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On Axiomatizability of Non-Commutative L_p-Spaces

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Abstract. It is shown that Schatten p-classes of operators between Hilbert spaces of different (infinite) dimensions have ultrapowers which are (completely) isometric to non-commutative L_p-spaces. On the other hand, these Schatten classes are not themselves isomorphic to non-commutative L_p spaces. As a consequence, the class of non-commutative L_p-spaces is not axiomatizable in the first-order language developed by Henson and Iovino for normed space structures, neither in the signature of Banach spaces, nor in that of operator spaces. Other examples of the same phenomenon are presented that belong to the class of corners of non-commutative L_p-spaces. For $p = 1$ this last class, which is the same as the class of preduals of ternary rings of operators, is itself axiomatizable in the signature of operator spaces.

Introduction

When model theory is applied to a given class of mathematical structures, a natural first question is whether the class is axiomatizable by sentences from a first-order language. Often this depends on the way in which the structures are viewed; that is, the answer depends on which language (signature) is used.

In this paper we consider the first-order axiomatizability of a class of structures from functional analysis with respect to several natural choices of signature. This requires us to use a modification of first-order logic. Usually, when objects from analysis or topology are considered, this is necessary, since the definitions necessarily involve non-first-order mathematical concepts such as completeness with respect to a metric. Such adaptations of model theory to analysis go back to work done in the late 1960s and early 1970s by J.-L. Krivine and D. Dacunha-Castelle [DCK] (who introduced the use of ultraproducts in Banach space theory) and by W. Luxemburg (who introduced the more-or-less equivalent tool of nonstandard hulls). This work was pursued by C. W. Henson, L. C. Moore, J. Stern, S. Heinrich, J. Iovino, and others.

Henson [He] introduced a suitable modification of first-order logic for the structures considered in this work. Recently a systematic introduction to this logic for normed space structures was given by Henson and Iovino [HI]. The important features of this theory are the use of a special first-order language consisting of *positive bounded formulas* and the introduction of a concept of *approximate satisfaction.* Examples of the structures to which this logic applies include normed spaces, normed

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lattices, operator spaces, and the like.

Many classical results from model theory have counterparts in the theory discussed in [HI]; in particular, a well-known result characterizing axiomatizable classes of structures using ultraproducts has its analogue, which we briefly explain now. The interest of this characterization is that it only uses tools which are familiar to specialists in functional analysis, and involves no other technical aspects of formal logic.

A class $\mathcal{C}$ of normed space structures of a given kind (Banach spaces, Banach lattices, operator spaces, *etc.*) is *axiomatizable* if there exists a set Φ of positive bounded sentences from the corresponding language, such that a normed space structure belongs to $\mathcal{C}$ if and only if it approximately satisfies all the sentences φ in Φ.

A necessary and sufficient condition for a class $\mathcal{C}$ of normed space structures of a given kind to be axiomatizable is that $\mathcal{C}$ is closed under 1-isomorphisms[1] and ultraproducts, and that the complementary class is closed under ultrapowers. (See [He, HI]. Note that we consider here only classes of structures that are obviously *uniform* in the sense of [HI].)

If $\mathcal{E}, \mathcal{F}$ are two normed space structures of the same kind, we say that $\mathcal{E}$ is an *ultraroot* of $\mathcal{F}$ if $\mathcal{F}$ is 1-isomorphic to some ultrapower of $\mathcal{E}$. Then a class $\mathcal{C}$ is axiomatizable if and only if it is closed under 1-isomorphisms, ultraproducts and ultraroots.

In this note we discuss the axiomatizability of the class of non-commutative L_p-spaces. For comparison and background, we first recall the case of commutative (*i.e.,* ordinary) L_p-spaces, $1 \leq p < \infty$. Since these spaces are characterized as Banach lattices by Bohnenblust's axiom:

$$\forall x \forall y \big(|x| \wedge |y| = 0 \Longrightarrow \|x+y\|^p = \|x\|^p + \|y\|^p \big)$$

(see [LT, Theorem 1b2] or [L, Ch. 5,§15, Theorem 3]), the class of (Banach lattices isomorphic to) L_p-spaces is trivially closed under ultraproducts and substructures, hence is axiomatizable in the language of Banach lattices. (Note that Bohnenblust's axiom is *not*, by itself, a positive bounded sentence in the sense of [HI]; however, it is not too hard to find a sequence of sentences of this language which is equivalent to Bohnenblust's axiom). The situation is more difficult if we examine the class of (Banach spaces isometric to) L_p-spaces and consider its axiomatizability in the language of Banach spaces. However, the answer has been known since the 1970s to be positive in this case, too. This is due to the isometric characterization of L_p-spaces as $\mathcal{L}_{p,1+}$ spaces in the sense of Lindenstrauss and Pełczyński, and to the fact that the class of $\mathcal{L}_{p,1+}$ spaces is easily seen to be closed under ultraroots.

The classical L_p-spaces have a natural counterpart in the non-commutative setting, where the Boolean algebra of μ-measurable sets (up to μ-negligible sets) relative to some measure space (Ω, Σ, μ) is replaced by some weak-operator closed lattice of (orthogonal) projections in some Hilbert space H; equivalently the algebra $L_\infty(\Omega, \Sigma, \mu)$ is replaced by some von Neumann algebra M. The non-commutative

[1]By 1-*isomorphisms* we mean surjective linear isometries which preserve the additional structure of the given kind of normed structures (*e.g.*, lattice isomorphisms in the case of Banach lattices, completely isometric maps in the case of operator spaces, *etc.*)

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analog of the space $L_1(\Omega, \Sigma, \mu)$ (*i.e.,* the predual of $L_\infty(\Omega, \Sigma, \mu)$) is then the unique predual M_* of M. The non-commutative analog of the space $L_p(\Omega, \Sigma; \mu)$ was described in the 1950s by Dixmier when the von Neumann algebra is semi-finite, (*i.e.,* can be equipped with a normal faithful semi-finite trace τ, like the usual trace in the case $M = B(H)$), see [Di], and by various authors in the 1970s in the much harder case where M is not semi-finite (we refer to [H, T]). In fact, in the main example described in Section 3, we only use the basic example $M = B(H)$, in which case $L_1(M)$ is simply the trace class $S_1(H)$, while $L_p(M)$ is the Schatten class $S_p(H)$. These are the non-commutative analogs of the spaces ℓ_1, resp. ℓ_p. The class of non-commutative L_p-spaces is closed under ultraproducts (see [G] when $p = 1$, and [R] when $1 < p < \infty$), so it makes sense to ask if it is axiomatizable.

In this note we show that for $1 \le p < \infty$, $p \ne 2$, the class of non-commutative L_p-spaces is *not* closed under ultraroots, and hence it is not axiomatizable, whether considered as a class of Banach spaces or as a class of operator spaces. In fact, we show that for all infinite dimensional Hilbert spaces H, K, and $p \in [1, \infty)$, the Schatten classes $S_p(H, K)$, $S_p(H)$, $S_p(K)$ have 1-isomorphic ultrapowers (relative to some common ultrafilter). But if H and K are not isometric, then $S_p(H, K)$ is not isomorphic to a non-commutative L_p space (not even if general non-isometric isomorphisms are allowed). Hence such Schatten p-classes are counterexamples to the closedness under ultraroots of the class of non-commutative L_p-spaces; consequently these classes are not axiomatizable, neither in the language of Banach spaces nor in that of operator spaces.

