Can. J. Math., Vol. XXXII, No. 4, 1980, pp. 831-838

BOXES IN $\mathbf{R}^n$—A 'FRACTIONAL' THEOREM

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1. Statement of results. A *box* in Euclidean k-dimensional space $\mathbf{R}^k$ is a set of the type

$$\{\bar{x} = (x_1, \ldots, x_k) \in \mathbf{R}^k: a_i \leqq x_i \leqq b_i, i = 1, 2, \ldots, k\}.$$

A family of boxes, unless stated otherwise, is finite.

The object of this paper is to study some intersectional properties of boxes in $\mathbf{R}^k$.

A box is a convex set and for convex sets one has an intersectional theorem:

HELLY'S THEOREM. *For a finite family $\mathscr{A}$ of at least $k + 1$ convex sets in $\mathbf{R}^k$ the intersection $\cap \mathscr{A}$ is non-empty provided that for any subfamily $\mathscr{B}$ of $\mathscr{A}$ with at least $k + 1$-members the intersection $\cap \mathscr{B}$ is not empty.*

Helly's theorem appears in [**7**]; consult [**5**] for general properties of convex sets in $\mathbf{R}^k$ and [**3**] for Helly type theorems.

For boxes a similar well known result holds:

THEOREM. *Any family $\mathscr{A}$ of boxes in $\mathbf{R}^k$ has non-empty intersection provided that any two members of $\mathscr{A}$ have non-empty intersection.*

A simple proof is as follows: Let $\mathscr{A} = \{A_1, \ldots, A_n\}$ where

$$A_j = \{\{x_1, \ldots, x_k\}: a_i^j \leqq x_i \leqq b_i^j, i = 1, \ldots, k\} \text{ for } 1 \leqq j \leqq n.$$

Let $c_i = \min \{b_i^j: 1 \leqq j \leqq n\}$, then $(c_1, \ldots, c_k) \in A_j$ for each j, $1 \leqq j \leqq n$ so that $\cap \mathscr{A} \neq \emptyset$. Two related theorems which are 'fractional' in the sense that a fraction of all the $k + 1$ (or 2) membered subfamilies have non-empty intersection are:

THEOREM A. *For each $0 \leqq \alpha \leqq 1$, $\beta = 1 - \sqrt{1 - \alpha}$ and for any finite family $\mathscr{A}$ of n segments on the line: If the number of 2-membered subfamilies $\mathscr{C}$ of $\mathscr{A}$ for which $\cap \mathscr{C} \neq \emptyset$ is at least $\alpha \cdot \binom{n}{2}$ then there is a $\mathscr{B} \subset \mathscr{A}$ with $\cap \mathscr{B} \neq \emptyset$ and $|\mathscr{B}| \geqq \beta \cdot n$. Furthermore $\beta = 1 - \sqrt{1 - \alpha}$ cannot be replaced by a larger number.*

THEOREM B. *For each $0 < \alpha < 1$ there is a $0 < \beta = \beta(k, \alpha) < 1$ such that for any finite family $\mathscr{A}$ of n convex sets in $\mathbf{R}^k$ with $n \geqq k + 1$: If the*

Received September 12, 1978 and in revised form January 10, 1979. The preparation of this paper was facilitated by a grant from the National Research Council of Canada.

number of $k+1$*-membered subfamilies* $\mathscr{C}$ *of* $\mathscr{A}$ *for which* $\cap\mathscr{C} \neq \emptyset$ *is at least* $\alpha\cdot\binom{n}{k+1}$ *then there is a* $\mathscr{B} \subset \mathscr{A}$ *with* $|\mathscr{B}| \geqq \beta\cdot n$ *and* $\cap\mathscr{B} \neq \emptyset$. (*Furthermore* $\beta \to 1$ *as* $\alpha \to 1$.)

