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Perturbation and Solvability of Initial L^p Dirichlet Problems for Parabolic Equations over Non-cylindrical Domains

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Abstract. For parabolic linear operators L of second order in divergence form, we prove that the solvability of initial L^p Dirichlet problems for the whole range $1 < p < \infty$ is preserved under appropriate small perturbations of the coefficients of the operators involved. We also prove that if the coefficients of L satisfy a suitable controlled oscillation in the form of Carleson measure conditions, then for certain values of $p > 1$, the initial L^p Dirichlet problem associated with $Lu = 0$ over non-cylindrical domains is solvable. The results are adequate adaptations of the corresponding results for elliptic equations.

1 Introduction

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open bounded set, $n \geq 3$, with some additional features that we will describe later in the paper. Consider operators of the form

$$Lu = \operatorname{div}(A\nabla u) - \frac{\partial u}{\partial t} \tag{1.1}$$

where $(X, t) \in \mathbb{R}^n \times \mathbb{R}$, and $A(X, t) = (a_{i,j}(X, t))$ is a symmetric matrix of real-valued functions that satisfies a standard ellipticity condition of the form

$$\lambda_1 |\xi|^2 < \sum_{i,j} a_{ij}(X, t)\xi_i \xi_j < \lambda_2 |\xi|^2 \tag{1.2}$$

for certain $0 < \lambda_1 < \lambda_2 < \infty$, and every $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. The constants λ_1, λ_2 are referred to as *ellipticity constants of L*.

When Ω is a cylindrical region, solutions to $Lu = 0$ are taken in the weak sense, as we now recall from standard literature. First, $V(\Omega)$ is defined as the space of functions $u \in L^2(\Omega)$ such that $|\nabla_X u| \in L^2(\Omega)$, $u(\cdot, t) \in L^2(\Omega_t)$ for all t, where $\Omega(t) = \{(Y, s) \in \Omega : s = t\}$, and such that the norm on V given by

$$\|u\|_V^2 = \int_\Omega |\nabla_X u(X, t)|^2 dXdt + \sup_t \int_{\Omega(t)} u^2(X, t)dX$$

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is finite. Here ∇_X denotes the gradient with respect to X variables only.

With this notation and these definitions, a weak solution u associated with $Lu = 0$ is an element $u \in V$ such that

$$(1.3)\qquad \int_\Omega \big\langle A(X,t)\nabla_X u(X,t), \nabla\varphi(X,t)\big\rangle\, dXdt - \int_\Omega u(X,t)\frac{\partial\varphi}{\partial t}(X,t)dXdt = 0$$

for every $\varphi \in C_0^1(\Omega)$.

The adjoint of an operator L as described above is given by

$$(1.4)\qquad L^* u = \operatorname{div}(A\nabla u) + \frac{\partial u}{\partial t},$$

and adjoint solutions will have a definition similar to (1.3): $v \in V$ is a weak solution of $L^* v = 0$ if for every $\varphi \in C_0^1(\Omega)$ one has

$$\int_\Omega \langle A(X,t)\nabla_X v(X,t), \nabla\varphi(X,t)\rangle dXdt + \int_\Omega v(X,t)\frac{\partial\varphi}{\partial t}(X,t)dXdt = 0.$$

As usual, to make sense of boundary values for weak solutions one introduces the subspace V_0 as the closure of $C_0^1(\Omega)$ (the class of C^1 functions vanishing on $\partial_p\Omega$) with respect to the norm of $V(\Omega)$.

In our non-cylindrical setting, we may assume that the coefficients are of class C^∞ and still obtain estimates depending only on the ellipticity constants of the operator L and dimension n. This way standard limit arguments may be applied to conclude estimates that hold when the coefficients are measurable and bounded.

In this work we deal with the questions of perturbation and solvability of initial L^p Dirichlet problems associated with $Lu = 0$, $1 < p < \infty$ on Ω, where Ω is a certain type of non-cylindrical domains.

In a sense, this work may be viewed as a natural continuation of the research started in [25], where we aimed to provide an example of a wide class of operators as L, generalizing the heat operator, and for which one could prove a mutual absolute continuity of caloric measure and surface measure in the A_∞ sense (as described below), so that the initial L^p Dirichlet problems can be solved.