These counterexamples are discrete in the sense that they can be described as "corners" in a non-commutative space associated with a "discrete" (type I) von Neumann algebra. In Section 4 we give other counterexamples which are non discrete, basically of the form $L_p(\mathcal{M})$ with $\mathcal{M} = B(H, K)\bar{\otimes}\mathcal{A}$, where $\mathcal{A}$ is an arbitrary σ-finite von Neumann algebra. In principle, reading this section requires knowledge of the theory of general non-commutative L_p-spaces. However, only a few features of this theory are really used in the proofs; indeed, they can easily be followed by the reader keeping the more familiar $L_p(\mathcal{M}, \tau)$-spaces in mind.

All these counterexamples are corners in non-commutative L_p-spaces. In the case $p = 1$, this class of spaces is exactly the well-known class of preduals of ternary rings of operators (TRO). Following a suggestion from Z.-J. Ruan, for which we express our appreciation, we show in Section 5 that the class of TRO preduals *is* axiomatizable (in the language of operator spaces). The question of axiomatizability of the class of corners in non-commutative L_p-spaces is left open for $p > 1$; it would be easily settled in the affirmative if one knew the analogue of the result of Ng and Ozawa [NO] (stating that the class of TRO-preduals is closed under completely contractive projections).

The counterexamples presented in Section 3 have a corresponding version in the case $p = \infty$, showing that the class of Banach spaces (resp., operator spaces) that are 1-isomorphic to C^*-algebras is not closed under ultraroots, and hence it is not axiomatizable. Indeed, we show that for all infinite dimensional Hilbert spaces H and K, the spaces of compact operators $S_\infty(H, K)$, $S_\infty(H)$, $S_\infty(K)$ have 1-isomorphic ultrapowers (relative to some common ultrafilter); but if H and K are not isometric, then $S_\infty(H, K)$ is not isomorphic to a C^*-algebra.

1 Basic Definitions of Model Theory for Normed Space Structures

For simplicity of exposition, we consider normed space structures of the following type (simpler than those considered in [HI]). Such a structure $\mathcal{E}$ consists of the following items:

(i) a normed space E over the scalar field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$;

(ii) collections of functions $(F_i)_{i\in I}$ and $(G_j)_{j\in J}$ of the form:

$$F_i \colon \mathbb{K}^{m_i} \times E^{n_i} \to \mathbb{K}, \quad G_j \colon \mathbb{K}^{m_j} \times E^{n_j} \to E,$$

each of which is uniformly continuous on every bounded subset of its domain; m_i+n_i is the arity of the function. The functions of arity 0 are the *constants*, and the others are the *operations* of $\mathcal{E}$. We write $\mathcal{E} = \{E, F_i, G_j \mid i \in I, j \in J\}$. The *signature* L of the normed space structure $\mathcal{E}$ consists of the data I, J, $(m_i, n_i)_{i\in I\cup J}$.

Certain operations and constants are required in all normed space structures: the algebraic operations on $\mathbb{K}$, the absolute value on $\mathbb{K}$, addition on E and the scalar multiplication operation of $\mathbb{K}$ on E, the norm on E; among the constants occur the additive identity of E and the rational numbers. If $\mathbb{K} = \mathbb{C}$, an additional operation is the conjugation operation on $\mathbb{C}$, and an additional constant is the number $i = \sqrt{-1}$.

Basic examples include the following:

- Normed spaces over $\mathbb{K}$: with the minimal set of functions described above.
- Normed lattices over $\mathbb{R}$: to the minimal set of functions one adds the lattice operations $\vee$ and $\wedge$ on E.
- Operator spaces: besides the minimal set of operations, the signature includes, for each n, the norm on the space $M_n(E)$ of $n\times n$ matrices with entries in E; this norm is seen as a function $E^{n^2} \to \mathbb{R} \subset \mathbb{C}$.

If $\mathcal{E}$, $\mathcal{F}$ have the same signature, an *isomorphism* T from $\mathcal{E}$ onto $\mathcal{F}$ is a bijective map $T\colon E \to F$ which preserves the functions F_i and G_j. Such an isomorphism is automatically linear and isometric; in the case of operator spaces it is completely isometric. We say that $\mathcal{E}$ is a *substructure* of $\mathcal{F}$ if $E \subset F$ and the operations of $\mathcal{F}$ extend the corresponding operations of $\mathcal{E}$.

As in ordinary model theory, the formulas of the language are written with symbols which are variables, function symbols, and logical symbols (logical connectives and quantifiers). Each variable is of scalar or vector type; the function symbols $(f_i)_{i\in I}$ and $(g_j)_{j\in J}$ associated to a given signature L formally connect arguments of real and vector types (in numbers as prescribed by the signature L) to values of real type (in the case of $(f_i)_{i\in I}$) or vector type (in the case of $(g_j)_{j\in J}$). More generally, the function symbols may also be used for connecting already constructed terms of real or vector type, to construct terms (scalar- or vector-valued) of higher complexity, via the formal counterpart of "substitution".

The building blocks of the language are the atomic formulas; these have the form $t \le r$ or $t \ge r$, where t is a real valued term and r a rational constant. The language of positive bounded formulas uses only the positive connectives $\vee$ and $\wedge$ and the "bounded quantifiers" $\forall_r$ and $\exists_r$, where $\forall_r x\, \varphi(x, y)$ means $\forall x\, (\|x\| \le r \to \varphi(x, y))$, while $\exists_r x\, \varphi(x, y)$ means $\exists x\, (\|x\| \le r \wedge \varphi(x, y))$. We have the same notion of satisfaction of a sentence φ (a formula without free variables) by a normed structure $\mathcal{E}$ (and the same notation $\mathcal{E} \models \varphi$) as in ordinary model theory, by interpreting each function symbol as the given function of the structure and the logical symbols with their usual meaning. Similarly, $E \models \varphi[a_1, \dots, a_n]$, has the usual interpretation, where $\varphi(x_1, \dots, x_n)$ is a formula with n variables and $a_1, \dots, a_n$ are elements of $\mathcal{E}$. (By *elements* of $\mathcal{E}$ are meant the elements of $\mathbb{K}$ and of E).

A new feature of the model theory presented in [HI, He] is the notion of *approximate satisfaction*. It requires the definition of the set of approximations of a given formula φ. Such an approximation is obtained by relaxing all the constraints appearing in φ; so atomic formulas of the form $t \le r$ (resp., $t \ge r$) are replaced by $t \le r'$ for some $r' > r$, (resp., $t \ge r'$ for some $r' < r$), while the bounded quantifiers $\forall_r$ (resp., $\exists_r$) are replaced by $\forall_{r'}$ with some $r' < r$ (resp., $\exists_{r'}$ with some $r' > r$). Then $\mathcal{E}$ is said to *approximately satisfy* a sentence φ (and we write $\mathcal{E} \models_{\mathcal{A}} \varphi$) if and only if $\mathcal{E} \models \varphi'$ for every approximation φ' of φ; one similarly defines $\mathcal{E} \models_{\mathcal{A}} \varphi[a_1, \dots, a_n]$.

2 A Criterion for the Existence of Isomorphic Ultrapowers

The aim of this section is to state and prove a criterion for two Banach spaces (or more sophisticated Banach space structures) to have (isometrically) isomorphic ultrapowers. Let us emphasize that this result gives a sufficient condition which is by no means necessary.

Proposition 2.1 *Let $\mathcal{F}$ be a normed space structure and $\mathcal{E}$ be a substructure of $\mathcal{F}$. Assume that for every finite system $(a_1, \dots, a_n)$ of elements of $\mathcal{E}$, every element $b \in \mathcal{F}$, and every real number $\varepsilon > 0$, there is an automorphism T of $\mathcal{F}$ and an element $c \in \mathcal{E}$ such that*

$$\|Ta_i - a_i\| < \varepsilon, \quad i = 1, \dots, n, \quad \text{and} \quad \|Tb - c\| < \varepsilon.$$

Then there is an ultrafilter $\mathcal{U}$ such that the corresponding ultrapowers $\mathcal{E}_{\mathcal{U}}$ and $\mathcal{F}_{\mathcal{U}}$ are (isometrically) isomorphic.