For Theorems A and B see [**1**] and [**8**] respectively. The main result to be proved is a generalization of Theorem A:

THEOREM C. *Let* k *be a positive integer and let* α *satisfy* $0 \leqq \alpha < 1/k$ *with* $\beta = 1 - \sqrt{1-k\alpha}$. *For any family* $\mathscr{A}$ *of* n *boxes in* $\mathbf{R}^k$: *If the number of 2-membered subfamilies* $\mathscr{C}$ *of* $\mathscr{A}$ *for which* $\cap\mathscr{C} \neq \emptyset$ *is at least* $\left(\frac{k-1}{k}+\alpha\right)\cdot\frac{n^2}{2}$ *then there exists a* $\mathscr{B} \subset \mathscr{A}$ *with* $|\mathscr{B}| \geqq [\beta\cdot n]$ *and* $\cap\mathscr{B} \neq \emptyset$. *Furthermore* $\beta = 1 - \sqrt{1-k\alpha}$ *cannot be replaced by a larger number.*

Note that for fixed $k > 1$ when $\alpha = 0$, $\beta = 0$ in spite of the fact that at least $\frac{k-1}{k}\cdot\frac{n^2}{n}$ of the pairs of boxes have nonempty intersection.

We now discuss a result in graph theory. For properties of graphs one may consult [**2**] or [**6**]. The set of edges and the set of vertices of a graph $\mathscr{P}$ are denoted by $E(\mathscr{P})$ and $V(\mathscr{P})$ respectively. The graph $\mathscr{P}'$ is a (maximal) *subgraph* of $\mathscr{P}$ if it is obtained from $\mathscr{P}$ by removing a set of vertices U and all the edges incident to at least one member of U. The graph $\mathscr{A}$ contains the graph $\mathscr{B}$ if $V(\mathscr{A}) \supset V(\mathscr{B})$ and $E(\mathscr{A}) \supset E(\mathscr{B})$. For a subset S of $\mathscr{P}$, $E(S)$ denotes the set of edges of $\mathscr{P}$ which are incident to at least one vertex of S. A *complete* (k-)*graph* is a graph (with k-vertices), such that any two of its vertices are joined by an edge. A subset S of $V(\mathscr{P})$ is called a *special* (k-)*set* of $\mathscr{P}$ if ($|S| = k$ and)

(1) The subgraph of $\mathscr{P}$ with vertices $S \cup \{v \in V(\mathscr{P}): v$ is incident to all the vertices of $S\}$ is a complete graph.

A graph $\mathscr{P}$ is called *k-complete* if

(2) It contains a special k-set provided that it contains a complete k-graph and any subgraph of $\mathscr{P}$ also satisfies (2)

The *intersection graph* of a family of sets $\mathscr{A}$ is a graph whose vertices are members of $\mathscr{A}$, two vertices being joined if the corresponding members of $\mathscr{A}$ have nonempty intersection.

The theorem on k-complete graphs to be used is

THEOREM D. *Let* α *satisfy* $0 \leqq \alpha < 1/k$. *A* k*-complete graph with* n *vertices and at least* $\frac{n^2}{2}\left(\frac{k-1}{k}+\alpha\right)$ *edges contains a complete graph with at least* $[(1-\sqrt{1-k\alpha})\cdot n]$ *vertices. Furthermore, the number* $1-\sqrt{1-k\alpha}$ *cannot be replaced by a larger constant.*

https://doi.org/10.4153/CJM-1980-063-4 Published online by Cambridge University Press

The motivation for studying k-complete graphs is

THEOREM E. *Let $\mathscr{A}$ be a (finite) family of boxes in $\mathbf{R}^k$. The intersection graph of $\mathscr{A}$ is a k-complete graph.*

Turán's theorem for graphs [**10**] gives the number of edges, for a graph with n vertices, which 'force' the graph to contain a complete graph with k vertices. The following slightly weaker form of Turán's original theorem will be used.

