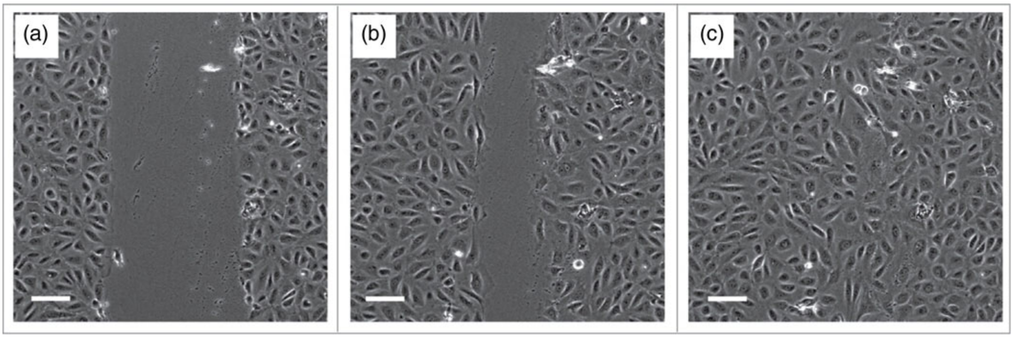

Figure 7. *Images from a scratch assay experiment at different time points. Human umbilical vein endothelial cells were plated on gelatin-coated plastic dishes, wounded with a p20 pipette tip, and then imaged overnight using a microscope equipped with point visiting and live-cell apparatus. Scale bar = 120 μm. This figure is taken from Jonkman et al. [14].*

by starting from compactly supported initial data. The question of the minimal speed is quite intricate given the nonlinear nature of the equation, and we leave it for future works.

6. Application to wound healing

The wound healing assay is used in a range of disciplines to study the coordinated movement of a cell population (Figure 7). We refer to the paper of Jonkman et al. [14] for a review on this topic. In this paper, we consider the cell-cell repulsion described by nonlinear diffusion, but cell-cell attraction also occurs and this problem was recently considered by Webb [20] (see also the references therein for more results).

In this section, we set the parameters of the system (A.1) as follows

$$\chi = \lambda = 4 \text{ and } \sigma = \kappa = 1.$$

Initial distribution for imperfect wound: We choose a bounded interval $[-K, K]$ and an initial distribution $u_0 \in C([-K, K])$ as follows

$$u_0(x) = \frac{1}{2}\left(\frac{2\,\mathrm{e}^{-\beta(x+K)}}{1+\mathrm{e}^{-\beta(x+K)}}\right) + \frac{1}{2}\left(\frac{2\,\mathrm{e}^{-\beta(K-x)}}{1+\mathrm{e}^{-\beta(K-x)}}\right). \tag{6.1}$$

In Figure 8, we plot $x \to u_0(x)$ with the parameter values $\beta = 0.5$, and $K = 20$, and $x \to u(7, x)$ the solution of system (A.1) at $t = 7$ days.

In Figure 9, we run a simulation from $t = 0$ until $t = 7$ of the model (A.1). We observe that two travelling waves moving in opposite directions appear almost immediately after the starting time $t = 0$. They merge together to give a flat distribution approximately on day 2.

Initial distribution for perfect wound: We choose a bounded interval $[-K, K]$ and an initial distribution $u_0 \in C([-K, K])$ as follows

$$u_0(x) = \frac{1}{2}\left(\max\left(1 - \beta\,(x + K)\,, 0\right)\right) + \frac{1}{2}\left(\max\left(1 - \beta\,(K - x)\,, 0\right)\right). \tag{6.2}$$

In Figure 10, we plot $x \to u_0(x)$ with the parameter values $\beta = 0.07$, and $K = 20$, and $x \to u(7, x)$ the solution of system (A.1) at $t = 7$ days.