The proof of this result is based on two results of model theory: the first one is an adaptation by Henson and Iovino of a deep classical result by Shelah and Keisler that gives a characterization of structures with isomorphic ultrapowers; the second is the adaptation of the well known Tarski–Vaught test to the model theory of normed structures.

Say that two structures $\mathcal{E}$ and $\mathcal{F}$ are *approximately elementary equivalent* ($\mathcal{E} \equiv_{\mathcal{A}} \mathcal{F}$) if and only of they satisfy approximately the same positive bounded sentences. Note that, in particular, isomorphic structures are approximately elementary equivalent (in fact elementary equivalent, in the ordinary model-theoretic sense). Then a theorem of Henson and Iovino [HI, Theorem 10.7] states that a necessary and sufficient condition for $\mathcal{E}$ and $\mathcal{F}$ to have isomorphic ultrapowers is that they are approximately elementary equivalent.

If $\mathcal{E}$ is a substructure of $\mathcal{F}$, then $\mathcal{E}$ is an *approximate elementary substructure* of $\mathcal{F}$ (notation: $\mathcal{E} \preceq_{\mathcal{A}} \mathcal{F}$) if and only if both satisfy approximately the same *formulas* where free variables are replaced by parameters from $\mathcal{E}$. *A fortiori* they satisfy the same sentences, so they are approximately elementary equivalent, but the converse is not true. The Tarski–Vaught test is a sufficient condition for a substructure to be an approximate elementary one.

Proposition 2.2 (Tarski–Vaught test: [HI, Proposition 6.6]) *Let $\mathcal{E}$, $\mathcal{F}$ be two normed space L-structures with $\mathcal{E} \subseteq \mathcal{F}$,* i.e., *$\mathcal{E}$ is a substructure of $\mathcal{F}$. Then $\mathcal{E}$ is an approximate elementary substructure of $\mathcal{F}$ if and only if for every positive bounded L-formula $\varphi(x_1, x_2, \dots, x_n, y)$ and every approximation φ' of φ, the following holds: if $a_1, \dots, a_n$ are scalars or elements of E and b is an element of F such that $\mathcal{F} \models \varphi[a_1, \dots, a_n, b]$, then there exists an element c of E such that $\mathcal{F} \models \varphi'[a_1, \dots, a_n, c]$.*

Proof of Proposition 2.1 We verify the Tarski–Vaught test. Let $\varphi(x_1, \dots, x_n; y)$ be a positive bounded L-formula, $a_1, \dots, a_n$ be elements of E, b be an element of F such that $\mathcal{F} \models \varphi[a_1, \dots, a_n, b]$. Let $C > 0$ be a constant such that $\|b\| \leq C$ and $\|a_i\| \leq C$ for all $i = 1, \dots, n$. By the perturbation lemma [HI, Proposition 9.1], for every approximation φ' of φ there exists $\varepsilon > 0$ such that if $c_1, \dots, c_n, c \in F$ and $d_1, \dots, d_n, d \in F$ all have norm $\leq C$ and verify $\|c_i - d_i\| < \varepsilon$ (for $i = 1, \dots, n$) and $\|c - d\| < \varepsilon$ and $\mathcal{F} \models \varphi[d_1, \dots, d_n, d]$, then $\mathcal{F} \models \varphi'[c_1, \dots, c_n, c]$. By hypothesis there is an automorphism T of $\mathcal{F}$ such that $\|Ta_i - a_i\| < \varepsilon$ and $\|Tb - c\| < \varepsilon$ for some $c \in E$. Since T is an automorphism of $\mathcal{F}$, the fact that $\mathcal{F} \models \varphi[a_1, \dots, a_n, b]$ implies that $\mathcal{F} \models \varphi[Ta_1, \dots, Ta_n, Tb]$; hence $\mathcal{F} \models \varphi'[a_1, \dots, a_n, c]$. ■

The relation on normed space structures given by the existence of isomorphic ultrapowers is an equivalence relation: this fact is by no means evident from the definition of this relation, but becomes clear using the theorem of Henson and Iovino, since the relation of approximate elementary equivalence is obviously an equivalence relation. Hence if $\mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ are normed space L-structures, and $\mathcal{U}, \mathcal{V}$ are ultrafilters such that $\mathcal{E}_{\mathcal{U}}$ is isomorphic to $\mathcal{F}_{\mathcal{U}}$ and $\mathcal{F}_{\mathcal{V}}$ is isomorphic to $\mathcal{G}_{\mathcal{V}}$, there exists an ultrafilter $\mathcal{W}$ such that $\mathcal{E}_{\mathcal{W}}$ is isomorphic to $\mathcal{G}_{\mathcal{W}}$. In fact, we have the following far-reaching result (which follows from of [HI, Theorem 10.8]).

Theorem 2.3 *Let $\mathcal{C}$ be a* set *of normed space L-structures. There exists an ultrafilter $\mathcal{U}$ such that for any $\mathcal{E}, \mathcal{F} \in \mathcal{C}$ that have isomorphic ultrapowers, the ultrapowers $\mathcal{E}_{\mathcal{U}}$ and $\mathcal{F}_{\mathcal{U}}$ are isomorphic.*

3 Ultraroots of Noncommutative L_p-Spaces: A Counterexample

If H, K are Hilbert spaces, and $1 \leq p < \infty$, we denote by $S_p(H, K)$ the Schatten p-class of operators $H \to K$. An operator $x \in B(H, K)$ belongs to $S_p(H, K)$ if and only if $|x| = (x^*x)^{1/2}$ belongs to the ordinary Schatten p-class $S_p(H)$ (equivalently $|x^*| = (xx^*)^{1/2} \in S_p(K)$), and $\|x\|_{S_p(H,K)} = \big\| |x| \big\|_{S_p(H)} = \big\| |x^*| \big\|_{S_p(K)}$. For $p = \infty$, we adopt the usual convention that $S_\infty(H, K)$ is the space of compact operators from H into K.

Theorem 3.1 *Let H_1, H_2, K_1, K_2 be infinite-dimensional Hilbert spaces and $1 \leq p \leq \infty$. Then there is an ultrafilter $\mathcal{U}$ such that the ultrapowers $S_p(H_1, K_1)_{\mathcal{U}}$ and $S_p(H_2, K_2)_{\mathcal{U}}$ are isometrically isomorphic.*

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Proof *Step 1:* Assume first that $H_1 = H_2 = H$. Clearly we may suppose that the Hilbertian dimensions of K_1, K_2 satisfy $\dim K_1 \le \dim K_2$; then K_1 is isometrically embeddable into K_2, so we may assume that $K_1 \subset K_2$. Now we have a natural isometric linear embedding $S_p(H,K_1) \hookrightarrow S_p(H,K_2)$ (namely, $x \mapsto jx$ where j is the inclusion of K_1 into K_2), and we may consider that $S_p(H,K_1)$ is a subspace of $S_p(H,K_2)$. We proceed now to verify that the hypotheses of Proposition 2.1 are fulfilled.