TURÁN'S THEOREM. *Any graph with n vertices and at least* $\frac{n^2}{2}\cdot\frac{k-2}{k-1}+1$ *edges contains a complete k-graph.*

Proofs of Theorems C, D and E will be presented in the next section. The last section contains Turán type problems and remarks.

Finally, "the family $\mathscr{A}$ intersects" and "C_1 and C_2 intersect" mean that $\bigcap \mathscr{A} \neq \emptyset$ and that $C_1 \cap C_2 \neq \emptyset$ respectively.

2. Proof of theorems.

Theorem E shall be proved first, then Theorems D and C.

Proof of Theorem E. For a box $\phi \neq Q = \{(x_1, \ldots, x_k); a_i \leqq x_i \leqq b_i, i = 1, \ldots, k\}$ let $f(Q) = (b_1, \ldots, b_n)$.

For two points $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{b} = (b_1, \ldots, b_k)$ define an order relation:

(3) $\mathbf{a} \succeq \mathbf{b}$ if either $\mathbf{a} = \mathbf{b}$ or for i the smallest integer such that $a_i \neq b_i$, $a_i > b_i$.

Note that if A and B are boxes and $A \subset B$ then $f(B) \geqq f(A)$ and also that $f(C) \in C$ for any box $C \neq \emptyset$.

From the definition of the function f if follows that

(4) If $f(\bigcap \mathscr{G}) = \mathbf{c}$ for a family $\mathscr{G}$ of at least k boxes in $\mathbf{R}^k$, then there is a $\mathscr{B} \subset \mathscr{G}$ with $|\mathscr{B}| = k$ and $f(\bigcap \mathscr{B}) = \mathbf{c}$.

Let $\mathscr{A}$ be a family of boxes in $\mathbf{R}^k$ and assume the existence of a $\mathscr{C} \subset \mathscr{A}$ with $\bigcap \mathscr{C} \neq \emptyset$ and $|\mathscr{C}| = k$. We have to show the existence of a $\mathscr{C}_0 \subset \mathscr{A}$ with $|\mathscr{C}_0| = k$ and $\bigcap \mathscr{C}_0 \neq \emptyset$ and such that

(5) If $\mathscr{B} = \{A \in \mathscr{A}: A \cap (\bigcap \mathscr{C}_0) \neq \emptyset\}$ then $\bigcap \mathscr{B} \neq \emptyset$.

Let $\mathscr{C}_0 \subset \mathscr{A}$ be such that $|\mathscr{C}_0| = k$, $\bigcap \mathscr{C}_0 \neq \emptyset$ and

(6) $f(\bigcap \mathscr{C}) \succeq f(\bigcap \mathscr{C}_0)$ for any $\mathscr{C} \subset \mathscr{A}$ with $|\mathscr{C}| = k$ and $\bigcap \mathscr{C} \neq \emptyset$.

Let $\mathscr{B}$ be as in (5) and suppose that $A \in \mathscr{B} \backslash \mathscr{C}_0$ with $\mathscr{D} = \mathscr{C}_0 \cup \{A\}$. By (4) there is a $\mathscr{D}' \subset \mathscr{D}$ with $\bigcap \mathscr{D}' \neq \emptyset$, $|\mathscr{D}'| = k$ and $f(\bigcap \mathscr{D}') = f(\bigcap \mathscr{D})$. But $\bigcap \mathscr{D} \subset \bigcap \mathscr{C}_0$ so that $f(\bigcap \mathscr{C}_0) \succ f(\bigcap \mathscr{D}) = f(\bigcap \mathscr{D}')$. By

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(6) $f(\cap\mathscr{D}') \geq f(\cap\mathscr{C}_0)$ so that $f(\cap\mathscr{D}) = f(\cap\mathscr{C}_0)$ and $f(\cap\mathscr{C}_0) \in \cap\mathscr{D} \subseteq A$. Since this is true for any $A \in \mathscr{B}$, the intersection $\cap\mathscr{B} \supset \{f(\cap\mathscr{C}_0)\} \neq \emptyset$ and (5) holds. This completes the proof of Theorem E.