In that generalization we considered a class $\mathcal{L}$ of operators as L described in (1.1), over a basic parabolic Lipschitz domain $\Omega(\psi)$ (defined in the bulk of the paper) and their *pull back operator* $\widetilde{L}_1$ under a parabolic adaptation of the so-called *Dahlberg–Kenig–Stein mapping* from $\Omega(\psi)$ to $\mathbb{R}^{n+1}$.

When taking L equal to the heat operator, this pull-back operator has principal part as in (1.1) and a drift term. Moreover, all the new coefficients satisfy a Carleson measure condition that allowed the application of a technique introduced for elliptic operators in [20] through the use of certain square function estimates.

One of our main results in this paper states that under some technical conditions of Carleson measure type on the oscillation of the coefficients of the matrix $A(X,t)$ over Ω one can obtain such a solvability (see Theorems 3.2 and 3.3) . Results of this type have been obtained for the corresponding elliptic equations, for instance, in [5, 6, 21].

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We also prove a parabolic version of a perturbation result for L^p Dirichlet problems originally proved in [9] (see Theorem 3.1) that to our knowledge has not been previously adapted to parabolic equations.

Although details are provided for divergence form linear equations, some ideas are applicable to non-divergence parabolic linear equations (see Theorems 6.1 and 6.3).

The works on equations of elliptic type provide not only the motivation to study the corresponding questions for the parabolic equations, but also, to some extent, some guideline of the techniques that could be applied.

However, it is important to point out some instances in which the parabolic analogues of well-known results for the elliptic case turn out to have non-trivial applications or adaptations (*e.g.,* the doubling property of parabolic measure [14, 27] or the elliptic-type Harnack principle [11, 13, 14]); or that some technical issues arise given the regularity required in time-variable (for instance the need of introducing a special regularity entailed in the definition of parabolic Lipschitz graphs provided some paragraphs below, see *e.g.,* [16, 22, 23]). Some of these issues are present in the results contained in this work.

2 Description of Parabolic Setting and of the Initial L^p Dirichlet Problem

The precise statement of our main results requires the introduction of some technical definitions that we provide in this section. In the next paragraphs we have included a fairly complete description of the non-cylindrical domains as well as of the initial L^p Dirichlet problem.

Parabolic Homogeneity

There are some pioneering works providing a thorough description of the *parabolic homogeneity* associated with $\mathbb{R}^{n+1}$ (see *e.g.,* [4, 10, 12] and the references therein). Here we just recall some basic notions used at several stages of this paper.

Within this work, the *parabolic distance* between (X,t), $(Y,s) \in \mathbb{R}^{n+1}$ is given by the expression $d(X,t;Y,s) = |X-Y| + |t-s|^{1/2} \equiv \|X-Y, t-s\|$. This last expression defines what we call the *parabolic norm* of points in $\mathbb{R}^{n+1}$, and it may also be applied to points $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

Given $(X,t) \in \mathbb{R}^n \times \mathbb{R} \equiv \mathbb{R}^{n+1}$, denote by $C_r(X,t)$ the *cylinder*

$$\{(Y,s) \in \mathbb{R}^{n+1} : |X-Y| < r,\ |t-s| < r^2\}.$$

The *parabolic ball* of radius $r > 0$ centered at (X,t) is $Q_r(X,t) = \{(Y,s) \in \mathbb{R}^{n+1} : |X-Y| + |t-s|^{1/2} < r\}$. Given $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we define the *parabolic cube in* $\mathbb{R}^n$ by

$$B_r(x,t) = \{(y,s) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x_i - y_1| < r, \quad |t-s|^{1/2} < r, \quad i = 1, \dots, n-1\}.$$

The *parabolic boundary* of an open connected set $\Omega \in \mathbb{R}^{n+1}$, denoted by $\partial_p \Omega$, consists of points $(Q,s) \in \partial\Omega$ (the topological boundary of Ω) such that for every

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$r > 0$ one has $\mathcal{C}_r(Q,s) \setminus \Omega \neq \varnothing$. Here the *parabolic cylinder* of radius $r > 0$ and centered at (X,t) is defined as

$$\mathcal{C}_r(X,t) = \{(Y,s) \in \mathbb{R}^{n+1} : |X - Y| < r,\ 0 < t - s < r^2\}.$$

Time-varying Graphs

We now describe what has become to be known as the "good graphs" for both initial L^p Dirichlet problems and boundedness of parabolic singular integrals, as considered in previous works (see *e.g.,* [15, 16, 19, 22] and the references therein).