Let $a_1,\dots,a_n$ be an n-tuple in $S_p(H,K_1)$, b an element of $S_p(H,K_2)$ and $\varepsilon > 0$. There exist finite rank operators $a'_1,\dots,a'_n \in S_p(H,K_1)$ and $b' \in S_p(H,K_2)$ such that $\|a'_i - a_i\| < \varepsilon$, $i = 1,\dots,n$ and $\|b' - b\| < \varepsilon$. Let $F = \operatorname{span}\{a'_1,\dots,a'_n,b'\}$. Let $L = \sum_i R(a'_i)$ (where $R(a'_i)$ denotes the range of the operator a'_i), $M = L + R(b')$ and $N = M \ominus L$. Let $N' \subset K_1$ such that $N' \perp L$ and $\dim N' = \dim N$. Let $G = K_2 \ominus M$ and $G' = K_2 \ominus (L \oplus N')$. We have $K_2 = L \oplus N \oplus G = L \oplus N' \oplus G'$. Note that G and G' have the same Hilbertian dimension (that of K_2). Hence there is a unitary u of K_2 such that $u|_L = \mathrm{id}\,|_L$, $u(N) = N'$ and $u(G) = G'$.

Let $T = L_u$ be the left composition operator on $S_p(H,K_2)$ associated with u (that is $T(a) = ua$ for every $a \in S_p(H,K_2)$. Then T is a surjective isometry of $S_p(H,K_2)$, $T(a'_i) = a'_i$, $i = 1,\dots,m$ and $c = T(b') \in S_p(H,K_1)$. Since $\|T(b) - T(b')\| = \|b - b'\| < \varepsilon$, the hypotheses of Proposition 2.1 are verified (taking $c = T(b')$).

Step 2: Assume now that $K_1 = K_2 = K$, while H_1, and H_2 may be different. Now we may suppose that $H_1 \subset H_2$. We have an isometric embedding $S_p(H_1,K) \hookrightarrow S_p(H_2,K)$ defined by $x \mapsto x\pi$, where π is the orthogonal projection from H_2 onto H_1. Given operators $a_1,\dots,a_n \in S_p(H_1,K)$, $b \in S_p(H_2,K)$, and $\varepsilon > 0$, we apply the construction of Step 1 above to the adjoint operators $a_1^*,\dots,a_n^* \in S_p(K,H_1)$ and $b^* \in S_p(K,H_2)$; this yields a unitary u of H_2 and an operator $c \in S_p(K,H_1)$ such that $\|ua_j^* - a_j^*\| < \varepsilon$, $j = 1,\dots,n$ and $\|ub^* - c\| < \varepsilon$. Then $c^* \in S_p(H_1,K)$ (note that c, as an element of $S_p(K,H_2)$, equals jc_0, where $c_0 \in S_p(K,H_1)$ and j is the inclusion map from H_1 into H_2; hence $c^* = c_0^* j^* = c_0^*\pi$ is indeed in the canonical image of $S_p(H_1,K)$ in $S_p(H_2,K)$). Moreover $\|a_j u^* - a_j\| < \varepsilon$, $j = 1,\dots,n$ and $\|bu^* - c^*\| < \varepsilon$. Finally $T = R_{u^*}\colon a \mapsto au^*$ defines a suitable automorphism of $S_p(H_2,K)$ (for obtaining the hypotheses of Proposition 2.1 in this case).

Step 3: For the general case, let H_1, H_2, K_1, K_2 as in the assumptions of Theorem 3.1. By Step 1, $S_p(H_1,K_1)$ and $S_p(H_1,K_2)$ have (isometrically) isomorphic ultrapowers; and by Step 2, $S_p(H_1,K_2)$ and $S_p(H_2,K_2)$ have isomorphic ultrapowers, too. Hence by transitivity of the relation "to have isomorphic ultrapowers" (see §2), so do $S_p(H_1,K_1)$ and $S_p(H_2,K_2)$. ■

Remark 3.2 When the Schatten classes are equipped with their usual operator space structures obtained by complex interpolation [Pi], it is immediate that the operator T constructed above is completely isometric. Hence the Schatten spaces considered in Theorem 3.1 have in fact (for some ultrafilter) completely isometric ultrapowers.

Corollary 3.3 *If H and K are infinite-dimensional Hilbert spaces, there exists an ultrafilter $\mathcal{U}$ such that for every $1 \le p \le \infty$ the ultrapower $S_p(H,K)_{\mathcal{U}}$ is (completely iso-*

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metrically) isomorphic to $S_p(H)_{\mathcal{U}}$ and to $S_p(K)_{\mathcal{U}}$, hence to a non commutative L_p-space if $p < \infty$, resp., to a C^*-algebra if $p = \infty$.

Proof This is a consequence of Theorem 3.1 and Theorem 2.3. For the last statement see [R] in the case $p < \infty$. ■

Exceptionally in the following statement, the isomorphisms are not required to be isometric. That is, in this result *isomorphism* means *bijective bounded linear map with bounded inverse.*

Proposition 3.4 *Let $1 \leq p < \infty$, $p \neq 2$ and H, K be infinite dimensional Hilbert spaces, with $\dim H < \dim K$. Then $S_p(H, K)$ is not isomorphic as a Banach space to a non-commutative L_p-space associated with a von Neumann algebra. Similarly $S_\infty(H, K)$ is not isomorphic as a Banach space to a C^*-algebra.*

Proof Suppose otherwise. Let $\mathcal{M}$ be a von Neumann algebra such that the non-commutative L_p-space $L_p(\mathcal{M})$ is isomorphic as a Banach space to $S_p(H, K)$. By duality we may assume that $1 \leq p < 2$. Note that $L_p(\mathcal{M})$ contains isometrically the Lebesgue space $L_p([0, 1])$, unless $\mathcal{M}$ is a type I von Neumann algebra with atomic center. (If $\mathcal{M}$ has a type II or type III part, then $L_p(\mathcal{M})$ contains a subspace isometric to $L_p(\mathcal{R})$, where $\mathcal{R}$ is the hyperfinite II_1-factor, see [M]; it is well known that $L_p(\mathcal{R})$ contains a subspace isometric to $L_p([0, 1])$. On the other hand, if $\mathcal{M}$ has type I, it is immediate that $L_p(\mathcal{M})$ contains $L_p(\mathcal{Z})$, where $\mathcal{Z}$ is the center of $\mathcal{M}$). But $L_p([0, 1])$ contains isometric copies of the spaces ℓ_r, $p < r < 2$, while $S_p(H, K)$ does not contain these Banach spaces isomorphically. (In fact every infinite dimensional subspace of $S_p(H, K)$ contains ℓ_p or ℓ_2 isomorphically, see [AL, Theorem 1].) Hence if $S_p(H, K)$ is (Banach) isomorphic to $L_p(\mathcal{M})$, then $\mathcal{M}$ is a type I von Neumann algebra with atomic center. In other words $\mathcal{M} = \left(\bigoplus_{i\in I} B(H_i)\right)_{\ell_\infty}$, where the H_i are Hilbert spaces and consequently $L_p(\mathcal{M}) = \left(\bigoplus_{i\in I} S_p(H_i)\right)_{\ell_p}$.

If $\dim H_i \leq \dim H$ for all $i \in I$ and $\#I \leq \dim H$, then the density character of $L_p(\mathcal{M})$ is at most $(\dim H)^2 = \dim H$, while the density character of $S_p(H, K)$ equals $\dim K$ (since $S_p(H, K)$ contains K isometrically); this is a contradiction. Hence either $\#I > \dim H$ or one of the H_i has Hilbertian dimension strictly greater than $\dim H$. In both cases, $L_p(\mathcal{M})$ contains a subspace isometric to a space $\ell_p(\Gamma)$, where Γ is an index set of cardinality $\#\Gamma > \dim H$ (recall that for every Hilbert space L, $S_p(L)$ contains a subspace isometric to $\ell_p(\dim L)$); consequently $S_p(H, K)$ contains isomorphically $\ell_p(\Gamma)$.