Proof of Theorem D. The proof is based on the following two observations:

1. Let $\mathscr{H}$ be a graph with l vertices which does not contain a complete m-graph and let S be a special k-set of $\mathscr{H}$. Then

$$(7) \qquad |E(S)| \leqq \binom{k}{2} + (l-k)(k-1) + m - k - 1.$$

and

2. A subgraph of a k-complete graph is a k-complete graph.

Inequality (7) holds since otherwise there would be at least $m - k$ vertices of $V(\mathscr{H})\backslash S$ which are joined by edges to all of the vertices of S. This would imply, since S is a special k-set of $\mathscr{H}$, that $\mathscr{H}$ contains a complete graph with $(m - k) + k = m$ vertices, contradicting the assumption. The statement in 2 is implied by the definition of a k-complete graph and the fact that a subgraph of a subgraph of $\mathscr{H}$ is a subgraph of $\mathscr{H}$.

Let $\mathscr{P}$ be a k-complete graph with n vertices and at least $\frac{n^2}{2}\cdot\left(\frac{k-1}{k} + \alpha\right)$ edges with $0 \leqq \alpha < 1/k$. Suppose that $\mathscr{P}$ does not contain a complete m-graph. We shall later show that this implies

$$(8) \qquad |E(\mathscr{P})| < (n^2k - (n-m)^2)/2k.$$

Let $m_0 = [n\cdot(1 - \sqrt{1 - k\alpha})]$. It is easy to check that

$$(9) \qquad \frac{1}{2}n^2\cdot\left(\frac{k-1}{k} + \alpha\right) \geqq \frac{1}{2k}(n^2k - (n-m_0)^2).$$

Since $|E(\mathscr{P})| \geqq \frac{1}{2}n^2\cdot\left(\frac{k-1}{k} + \alpha\right)$, the inequality (9) implies that

$$|E(\mathscr{P})| \geqq (n^2k - (n-m_0)^2)/2k.$$

Consequently $\mathscr{P}$ must contain a complete m_0-graph, for otherwise by (8)

$$|E(\mathscr{P})| < (n^2k - (n-m_0)^2)/2k$$

a contradiction.

It remains to prove inequality (8). Assume that $\mathscr{P}$ does not contain a complete m-graph. We shall also assume that $n - m + 1$ is divisible by k so that $n - m + 1 = q\cdot k$ for an integer q. The case where this is not true is treated similarly but the calculations are slightly more involved. Define a sequence $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_{q+1}$ of k-complete graphs:

Let $\mathscr{P}_1 = \mathscr{P}$ and for $1 \leqq 1 \leqq q$ let S_i be a special k-set of $\mathscr{P}_i$ and let $\mathscr{P}_{i+1}$ be the subgraph of P_i with vertices $V(\mathscr{P}_i)\backslash S_i$. Note that by Turán's theorem [**10**] a graph with n vertices and more than $\frac{n^2}{2}\cdot\binom{k-2}{k-1}$ edges contains a complete k-graph and therefore a graph with n vertices and at least $\frac{n^2}{2}\cdot\left(\frac{k-1}{k}\right)$ edges contains a complete k-graph. From this remark and the inequality

$$|E(\mathscr{P})| \geqq \frac{n^2}{2}\left(\frac{k-1}{k}+\alpha\right)$$

it follows that each $\mathscr{P}_i$ contains a complete k-graph as claimed.