From now on, we adopt the convention that points in $\mathbb{R}^{n+1}$ may be denoted by $(x_0,x,t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$, to stress that in graph coordinates x_0 is the variable depending on (x,t). This particular way to denote graph coordinates for problems associated with heat equation goes back at least to [22].

A function $\psi\colon \mathbb{R}^n \to \mathbb{R}$ is a $\mathrm{Lip}(1,1/2)$ *function with constant* $A_0 > 0$ if for $(x,t),(x,s) \in \mathbb{R}^n$, $|\psi(x,t) - \psi(y,s)| \leq A_0\|x - y, t - s\|$. The function ψ is called a *parabolic Lipschitz function with constant* A_1 if it satisfies the following two conditions:

- ψ satisfies a *Lipschitz condition in the space variable*
$$|\psi(x,t) - \psi(y,t)| \leq A_1|x - y| \quad \text{uniformly on } t \in \mathbb{R};$$
- for every interval $I \subseteq \mathbb{R}$, every $x \in \mathbb{R}^n$,
$$\frac{1}{|I|}\int_I\int_I \frac{|\psi(x,t) - \psi(x,s)|^2}{|s - t|^2}\, dtds \leq A_1 < \infty.$$

This last condition can be recalled as a *BMO-Sobolev scale* in the t-variable, by results in [30]. It roughly states that a half order derivative of $\psi(x,t)$ with respect to t variable is in BMO. See more details in [16].

A *basic parabolic Lipschitz domain* is a domain of the form

$$\Omega(\psi) = \left\{ (x_0,x,t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : x_0 > \psi(x,t) \right\}$$

for some parabolic Lipschitz function ψ. This function is often taken with compact support.

We adopt the notation $\mathbf{X}$, $\mathbf{Y}$, etc. for points in $\mathbb{R}^{n+1}$ whenever both the time variable t and the graph variable x_0 are irrelevant for the argumentation. This notation becomes particularly handy in Section 5.

Parabolic Starlike Cylinders

To describe domains given locally by parabolic Lipschitz graphs we adapt definitions from [3], which are also used in [24].

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open connected set such that $\partial_p\Omega = \partial\Omega$ and let $A_1, r_0 > 0$. Define the *local cylinder with constants* A_1, r_0 as

$$\mathcal{Z} = \{(x_0,x,t) \in \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R} : |x_i| < r_0,\ i = 1,2,\dots,n-1,\ |x_0| < 2nA_1r_0,\ t \in \mathbb{R}\}.$$

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Here, $x \in \mathbb{R}^{n-1}$ is viewed as the $(n-1)$-tuple $x = (x_1, x_2, \dots, x_{n-1})$. We denote by $2\mathcal{Z}$ the concentric double of $\mathcal{Z}$, and set $\operatorname{diam}\Omega = \sup_{s \in \mathbb{R}} \Omega(s)$, where, as before, $\Omega(s) = \{(X,t) \in \Omega : t = s\}$.

We say that Ω is an *infinite starlike parabolic cylinder with constants* A_1, r_0, if there exist local cylinders $\{\mathcal{Z}_i : i = 1, 2, \dots, N\}$, with constants A_1, r_0, that are obtained from $\mathcal{Z}$ through rigid motions in the space variables, and parabolic Lipschitz functions $\{\psi_i : i = 1, 2, \dots, N\}$ with constant A_1, defined on the transformation of $\mathbb{R}^n$ through the same rigid motion defining $\mathcal{Z}_i$, and such that the following conditions hold:

- $2\mathcal{Z}_i \cap \partial\Omega = \{(x_0, x, t) : x_0 = \psi_i(x,t)\} \cap 2\mathcal{Z}_i$, $i = 1, 2, \dots, N$;
- $2\mathcal{Z}_i \cap \Omega = \{(x_0, x, t) : x_0 > \psi_i(x,t)\} \cap 2\mathcal{Z}_i$, $i = 1, 2, \dots, N$;
- $\partial\Omega$ is covered by $\bigcup_{i=1}^{N} \mathcal{Z}_i$;
- $\Omega(t)$ is bounded for every $t \in \mathbb{R}$;
- there exists $\mathcal{X}_0 \in \mathbb{R}^n$ and $\rho_0 > 0$ such that $\{X \in \mathbb{R}^n : |X - \mathcal{X}_0| < \rho_0\} \times \mathbb{R} \subset \Omega$.

If Ω is an infinite starlike parabolic cylinder, then for $T > 0$ we define the *bounded parabolic cylinder of height* T as $\Omega_T = \{(X,t) \in \Omega : 0 < t < T\}$. The *lateral boundary of* Ω_T is denoted by $S_T \equiv \partial_p \Omega_T \cap \partial\Omega$. Also set $\Xi = (\mathcal{X}_0, T)$, which will be recalled as the *parabolic center of* Ω_T.

Suppose $(Q,s) = (q_0, q, s) \in \partial\Omega$ and $0 < r < r_0$. The *Carleson region* is defined as

$$\Psi_r(Q,s) \equiv \left\{ (x_0, x, t) \in \Omega : |x - q| + |t - s|^{1/2} < r,\ |x_0 - q_0| < N_0) \right\},$$

and the *surface cube* is defined as $\Delta_r(Q,s) \equiv \overline{\Psi_r(Q,s)} \cap \partial\Omega$.

Here N_0 is chosen such that, with respect to the corresponding local cylinder, one has that $\Psi_r(Q,s)$ contains the set

$$\{(x_0, x, t) : \psi(x,t) \in \Delta_r(Q,s),\ \psi(x,t) < x_0 < \psi(x,t) + r\}.$$

This choice guarantees a sort of *parabolic starlike* property of points in $\Delta_r(Q,s)$ with respect of either of the following points:

$$\overline{\mathcal{A}}(\Delta) = \overline{\mathcal{A}}_r(Q,s) \equiv (q_0 + 2N_0 r, q, s + 2r^2),$$
$$\underline{\mathcal{A}}(\Delta) = \underline{\mathcal{A}}_r(Q,s) \equiv (q_0 + 2N_0 r, q, s - 2r^2).$$

Notice that if $\Delta = \Delta_r(Q,s)$ is as described before, then the notation $\Psi(\Delta)$ still makes sense with an obvious meaning. If Δ is any surface cube such that its closure satisfies $\overline{\Delta} \subset S_T$, to shorten notation we write $\Delta \Subset S_T$.

Given $(X,t) \in \Omega$ we write $\delta(X,t) = \operatorname{dist}(X,t;\partial\Omega)$, and defining for $(Q,s) \in S_T$,

$$\widetilde{\Gamma}_\alpha(Q,s) = \{(X,t) \in \Omega : \operatorname{dist}(X,t;Q,s) \leq (1+\alpha)\operatorname{dist}(X,t;\partial\Omega)\},$$

we denote by $\Gamma_\alpha(Q,s)$ the *non-tangential region* defined as the truncation of $\widetilde{\Gamma}_\alpha(Q,s)$ at height $\rho(Q) \equiv |Q - \mathcal{X}_0| + \rho_0/2$. Unless we state otherwise, the aperture $\alpha > 0$ is chosen such that $\Gamma_\alpha(Q,s) \subset \Omega$ for every $(Q,s) \in \partial\Omega$ (see [3, p. 572]).