Let $(x_\gamma)_{\gamma\in\Gamma}$ be a Γ-indexed isomorphic ℓ_p-basis in $S_p(H, K)$. We may assume that

$$\Bigl\| \sum_\gamma \lambda_\gamma x_\gamma \Bigr\| \geq \Bigl(\sum_\gamma |\lambda_\gamma|^p \Bigr)^{1/p}$$

for every finitely supported family $(\lambda_\gamma)_{\gamma\in\Gamma}$ of complex numbers. Let $(e_j)_{j\in J}$ be an orthonormal basis of H, and for every $F \subset J$ let p_F be the orthogonal projection onto

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$\overline{\text{span}}[e_j \mid j \in F]$. Let $0 < \alpha < 1$. For every $\gamma \in \Gamma$ there exists a finite subset F_γ of J such that

$$\|x_\gamma p_{F_\gamma}^\perp\| < \alpha.$$

Since $\#\Gamma > \#J = \#\mathcal{F}(J)$ (the set of finite subsets of J), there is $F_0 \in \mathcal{F}(J)$ for which the inequality

$$\|x_\gamma p_{F_0}^\perp\| < \alpha$$

is valid for every γ in an infinite subset Γ' of Γ. Since $S_p(H, K)$ has Rademacher type p (see [LT] for a definition), we have for every finitely supported system $(\lambda_\gamma)_{\gamma\in\Gamma'}$ of complex numbers:

$$\mathbb{E}_\varepsilon \Big\| \sum_\gamma \varepsilon_\gamma \lambda_\gamma x_\gamma p_{F_0}^\perp \Big\|^p \le C^p \sum_\gamma |\lambda_\gamma|^p \|x_\gamma p_{F_0}^\perp\|^p \le C^p \alpha^p \sum_\gamma |\lambda_\gamma|^p,$$

where C is the type p constant of $S_p(H, K)$. (In fact $C = 1$, as can be shown using complex interpolation between the cases $p = 1$ and $p = 2$.)

Consequently we have

$$\begin{aligned}\Big(\mathbb{E}_\varepsilon \Big\| \sum_\gamma \varepsilon_\gamma \lambda_\gamma x_\gamma p_{F_0} \Big\|^p\Big)^{1/p} &\ge \Big(\mathbb{E}_\varepsilon \Big\| \sum_{\gamma\in\Gamma'} \varepsilon_\gamma \lambda_\gamma x_\gamma \Big\|^p\Big)^{1/p} \\ &\quad - \Big(\mathbb{E}_\varepsilon \Big\| \sum_{\gamma\in\Gamma'} \varepsilon_\gamma \lambda_\gamma x_\gamma p_{F_0}^\perp \Big\|^p\Big)^{1/p} \\ &\ge (1-\alpha)\Big(\sum_\gamma |\lambda_\gamma|^p\Big)^{1/p}.\end{aligned}$$

However the space $\overline{\text{span}}[x_\gamma p_{F_0} \mid \gamma \in \Gamma']$ is a subspace of $S_p(H_0, K)$, where $H_0 = R(p_{F_0})$; since H_0 is a finite dimensional Hilbert space, the Schatten p-class $S_p(H_0, K)$ is isomorphic to a Hilbert space, hence has type 2, *i.e.*

$$\Big(\mathbb{E}_\varepsilon \Big\| \sum_\gamma \varepsilon_\gamma \lambda_\gamma x_\gamma p_{F_0} \Big\|^p\Big)^{1/p} \le C\Big(\sum_\gamma |\lambda_\gamma|^2 \|x_\gamma p_{F_0}\|^2\Big)^{1/2} \le CM\Big(\sum_\gamma |\lambda_\gamma|^2\Big)^{1/2},$$

where $M = \sup_\gamma \|x_\gamma\| < \infty$. This clearly provides a contradiction. ■

Remark 3.5 Note that only the isometric version of Proposition 3.4 is needed to prove our non-axiomatizability results, and it has a somewhat simpler proof. In particular in the isometric setting, the fact that the algebra $\mathcal{M}$ is necessarily of type I is an immediate consequence of Marcolino's result [M] stating that only type I algebras have associated L_p-spaces which are stable in the Krivine–Maurey sense. On the other hand, by the Clarkson inequality [MC], a subspace of a Schatten p-class which is isometric to an $\ell_p(\Gamma)$ space is generated by a basis (x_γ) consisting of pairwise disjoint elements, *i.e.*, $x_\gamma = p_\gamma x_\gamma q_\gamma$ where (p_γ) (resp., (q_γ)) is a system of pairwise disjoint projections; consequently $\#\Gamma \le \min(\dim H, \dim K)$, which yields the needed contradiction.

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Remark 3.6 If H, K are Hilbert spaces with different Hilbertian dimensions, then the space $B(H, K)$ of bounded operators from H to K is not linearly isometric to a C^*-algebra: this follows by duality from Proposition 3.4 and the fact that the predual of a von Neumann algebra is unique (up to linear isometry); see also the proof of the case $p = 1$ of Proposition 4.2. However, it is unknown to the authors if some ultrapower of $B(H, K)$ is isomorphic to a C^*-algebra.

4 Ultraroots of Noncommutative L_p-Spaces: Non-Discrete Counterexamples

The counterexample of Section 3 is discrete in the sense that it has the form $pL_p(\mathcal{N})q$ where $\mathcal{N}$ is a discrete (type I) von Neumann algebra, and p, q are projections in $\mathcal{N}$. We show here how to obtain non-discrete counterexamples.

Recall that two projections p, q in a von Neumann algebra $\mathcal{A}$ are called *equivalent* if there is a partial isometry u in $\mathcal{A}$ such that $uu^* = q$, $u^*u = p$. A projection p is said to be *properly infinite* if there exists an infinite family $(p_i)_{i\in I}$ of pairwise disjoint and equivalent projections such that $p = \sum_{i\in I} p_i$. The *central support* $c(p)$ of a projection p is the least central projection r in $\mathcal{A}$ such that $r \geq p$. We have also $c(p) = \bigvee\{upu^* \mid u \in \mathcal{A} \text{ unitary}\}$. A projection p is called *σ-finite* if every decomposition $p = \sum_i p_i$ of p into pairwise disjoint non-zero projections is at most countable. If $h \in L_p(\mathcal{A})$, we denote by $\ell(h)$ (resp., $r(h)$) its left support (resp., right support), *i.e.*, the least projection e in $\mathcal{A}$ such that $eh = h$ (resp., $he = h$). The left and right supports of an element h of $L_p(\mathcal{A})$ $(1 \leq p < \infty)$ are always σ-finite.

A *corner* in a non-commutative space $L_p(\mathcal{A})$ is a subspace of the form $\mathcal{S} = eL_p(\mathcal{A})f$, where e, f are projections in $\mathcal{A}$. The left support $\ell(\mathcal{S})$ (resp., right support $r(\mathcal{S})$) of a corner $\mathcal{S}$ is the least projection e (resp., f) such that $\mathcal{S} = e\mathcal{S}$ (resp., $\mathcal{S} = \mathcal{S}f$): then $\mathcal{S} = \ell(\mathcal{S})L_p(\mathcal{A})r(S)$. Note that $\ell(\mathcal{S})$ and $r(\mathcal{S})$ have the same central support, which we denote by $c(\mathcal{S})$ (because $eL_p(\mathcal{A})f = (0)$ if (and only if) $c(e) \perp c(f)$).

Proposition 4.1 *Let $\mathcal{A}$ be a von Neumann algebra and e, f be properly infinite projections in $\mathcal{A}$, with central support I. Let $1 \leq p < \infty$. Then there is an ultrafilter $\mathcal{U}$ such that $(eL_p(\mathcal{A})f)_\mathcal{U}$ and $L_p(\mathcal{A})_\mathcal{U}$ are isometric (in fact, completely isometric).*

Proof The proof follows the pattern of the proof of Theorem 3.1. We prove that $eL_p(\mathcal{A})f$ and $L_p(\mathcal{A})f$ have isomorphic ultrapowers and leave the rest of the proof to the reader. We use the following facts.