The graph $\mathscr{P}_i$ for $1 \leqq i \leqq q+1$ is a k-complete graph with $n - (i-1)\cdot k$ vertices and $\mathscr{P}_i$ does not contain a complete m-graph. From inequality (7) one obtains the sequence of inequalities

$$(10)\quad |E(\mathscr{P}_i)| - |E(\mathscr{P}_{i+1})| \leqq \binom{k}{2} + (n - i\cdot k)(k-1) + m - k - 1, \quad \text{for} \quad i = 1, 2, \ldots, q.$$

Adding the q inequalities results in

$$(11)\quad |E(\mathscr{P})| - |E(\mathscr{P}_{q+1})| \leqq \frac{q}{2}\left[2\binom{k}{2} + (2n - (q+1)\cdot k)\cdot(k-1) + 2m - 2k - 2\right].$$

Since $|V(\mathscr{P}_{q+1})| = n - q\cdot k = m - 1$ the number of edges of $\mathscr{P}_{k+1}$ is not more than $\binom{m-1}{2}$ so that by (11)

$$|E(\mathscr{P})| \leqq \binom{m-1}{2} + \frac{q}{2}((k-1)(2n-qk) + 2m - 2k - 2).$$

Using the equality $q = (n - m + 1)/k$ and a straightforward calculation results in equality (8). This concludes the proof of Theorem D without the last statement.

The proof of the last statement of Theorem D follows from the last statement of Theorem C and from Theorem E.

Proof of Theorem C. By Theorem E the intersection graph $\mathscr{P}$ of the family of boxes $\mathscr{A}$ is a k-complete graph with at least $\left(\frac{k-1}{k}+\alpha\right)\cdot\frac{n^2}{2}$ edges. By Theorem D, without the last statement, $\mathscr{P}$ contains a complete graph with at least $[(1 - \sqrt{1-k\alpha})\cdot n]$ vertices. Therefore $\mathscr{A}$ contains a subfamily $\mathscr{B}$ with

$$|\mathscr{B}| \geqq [(1 - \sqrt{1-k\alpha})\cdot n]$$

and such that any two boxes in the family intersect; consequently $\bigcap\mathscr{B} \neq \emptyset$.

https://doi.org/10.4153/CJM-1980-063-4 Published online by Cambridge University Press

It remains to show that $\beta = 1 - \sqrt{1 - k\alpha}$ cannot be replaced by a larger number.

Case 1. $k = 1$, $\alpha > 0$. Given $\alpha > 0$ construct for any n a family $\mathscr{A}_n$ of n segments on the line as follows: Start with a segment of length $[(1 - \sqrt{1 - \alpha})n] + 2$ and move it to the right $n - 1$ times by one unit each time. The n segments obtained are the members of $\mathscr{A}_n$. It is easy to check that the number of intersecting pairs in $\mathscr{A}_n$ is at least $\alpha n^2/2$ but for each $\beta > (1 - \sqrt{1 - \alpha})$ and n sufficiently large there are no $[\beta \cdot n]$ segments in $\mathscr{A}_n$ with nonempty intersection.

Case 2. $k > 1$, $\alpha > 0$. The construction is based on Case 1. Let $0 < \alpha < (k - 1)/k$ be given and let $\beta > 1 - \sqrt{1 - k\alpha}$. Choose $n = km$ sufficiently large and let

$$\mathscr{D} = \{\{x: a_i \leqq x \leqq b_i\}: i = 1, \ldots, m\}$$

be a family of m segments with at least $\frac{1}{2}m^2 \cdot k\alpha$ intersecting pairs but fewer than $[\beta m]$ intersecting segments. By Case 1 such a family exists. Now construct k families $\mathscr{A}_1, \ldots, \mathscr{A}_k$, of boxes in $\mathbf{R}^k$:

$$\mathscr{A}_j = \{A_i^j = \{\mathbf{x} \in \mathbf{R}^k: a_i \leqq x_j \leqq b_i\}: 1 \leqq i \leqq m\} \text{ for } 1 \leqq j \leqq k.$$

Finally let $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \ldots \cup \mathscr{A}_k$. The family $\mathscr{A}$ has the desired properties:

1. Any member of $\mathscr{A}_i$ intersects any member of $\mathscr{A}_j$ for $i \neq j$ giving $m^2k(k - 1)/2 = n^2(k - 1)/2k$ intersecting pairs. Two members of $\mathscr{A}_i$ $(1 \leqq i \leqq k)$ intersect if the corresponding members of $\mathscr{D}$ intersect, adding at least $k \cdot 1m^2k\alpha/2 = n^2\alpha/2$ intersections. Thus as least $\dfrac{n^2}{2}\left(\dfrac{k - 1}{k} + \alpha\right)$ pairs of members of $\mathscr{A}$ have nonempty intersection.