Parabolic (Caloric) Measure

For Ω_T as described above the *parabolic measure associated with L*, denoted by $\omega_L(\,\cdot\,; X,t)$ for $(X,t) \in \Omega_T$, is the unique Borel measure supported on S_T such that

$$u_f(X,t) = \int_{\partial_p\Omega} f(Y,s)\,d\omega_L(Y,s;X,t)$$

is the solution, in the Perron–Wiener–Brelot sense (see *e.g.*, [7]), of the Dirichlet-type of problem $Lu = 0$ on Ω_T, $u|_{S_T} = f$ for f continuous and supported on S_T. Observe that in particular $u(X,0) = 0$ for every $X \in \Omega(0)$. We denote by ω the parabolic measure $\omega_L(\,\cdot\,,\Xi)$

Analogous to the case of equations of elliptic type, the key property of caloric measure in order to solve an initial L^p Dirichlet problem associated with $Lu = 0$ is a weight property referred to as *reverse Hölder property*, which we now recall. Denote by σ the *surface measure* defined for any Borel set $F \subset \mathbb{R}^{n+1}$ as

$$\sigma(F) = \int_F d\sigma_t\,dt,$$

where σ_t is the $(n-1)$-dimensional Hausdorff measure of $F_t \equiv F \cap \mathbb{R}^n \times \{t\}$, and dt denotes integration with respect to 1-dimensional Hausdorff measure.

The caloric measure ω_L is in the class $A_\infty(S_T,\sigma)$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{\sigma(F)}{\sigma(\Delta)} < \delta \quad \text{implies} \quad \frac{\omega_L(F)}{\omega_L(\Delta)} < \epsilon$$

for every Borel set $F \subset \Delta$, and every surface cube $\Delta \Subset S_T$.

By the well-known general theory of Muckenhoupt weights, it turns out that this defines a *uniform mutual absolute continuity* between ω_L and σ. It is also well known (see *e.g.*, [24]) that $\omega_L \in A_\infty(\sigma)$ if and only if the Radon–Nikodým derivative $k(Q) = d\omega_L/d\sigma(Q)$ satisfies the following property: there exists $1 < q < \infty$ such that ω_L is in the *reverse Hölder class* $\mathrm{RH}_q(S_T)$, namely there is a constant C_q such that for every surface cube $\Delta \subset S_T$ the following *reverse Hölder inequality* holds

$$\left(\frac{1}{\sigma(\Delta)}\int_\Delta k^q(Q)\,d\sigma(Q)\right)^{1/q} \le C_q \frac{1}{\sigma(\Delta)}\int_\Delta k(Q)\,d\sigma(Q). \tag{2.1}$$

In fact, we can assume that the surface cubes have radii way smaller than r_0, depending on A_1, where these two constants are from the definition of the local cylinders of Ω.

Initial L^p Dirichlet Problem

We say that the *initial L^p Dirichlet problem associated with $Lu = 0$, $1 < p < \infty$, is solvable*, with L as described in (1.1), if for any $f \in C(S_T)$, the solution $u(X,t)$ of the initial Dirichlet problem $Lu = 0$ over Ω_T, $u = f$ on S_T, satisfies the estimate

$$\|Nu\|_{L^p(S_T)} \le C\|f\|_{L^p(S_T)},$$

with a constant $C > 0$ not depending on f. Here the *non-tangential maximal function of u* defined for $(Q, s) \in \partial\Omega$ is defined as

$$Nu(Q, s) = \sup\{|u(X, t)| : (X, t) \in \Gamma_\alpha(Q, s)\}.$$

Standard real-variable techniques and some well-known properties of parabolic measure and solutions associated with $Lu = 0$ (see *e.g.*, [24]) imply that the initial L^p Dirichlet problem, $1 < p < \infty$, is solvable if and only if $\omega \in \mathrm{RH}_q(S_T)$, with $1/p + 1/q = 1$.

From this viewpoint, the main goal in order to solve an initial L^p-Dirichlet problem is to obtain the corresponding reverse Hölder property (2.1) of caloric measure ω_L with respect to surface measure σ. This is the reason that in previous work (see *e.g.*, [16, 25]) the results are stated on basic parabolic Lipschitz domains $\Omega(\psi)$ with compactly supported ψ.

Another useful and fundamental result on the classes of weights we mentioned before, is that when $\log d\omega/d\sigma \in \mathrm{VMO}(d\sigma)$, $\omega \in \mathrm{RH}_q(S_T)$ for every $1 < q < \infty$ (see *e.g.*, [29]). Here $\mathrm{VMO}(d\sigma)$ can be recalled as the closure of the space of continuous functions on S_T with respect to the $BMO(d\sigma)$ norm (see [28]).