(i) If $(e_i)_{i\in I}$ is a family of pairwise disjoint projections in $\mathcal{A}$ with $\sum_{i\in I} e_i = I$, then for every $h \in L_p(\mathcal{A})$ and $\varepsilon > 0$ there exists a finite subset $F \subset I$ such that $\|e_F^\perp h\| < \varepsilon$, where $e_F = \sum_{i\in F} e_i$. (If not, one could find $\varepsilon > 0$ and a sequence (F_n) of mutually disjoint finite subsets of I such that $\|e_{F_n}h\| \geq \varepsilon$ for all n. However since the e_{F_n} are disjoint, it is a standard fact that

$$\|\sum_n e_{F_n}h\| \geq \Big(\sum_n \|e_{F_n}h\|^q\Big)^{1/q},$$

where $q = p \vee 2$, which yields a contradiction.)

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(ii) If e has central support I and is properly infinite, then every σ-finite projection π in $\mathcal{A}$ is equivalent to a projection $\pi' \le e$ (see [Di, III, 8, Corollary 5]).

Let $a_1, \dots, a_n \in eL_p(\mathcal{A})f$ and $b \in L_p(\mathcal{A})f$, and $\varepsilon > 0$. Write $e = \sum_{i \in I} e_i$ where (e_i) is an infinite family of pairwise disjoint and equivalent projections of $\mathcal{A}$. By fact (i), there is a finite subset $F \subset I$ such that $\|a_i - e_F a_i\| < \varepsilon$, $i = 1, \dots, n$ and $\|eb - e_F b\| < \varepsilon$. Set $G = I \setminus F$. By fact (i), the left support projection $\ell(e^\perp b)$ is σ-finite, and by fact (ii) it is equivalent to a subprojection e' of e_G. Let u be a partial isometry in $\mathcal{A}$ with $uu^* = e'$ and $u^*u = \ell(e^\perp b)$. Set $w = u + u^* + (e' + \ell(e^\perp b))^\perp$. Then w is a unitary of $\mathcal{A}$ such that $w\ell(e^\perp b) = e'$, $we' = \ell(e^\perp b)$ and $we_F = e_F$. Then we have

$$\|wa_i - a_i\| \le \|wa_i - we_F a_i\| + \|e_F a_i - a_i\| = 2\|a_i - e_F a_i\| < 2\varepsilon$$

and similarly $\|web - eb\| < 2\varepsilon$. Setting $c = eb + ue^\perp b$ we have $c \in eL_p(\mathcal{A})f$ and $\|wb - c\| < 2\varepsilon$. ■

Proposition 4.2 *Let $1 \le p < \infty$, $p \ne 2$; let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{S}$ a corner in $L_p(\mathcal{A})$ with left and right supports e, f. If $\mathcal{S}$ is isometric to a non-commutative L_p-space associated with a von Neumann algebra, then the reduced von Neumann algebras $e\mathcal{A}e$ and $f\mathcal{A}f$ are $*$-isomorphic.*

Proof We examine separately the cases $p = 1$ and $p > 1$.

Case $p = 1$. This case is probably well known (Z.-J. Ruan pointed out to us a similar argument in the operator space setting; see [Ru, §6]). If $T\colon \mathcal{S} \to L_1(\mathcal{N})$ is a surjective isometry, where $\mathcal{N}$ is a von Neumann algebra, then by duality $T'\colon \mathcal{N} \to \mathcal{S}' = f\mathcal{A}e$ is a surjective isometry. Note that under any $*$-isomorphisms of $\mathcal{N}$ and $\mathcal{A}$ with some C^*-subalgebras of some $B(H)$, the spaces $\mathcal{N}$ and $\mathcal{S}'$ both appear as TRO's, *i.e.,* subspaces of $B(H)$ closed under the triple product $\{x, y, z\} = xy^*z$. By a theorem of Harris [Ha], any surjective isometry between TRO's preserves the symmetrized triple product. Hence:

$$(1) \qquad T'(xy^*z + zy^*x) = (T'x)(T'y)^*(T'z) + (T'z)(T'y)^*(T'x) \quad \forall x, y, z \in \mathcal{S}^*.$$

Let $u = T'1$ be the image of the identity of $\mathcal{N}$. We then have $u = uu^*u$ (taking $x = y = z = 1$ in equation (1)). Hence u is a partial isometry in $\mathcal{A}$ with left projection $p = uu^*$ and right projection $q = u^*u$. Clearly $p \le f$ and $q \le e$. Moreover, for every $x \in \mathcal{N}$ we have (taking $y = z = 1$ in equation (1))

$$2T'x = (T'x)q + p(T'x).$$

Since T' is surjective, this means that

$$\forall a \in \mathcal{S}',\ a = (aq + pa)/2.$$

In particular $\|aq^\perp\| = \|paq^\perp\|/2 \le \|aq^\perp\|/2$, hence $aq^\perp = 0$, *i.e.,* $a = aq$ and also $a = pa$. Since this is true for every $a \in \mathcal{S}'$ we have $p = f$ and $q = e$. Consequently

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e and f are equivalent projections in $\mathcal{A}$ ($e = u^*u$, $f = uu^*$) and a $*$-isomorphism $\pi\colon e\mathcal{A}e \to f\mathcal{A}f$ can be defined by

$$\pi(a) = uau^* \ \forall a \in \mathcal{A}.$$

Case $p > 1$. We may localize to $\mathcal{S}$ the main argument of the paper [S], analyzing surjective isometries between two non-commutative L_p-spaces.

First let us introduce a few definitions. Among the sub-corners of a corner $\mathcal{C}$ are the *columns* $\mathcal{C}q$ (where q is a subprojection of $r(\mathcal{C})$), the *rows* $q\mathcal{C}$, (where q is a subprojection of $\ell(\mathcal{C})$), and the *central sections* $z\mathcal{C}$, where z is a central projection in $\mathcal{A}$: a row which is also a column is in fact a central section. For further use note that a column (resp., a row, resp., a central section) in a corner $\mathcal{C}$ can be written uniquely as $\mathcal{C}q$, resp., $p\mathcal{C}$, resp., $z\mathcal{C}$, where q is a subprojection of $r(\mathcal{C})$ (resp., p is a subprojection of $\ell(\mathcal{C})$, z is a central subprojection of $c(\mathcal{C})$).

Two results of [S] can be transposed immediately in the present context. The first one states that the image of a central section of a corner $\mathcal{S}_1$ by a surjective isometry onto another corner $\mathcal{S}_2$ is a central section of $\mathcal{S}_2$. The second one provides a determination of the images of columns (resp., rows) under surjective isometries. It states that such an image is the sum of a row and a column which are centrally disjoint. So if $T\colon \mathcal{S}_1 \to S_2$ is a surjective isometry between two corners in non-commutative L_p-spaces $L_p(\mathcal{A}_1)$, resp., $L_p(\mathcal{A}_2)$, then for every projection $q \leq r(\mathcal{S}_1)$ there exist a central projection z of $\mathcal{A}_2$ and projections $q_r \leq r(\mathcal{S}_2)$, $q_\ell \leq \ell(\mathcal{S}_2)$ such that

$$T(\mathcal{S}_1 q) = z\mathcal{S}_2 q_r + z^\perp q_\ell \mathcal{S}_2. \tag{2}$$

Moreover, as is shown in [S], the central projection z does not depend on q when $\mathcal{S}_1 q$ has no abelian central section, and this choice of z works also for a general $q \leq r(\mathcal{S}_1)$.