In Figure 11, we run a simulation from $t = 0$ until $t = 7$ of the model (A.1) for the parameter values $\sigma = 1$ and $\chi = 1$. We observe that two travelling waves moving in opposite directions appear almost

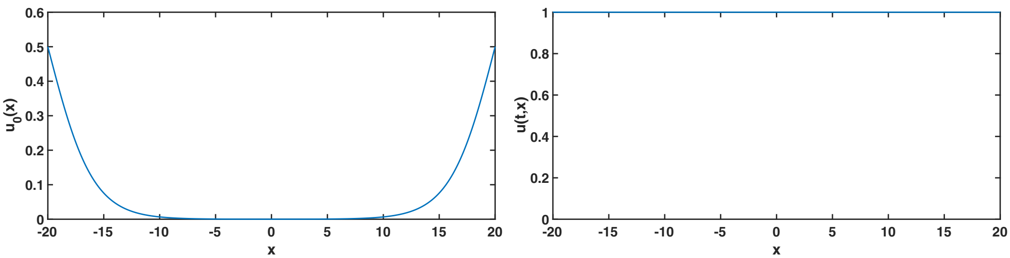

Figure 8. *On the left-hand side, we plot $x \rightarrow u_0(x)$ the initial distribution of system (A.1), obtained by using formula (6.1) with $\beta = 0.5$ and $K = 20$. On the right-hand side, we plot $x \rightarrow u(t,x)$ the solution of system (A.1) at $t = 7$ days.*

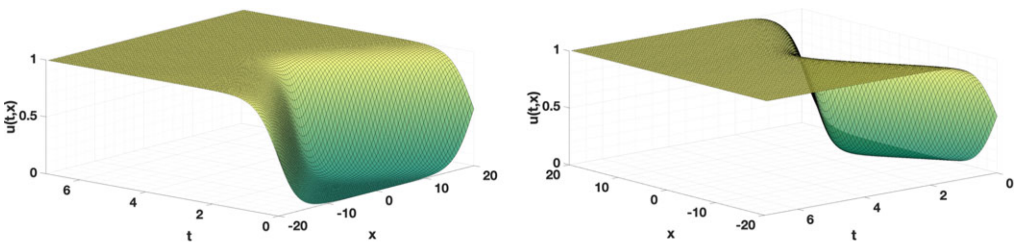

Figure 9. *In this figure, we plot the solution of the model (A.1) starting from the initial distribution (6.1) (with $\beta = 0.5$ and $K = 20$).*

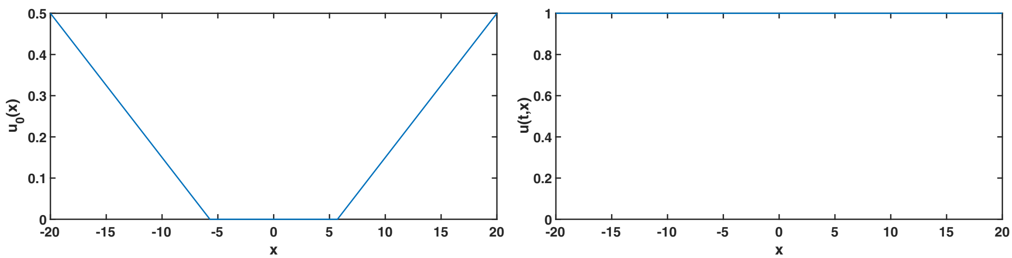

Figure 10. *On the left-hand side, we plot $x \rightarrow u_0(x)$ the initial distribution of system (A.1), obtained by using formula (6.2) with $\beta = 0.07$ and $K = 20$. On the right-hand side, we plot $x \rightarrow u(t,x)$ the solution of system (A.1) at $t = 7$ days.*

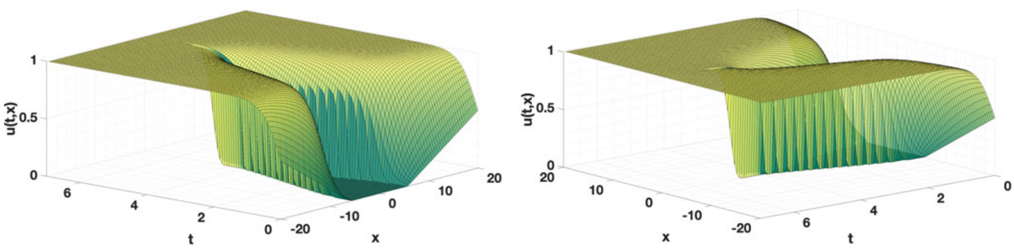

Figure 11. *In this figure, we plot the solution of the model (A.1) starting from the initial distribution (6.2) (with $\beta = 0.07$ and $K = 20$).*

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immediately after the starting time $t = 0$. They merge together to give a flat distribution approximately on day 5.