Throughout this work we make use at several stages of some basic properties of solutions and parabolic measure for divergence form operators, and at each occurrence we will provide the pertinent references. In subsequent sections we retain the notation introduced above.

3 Description of the Results

Now we introduce notation to describe the perturbation results in this paper. Given two operators L_0 and L_1 as in (1.1) with associated matrices $A_0(X, t)$ and $A_1(X, t)$ respectively, and $(X, t) \in \Omega$, we define $C(X, t) = C_{\delta(X,t)/4}(X, t)$ and set

$$\mathcal{E}(X, t) = A_0(X, t) - A_1(X, t),$$

$$\mathcal{A}(X, t) = \sup\left\{ |A_0(Y, s) - A_1(Y, s)|^2 : (Y, s) \in C(X, t)\right\}.$$

The following theorem may be referred to as the *main perturbation result* of this paper. For its proof (included in Section 5) we use a result from [24].

Theorem 3.1 *Fix $T > 0$, and let Ω be an infinite starlike parabolic cylinder with constants A_1, r_0, and with A_1 appropriately small. Let L_0 and L_1 be two operators as in (1.1) defined over Ω, and assume that*

$$\lim_{\mathrm{diam}\Delta \to 0} \left[\frac{1}{\sigma(\Delta)} \int_{\Psi(\Delta)} \frac{\mathcal{A}(X, t)}{\delta(X, t)}\, dXdt \right] = 0,$$

where $\Delta \Subset S_T$ denotes any surface ball on S_T. Suppose further that $\log(d\omega_0/d\sigma) \in \mathrm{VMO}(d\sigma)$. *Then* $\log(d\omega_1/d\sigma) \in \mathrm{VMO}(d\sigma)$.

The smallness assumption of A_1 is included in view of the result in [16, Corollary 1.18].

Now we refer to the following theorems as the *solvability results* for easy reference within this work. The arguments to prove them are given in Section 6.

Theorem 3.2 *Fix $T > 0$, and let Ω be an infinite starlike parabolic cylinder with constants A_1 and r_0, with A_1 so small that (4.2) holds. Let L be any operator as described in (1.1), with matrix coefficients A, defined over Ω_T. Suppose that there exists a constant $C > 0$ such that for every $(Q,s) \in S_T$ and every $r > 0$ such that $\Delta_r(Q,s) \Subset S_T$:*

$$\int_{\Psi_r(Q,s)} \Big(\sup_{(Z,\tau)\in C(X,t)} \delta(Z,\tau)|\nabla A(Z,\tau)|^2 + \sup_{(Z,\tau)\in C(X,t)} \delta(Z,\tau)^3 \Big| \frac{\partial A}{\partial \tau}(Z,\tau) \Big|^2 \Big) \, dXdt < C\sigma\big(\Delta_r(Q,s)\big) . \tag{3.1}$$

Then $k = d\omega/d\sigma \in \mathrm{RH}_p(\Omega_T)$ for some $p > 1$.

The proof of this theorem actually uses the main theorem in [25]. Also, the result addressed in Theorem 3.2 has a version in which certain oscillation property on the coefficients replaces the one on the gradient $|\nabla A|$, as we now describe.

Let

$$\mathcal{O}(X,t) = \sup\left\{ |A(Y,s) - \operatorname{avg}((Y,s);A)|^2 : (Y,s) \in C(X,t) \right\},$$

where $\operatorname{avg}((Y,s);A)$ is the matrix given by

$$\operatorname{avg}((Y,s);A) = \frac{1}{|C(Y,s)|} \int_{C(Y,s)} A(Z,\tau) dZ d\tau,$$

and where for any $n \times n$ matrix A the quantity $|A|$ denotes the Euclidian magnitude of the $n \times n$ dimensional vector given by the entries of A.