The argument of [S] is based on the preservation of two kinds of orthogonality for pairs of elements by isometries: the first one is defined as the orthogonality of left, resp., right, supports:

$$h \perp k \iff \ell(h) \perp \ell(k) \text{ and } r(h) \perp r(k).$$

This orthogonality has a purely metric formulation in L_p for $p \neq 2, \infty$ (the equality case in Clarkson's inequality, see [RX]) and is thus preserved by any isometry between two subspaces of non-commutative L_p-spaces. The second type of orthogonality used in [S] is related to Lumer's concept of semi-inner product. Recall that if X is a smooth Banach space, then for every non zero element $x \in X$ there is a unique functional $Jx \in X'$ such that $\|Jx\| = \|x\|$ and $\langle x, Jx\rangle = \|x\|^2$. Then Lumer's semi-inner product is defined by $[x, y] = \langle x, Jy\rangle$ and Lumer's semi-orthogonality by

$$x \top y \iff [x, y] = 0.$$

These concepts are preserved under isometries; this applies to subspaces of non-commutative L_p-spaces, provided $p \neq 1, \infty$.

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Now we adapt [S, Lemma 4.5] to the present context. Let $T\colon L_p(\mathcal{N}) \to \mathcal{S} = eL_p(\mathcal{A})f$ be a surjective isometry, and let $z \leq c(\mathcal{S})$ be a central projection in $\mathcal{A}$ verifying (2) (with $\mathcal{S}_1 = L_p(\mathcal{N})$ and $\mathcal{S}_2 = \mathcal{S}$). Let $\rho \in \mathcal{Z}(\mathcal{N})$ be a central projection in $\mathcal{N}$ such that $T^{-1}(z\mathcal{S}) = \rho L_p(\mathcal{N})$. Then we have

$$T(\rho L_p(\mathcal{N})q) = z\mathcal{S}q_r; \quad T(\rho^\perp L_p(\mathcal{N})q) = z^\perp q_\ell \mathcal{S}. \tag{3}$$

That is, T maps columns of $\rho L_p(\mathcal{N})$ to columns of $z\mathcal{S}$ and columns of $\rho^\perp L_p(\mathcal{N})$ to rows of $z^\perp\mathcal{S}$. Similarly there are central projections $\rho' \in \mathcal{N}$, $z' \in c(\mathcal{S})\mathcal{A}$ such that T^{-1} maps columns of $z'\mathcal{S}$ to columns of $\rho' L_p(\mathcal{N})$ and columns of $z'^\perp\mathcal{S}$ to rows of $\rho'^\perp L_p(\mathcal{N})$. Consequently, every column of $\rho\rho'^\perp L_p(\mathcal{N})$ is also a row, *i.e.,* is a central section of $\rho\rho'^\perp L_p(\mathcal{N})$; hence $\rho\rho'^\perp\mathcal{N}$ is commutative. Assume for the moment that $\mathcal{N}$ has no commutative central section. Then $\rho\rho'^\perp = 0$, *i.e.,* $\rho \subset \rho'$ and T^{-1} maps columns of $z\mathcal{S}$ to columns of $\rho L_p(\mathcal{N})$. Then necessarily T maps rows of $\rho L_p(\mathcal{N})$ to rows of $z\mathcal{S}$ (if not, a row of $\rho L_p(\mathcal{N})$ would be a column, *i.e.*, a central section, contradicting the hypothesis that $\mathcal{N}$ has no abelian summand). Similarly, using central projections $\rho'' \in \mathcal{N}$, $z'' \in c(\mathcal{S}) \cdot \mathcal{A}$ such that T^{-1} maps rows of $z''\mathcal{S}$ to rows of $\rho'' L_p(\mathcal{N})$ and rows of $z'^\perp\mathcal{S}$ to columns of $\rho'^\perp L_p(\mathcal{N})$, one sees that T^{-1} maps rows of $z^\perp\mathcal{S}$ to columns of $\rho^\perp L_p(\mathcal{N})$, and T maps rows of $\rho^\perp L_p(\mathcal{N})$ to columns of $z^\perp\mathcal{S}$.

Now observe that T maps rows of $\rho^\perp\rho' L_p(\mathcal{N})$ to columns of $z^\perp z'\mathcal{S}$ while T^{-1} maps columns of $z^\perp z'\mathcal{S}$ to columns of $\rho^\perp\rho' L_p(\mathcal{N})$; hence every row of $\rho^\perp\rho' L_p(\mathcal{N})$ is a column, and $\rho^\perp\rho' = 0$. Thus $\rho = \rho'$, $z = z'$, and similarly $\rho^\perp = \rho''$, $z^\perp = z''$. Finally, T and T^{-1} exchange the columns (resp., the rows) of $\rho L_p(\mathcal{N})$ with the columns (resp., the rows) of $z\mathcal{S}$, and the columns (resp., the rows) of $\rho^\perp L_p(\mathcal{N})$ with the rows (resp., the columns) of $z^\perp\mathcal{S}$.

The relation (3) defines one-to-one maps $\pi_r\colon q \mapsto q_r$ from $\mathcal{P}(\rho\mathcal{N})$ onto the set of projections of $\mathcal{A}$ which are dominated by $z \cdot f$ (*i.e.*, $\mathcal{P}(zf\mathcal{A}f)$) and $\pi_\ell\colon q \mapsto q_\ell$, from $\mathcal{P}(\rho^\perp\mathcal{N})$ onto $\mathcal{P}(z^\perp f\mathcal{A}f)$. It is shown in [S] how to extend π_r to a $*$-isomorphism from $\rho\mathcal{N}$ onto $zf\mathcal{A}f$, and π_ℓ to a $*$-anti-isomorphism from $\rho^\perp\mathcal{N}$ onto $z^\perp f\mathcal{A}f$. Similarly, considering the action of T on the rows of $\mathcal{S}$, one obtains a $*$-isomorphism π'_ℓ from $\rho\mathcal{N}$ onto $ze\mathcal{A}e$ and a $*$-anti-isomorphism π'_r from $\rho^\perp\mathcal{N}$ onto $z^\perp e\mathcal{A}e$. The compositions $\pi_r\pi_\ell'^{-1}$ and $\pi_\ell\pi_r'^{-1}$ are $*$-isomorphisms and their direct sum is the desired $*$-isomorphism from $e\mathcal{A}e$ onto $f\mathcal{A}f$.

In the case where $\mathcal{N}$ has a non trivial commutative central section $\rho_c\mathcal{N}$, then $T(\rho_c L_p(\mathcal{N}))$ is a central section $z_c\mathcal{S}$ in which all the rows and columns are central sections. It is not hard to see that $z_c e\mathcal{A}e$ and $z_c f\mathcal{A}f$ are then both abelian and that the one-to-one correspondance induced by T on their central sections extends to $*$-isomorphisms between them and $\rho_c\mathcal{N}$. ■

Definition 4.3 If κ is a cardinal number, say that a projection e is *κ-decomposable* if $e = \sum_{i\in I} e_i$ for some family $(e_i)_{i\in I}$ of σ-finite and mutually orthogonal projections in $\mathcal{A}$, where I has cardinality less or equal to κ.