It is observed that the speed of healing depends strongly on the imperfection of the wound. If we compare the two simulations, we see that the wound seems much larger in Figure 8 than in Figure 10. But the time required for healing is about 2 days in Figure 9 whereas it is about 5 days in Figure 11. Therefore, the imperfection of the wound has a strong influence on the healing time.

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Competing interests. The authors declare none.

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Appendix

An Upwind method applied to the numerical scheme

In Section 5, we use the following system of PDE to run the numerical simulations

$$\begin{cases} \partial_t u(t,x) = \chi\,\partial_x\left(u(t,x)\partial_x p(t,x)\right) + \lambda\, u(t,x)(1-u(t,x)/\kappa), & t\in(0,T], x\in[-K,K],\\ p(t,x)-\sigma^2\partial_{xx}p(t,x) = u(t,x), & t\in(0,T], x\in[-K,K],\\ \partial_x p(t,-K) = \partial_x p(t,+K) = 0, & t\in(0,T], \end{cases} \tag{A.1}$$

with

$$u(t,x) = u_0(x) \in L^\infty_+\left([-K,K],\mathbb{R}\right).$$

Now we use the finite volume method to consider equation (A.1). Our numerical scheme reads as follows:

$$u_i^{n+1} = u_i^n - \chi\frac{\Delta t}{\Delta x}\left(\phi(u_{i+1}^n,u_i^n) - \phi(u_i^n,u_{i-1}^n)\right) + \Delta t u_i^n(1-u_i^n), \quad i=1,2,\dots,M, \tag{A.2}$$

where the flux $\phi(u_{i+1}^n,u_i^n)$ for $i=0,\dots,M$ is defined as

$$\phi(u_{i+1}^n,u_i^n) = \left(v_{i+\frac{1}{2}}^n\right)^+ u_i^n - \left(v_{i+\frac{1}{2}}^n\right)^- u_{i+1}^n = \begin{cases} v_{i+\frac{1}{2}}^n u_i^n, & v_{i+\frac{1}{2}}^n \geq 0,\\ v_{i+\frac{1}{2}}^n u_{i+1}^n, & v_{i+\frac{1}{2}}^n < 0, \end{cases} \tag{A.3}$$

where

$$x^+ = \max(0,x), \text{ and } x^- = \max(0,-x),$$

and

$$v_{i+\frac{1}{2}}^n = -\frac{p_{i+1}^n - p_i^n}{\Delta x}, i=0,1,\dots,M, \tag{A.4}$$

where

$$v_{0+\frac{1}{2}}^n = v_{M+\frac{1}{2}}^n = 0.$$

Moreover, the vector P^n is defined by

$$P^n := \left(I - \frac{\sigma^2}{\Delta x^2}A\right)^{-1} U^n, \tag{A.5}$$

where

$$A = \begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1\\ & & & 1 & -1 \end{pmatrix}_{M\times M}. \tag{A.6}$$

Indeed, we have

$$p_i^n - \frac{\sigma^2}{\Delta x^2}\left(p_{i+1}^n - 2p_i^n + p_{i-1}^n\right) = u_i^n, \quad i=1,2,\dots,M, \tag{A.7}$$

and since we use the Neumann boundary condition, we must impose

$$p_0^n = p_1^n \text{ and } p_M^n = p_{M+1}^n.$$

Since the Neumann boundary condition corresponds to a no flux boundary condition, we have

$$\phi(u_1^n,u_0^n) = 0, \text{ and } \phi(u_{M+1}^n,u_M^n) = 0, \tag{A.8}$$

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which corresponds to $p_0^n = p_1^n$ and $p_{M+1}^n = p_M^n$. Therefore, the numerical scheme at the boundary becomes

$$
\begin{aligned}
u_1^{n+1} &= u_1^n - \chi \frac{\Delta t}{\Delta x}\phi(u_2^n, u_1^n) + \Delta t\, u_1^n(1 - u_1^n),\\
u_M^{n+1} &= u_M^n + \chi \frac{\Delta t}{\Delta x}\phi(u_{M+1}^n, u_M^n) + \Delta t\, u_M^n(1 - u_M^n).
\end{aligned}
\tag{A.9}
$$

Due to the boundary condition, we have the conservation of mass for equation (A.1) when the reaction term equals zero.

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