2. Suppose that some d members of $\mathscr{A}$ have nonempty intersection. Then there is an l $(1 \leqq l \leqq k)$ such that some d/k members of $\mathscr{A}_l$ have nonempty intersection and therefore some d/k members of $\mathscr{D}$ have nonempty intersection. By the construction of $\mathscr{D}$, $d/k < [\beta \cdot m]$. Therefore $d < k[\beta \cdot m]$ and $d < [\beta \cdot n]$.

3. The 'boxes' are not boxes, but it is easy to transform them into boxes without changing the intersection pattern.

Case 3. $\alpha = 0$. Let $a_1 < a_2 < \ldots < a_m$. Let

$$\mathscr{A}_j = \{A_i^j = \{\mathbf{x} \in \mathbf{R}^k: x_j = a_i\}: 1 \leqq i \leqq m\} \text{ for } 1 \leqq j \leqq k$$

and let

$$\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \ldots \cup \mathscr{A}_k \text{ with } n = k \cdot m.$$

The family $\mathscr{A}$ is a family of hyperplanes such that $k \cdot (k - 1) \cdot m^2/2 = n^2(k - 1)/2k$ pairs have nonempty intersection but no more than k members (one from each $\mathscr{A}_i$, $i = 1, \ldots, k$) have nonempty intersection.

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The hyperplanes may be transformed to boxes without changing the intersection pattern. This shows that in $\mathbf{R}^k$ for $\alpha = 0$ the best value of β is also 0. This completes the proof of Theorem C.

3. Problems and remarks. The main purpose of this section is to state two related problems in graph theory. The following result is due to Erdös [**4**]:

ERDÖS' LEMMA. *There is a* $0 < c < 1$ *such that any graph with n vertices and at least* $\left[\frac{n^2}{4}\right] + 1$ *edges contains* $c \cdot n$ *triangles with a common edge.*

Using this lemma it is simple to show that for any $\epsilon > 0$ there is a $\beta > 0$, such that any graph with n vertices and at least $\frac{1}{2}n^2(\frac{1}{2} + \epsilon)$ edges contains at least $\beta \cdot n^3$ triangles. It is natural to conjecture

CONJECTURE 1. *For* $k > 2$ *and for each* $\epsilon > 0$ *there is a* $\beta > 0$ $(\beta = \beta(k, \epsilon))$ *such that any graph with n vertices and at least* $\frac{n^2}{2}\left(\frac{k-2}{k-1} + \epsilon\right)$ *edges contains at least* $\beta \cdot n^k$ *complete k-graphs.*

If the following generalization of the Erdös lemma is true then Conjecture 1 is true.

CONJECTURE 2. *For* $k > 2$ *there is a* $0 < c = c(k)$ *such that any graph with n vertices and at least* $\frac{n^2}{2} \cdot \frac{k-2}{k-1} + 1$ *edges contains at least* $c \cdot n^{k-2}$ *complete* $k + 1$ *graphs with a common edge.*

Remarks. A 1-complete graph is one for which every subgraph has a *special vertex* (a vertex such that any two vertices which are both incident to it are joined by an edge). Lekkerkerker and Boland [**9**], in studying (and characterizing) interval graphs have shown that a graph is 1-complete if and only if it is a chord graph (a graph in which each circuit of length greater than three contains a chord). Is there a natural property of graphs which is equivalent to that of being 2-complete?

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