Corollary 4.4 *Let $1 \leq p < \infty$, $p \neq 2$, and $\mathcal{A}$ be a von Neumann algebra and $\mathcal{S} := eL_p(\mathcal{A})f$ be a corner with supports $e = r(\mathcal{S})$, $f = \ell(\mathcal{S})$. Assume that e is κ-decomposable, while f is not. Then $\mathcal{S}$ is not linearly isometric to a non-commutative L_p-space associated with a von Neumann algebra.*

Proof In this case, $e\mathcal{A}e$ and $f\mathcal{A}f$ are clearly not $*$-isomorphic. ■

Example 4.5 Let $\mathcal{N}$ be a von Neumann algebra. Given two Hilbert spaces H, K, let $\mathcal{S} = B(H,K)\bar{\otimes}\mathcal{N}$. If $\mathcal{H}$ is a Hilbert space on which $\mathcal{N}$ is represented (as a concrete von Neumann algebra of operators in $\mathcal{H}$) and $H\otimes\mathcal{H}$, $K\otimes\mathcal{H}$ are the usual Hilbertian tensor products, then $\mathcal{S}$ identifies with the weak-operator closed subspace of $B(H\otimes\mathcal{H}, K\otimes\mathcal{H})$ generated by the operators $a\otimes x$ with $x \in B(H,K)$, $a \in \mathcal{N}$. Alternatively, if $(e_i)_{i\in I}$ and $(f_j)_{j\in J}$ are orthogonal bases in H, resp., K, then the elements of $\mathcal{S}$ can be represented as (certain) infinite matrices $(a_{i,j})$ with entries in $\mathcal{N}$. If $L = H \oplus K$, P_H, P_K are the orthogonal projections from L onto H, resp., K and $\mathcal{A} = B(L)\bar{\otimes}\mathcal{N}$, then $\mathcal{S} = e\mathcal{A}f$ where $e = P_H \otimes I_{\mathcal{N}}$, $f = P_K \otimes I_{\mathcal{N}}$. Then $\mathcal{A}, e, f$ satisfy the hypotheses of Proposition 4.1. The hypotheses of Corollary 4.4 are satisfied if for some infinite cardinal κ, $\mathcal{N}$ is κ-decomposable, $\dim H \le \kappa$ and $\dim K > \kappa$.

5 Ultraroots of TRO-Preduals: Operator Space Version

Recall that a *ternary ring of operators* (TRO) is a subspace of some $B(H)$ space which is closed under the triple product operation $\{x, y, z\} = xy^*z$. An abstract characterization of these spaces was given by Zettl [Z]. We refer to the litterature cited in the introduction of [Ru] for more information. A W^*-TRO is a TRO which is a dual Banach space. By [EOR], every W^*-TRO can be represented as a corner $X = e\mathcal{A}e^{\perp}$ in a von Neumann algebra $\mathcal{A}$ (e is a projection in $\mathcal{A}$) and has a unique predual (which identifies with $e^{\perp}\mathcal{A}_*e$ under the duality $\langle\mathcal{A}, \mathcal{A}_*\rangle$). This point can be stated slightly more precisely: if E is a Banach space, the dual of which is linearly isometric to X, then the canonical images of E and of $X_* = e^{\perp}\mathcal{A}_*e$ in X^* coincide *as sets*; this is a consequence of the analogous statement for von Neumann algebras (known as Sakai's theorem) and the proof of [EOR, Theorem 2.1]. Hence the conjugate isometry of any linear isometry from X onto E^* induces a map from the canonical image of E in its bidual onto that of X_*. Consequently, if E is an operator space, the dual of which is completely isometric to X, then E is completely isometric to X_*. Such corners in non-commutative L_1-spaces form exactly the class of completely contractively complemented subspaces in non-commutative L_1-spaces [NO].

Proposition 5.1 *The class of TRO-preduals is closed under ultraproducts and ultraroots in the operator space category. In other words, it is axiomatizable in the language of operator spaces.*

Proof *Step 1:* Let $\mathcal{S} = \prod_{\mathcal{U}} \mathcal{T}_{i*}$ be an ultraproduct of TRO-preduals. Each $\mathcal{T}_{i*}$ is (completely isometrically) identified with a corner $p_i\mathcal{A}_{i*}q_i$ in the predual of a von Neumann algebra $\mathcal{A}_i$. Recall that $\prod_{\mathcal{U}} \mathcal{A}_{i*}$ can be identified completely isometrically with the predual of a von Neumann algebra $\mathcal{M}$ which contains the ultraproduct $\prod_{\mathcal{U}} \mathcal{A}_i$ as sub-C^*-algebra. In particular, the families of projections (p_i) and (q_i) define projections $\tilde{p}$ and $\tilde{q}$ in $\prod_{\mathcal{U}} \mathcal{A}_i$, hence in $\mathcal{M}$, and $\mathcal{S} = \tilde{p}\mathcal{M}_*\tilde{q}$ is a corner in a non-commutative L_1-space, *i.e.*, a TRO predual.

Step 2: Let E be an operator space with an ultrapower $E_{\mathcal{U}}$ which is completely isometric to the predual of a TRO V. Let $i\colon E \to E_{\mathcal{U}} = V_*$ be the diagonal embedding $x \mapsto \hat{x} = (x)^{\bullet}_{i\in I}$, and $w\colon E_{\mathcal{U}} \to E^{**}$ be the weak*-limit operator defined by

$$w(\tilde{x}) = w^*\text{-}\lim_{i,\mathcal{U}} x_i \text{ if } \tilde{x} = (x_i)^{\bullet}.$$

Then w is a complete contraction. Let $j_E\colon E \to E^{**}$ be the natural (completely isometric) embedding; then $wi = j_E$. Dualizing, we obtain complete contractions $i^*\colon V \to E^*$ and $w^*\colon E^{***} \to V$ such that $i^*w^* = j_E^*$. The map $j_E^*\colon E^{***} \to E^*$ is the canonical projection (the restriction map):

$$\begin{array}{ccc} E & \xrightarrow{\ i\ } & E_{\mathcal{U}} = V_* \\ {\scriptstyle \mathrm{id}_E}\downarrow & & \downarrow{\scriptstyle w} \\ E & \xrightarrow{\ j_E\ } & E^{**} \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\ i^*\ } & E^* \\ {\scriptstyle w^*}\uparrow & & \uparrow{\scriptstyle \mathrm{id}_{E^*}} \\ E^{***} & \underset{j_{E^*}}{\overset{j_E^*}{\rightleftarrows}} & E^* \end{array}$$

Consequently we have $i^*w^*j_{E^*} = j_E^*j_{E^*} = \mathrm{id}_{E^*}$. Hence $w^*j_{E^*}$ is a complete isometry from E^* onto a linear subspace F of V which is 1-completely complemented by the projection $w^*j_{E^*}i^*$.

By a well-known result of Youngson [Y], F is completely isometric to a TRO W. Since W is a dual Banach space, it is a $W*$-TRO, *i.e.,* W is TRO-isomorphic (hence completely isometric) to a corner $e\mathcal{A}e^{\perp}$ of a Von Neumann algebra $\mathcal{A}$. By unicity of the predual of a TRO in the operator space sense, E is necessarily completely isometric to W_*. ■

Problem For each $1 \le p \le \infty$, let T_p be the approximate theory of the class of all non-commutative L_p-spaces (when $p < \infty$) or of C^*-algebras (when $p = \infty$), considered as operator spaces. That is, T_p is the set of all positive bounded sentences φ in the language of operator spaces such that for every non-commutative L_p-space (resp., C^*-algebra) $\mathcal{E}$, one has $\mathcal{E} \models_{\mathcal{A}} \varphi$. Let $\mathcal{K}_p$ be the class of all operator spaces $\mathcal{E}$ such that $\mathcal{E} \models_{\mathcal{A}} T_p$. Then an operator space $\mathcal{E}$ is in $\mathcal{K}_p$ if and only if $\mathcal{E}$ is an ultraroot of some non-commutative L_p-space (resp., C^*-algebra). (See [HI, Remark 13.7].) We pose the problem of giving a mathematical description or characterization of the operator spaces in $\mathcal{K}_p$, for each p. Note that Proposition 5.1 implies that $\mathcal{K}_1$ is a class of TROs. This problem is also of interest when these spaces are simply considered as Banach spaces and the corresponding language is used.

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