Theorem 3.3 *Fix $T > 0$, and let Ω be an infinite starlike parabolic cylinder. Let L be an operator as described in (1.1) defined over Ω. Suppose that there exists a constant $C > 0$ such that for every $(Q,s) \in S_T$ and every $r > 0$ such that $\Delta_r(Q,s) \subset S_T$*

$$\int_{\Psi_r(Q,s)} \frac{\mathcal{O}(X,t)}{\delta(X,t)} dXdt < C\sigma(\Delta_r(Q,s)). \tag{3.2}$$

Then $k = d\omega/d\sigma \in \mathrm{RH}_p(\Omega_T)$ for some $p > 1$.

The proof of Theorem 3.3 follows from Theorem 3.2 and the perturbation result in [24], through the technique explained in detail in [5, pp. 377–378]. For completeness, we provide some details of the adaptation of this argument in Section 6.

In the next section we describe the main result in [25] and define the class $\mathcal{L}$ of operators as in (1.1) considered in some arguments proving the main theorems of this paper. It contains the motivation and initial point of the research reported here.

4 Operators with Drift Term and the Class $\mathcal{L}$

We start by describing the parabolic version of Dahlberg–Kenig–Stein mapping. Let $\mathbb{P} \in C_0^\infty$, supported in $\Sigma^{n-1} = \{(y,s) \in \mathbb{R}^n : |y,s| < 1\}$ with $\int_{\mathbb{R}^n} \mathbb{P}(x,t)dxdt = 1$, and for $\lambda > 0$ set

$$\mathbb{P}_\lambda(y,s) = \lambda^{-n-1}\mathbb{P}(y\lambda^{-1}, s\lambda^{-2}),$$
$$\mathbb{P}_\lambda f(x,t) = \int_{\mathbb{R}^n} \mathbb{P}_\lambda(x-y, t-s) f(y,s)dyds.$$

Note that $\lim_{\lambda\to 0} \mathbb{P}_{\gamma\lambda}\psi(x,t) = \psi(x,t)$ for any $\gamma > 0$. Let

$$\widetilde{\mathbb{R}}^{n+1}_+ = \{(\lambda, x, t) : \lambda > 0,\ x \in \mathbb{R}^n,\ t \in \mathbb{R}\},$$

and define the *parabolic version of Dahlberg–Kenig–Stein mapping* $\rho \colon \widetilde{\mathbb{R}}^{n+1}_+ \to \Omega(\psi)$ by

$$\rho(\lambda, x, t) = \big(\lambda + \mathbb{P}_{\gamma\lambda}\psi(x,t), x, t\big), \quad \rho(0,x,t) = \big(\psi(x,t), x, t\big), \tag{4.1}$$

where $\gamma > 0$ is chosen appropriately small.

As observed in [16, p. 364], if one takes parabolic Lipschitz functions with small constant A_1, then one can also choose γ sufficiently small so that

$$\frac{1}{2} < 1 + \frac{\partial}{\partial\lambda}\mathbb{P}_{\gamma\lambda} < \frac{3}{2}, \tag{4.2}$$

which incidentally also guarantees that $\rho : \widetilde{\mathbb{R}}^{n+1}_+ \to \Omega(\psi)$ defined by (4.1) is one-to-one.

We will now record a part of [16, Lemma 2.8] for easy future reference.

Lemma 4.1 *Let σ, θ be non-negative integers and $\phi = (\phi_1, \phi_2, \dots, \phi_{n-1})$ a multi index, where, as usual, $|\phi| = \phi_1 + \cdots + \phi_{n-1}$. Set $\ell = \sigma + |\phi| + \theta$. Suppose that ψ is a parabolic Lipschitz function with constant A_1.*

(i) *The measure*

$$d\nu = \Big(\frac{\partial^\ell \mathbb{P}_{\gamma\lambda}\psi}{\partial\lambda^\sigma dx^\phi \partial t^\theta}\Big)^2 \lambda^{(2\ell+2\theta-3)} dxdtd\lambda$$

is a Carleson measure on $\widetilde{\mathbb{R}}^{n+1}_+$ whenever either $\sigma + \theta \geq 1$ or $|\phi| \geq 2$; namely, the estimate

$$\nu(B_r(x,t) \times (0,r)) \leq Cr^{n+1}$$

holds for sufficiently small $r > 0$, and with constant $C > 0$ depending on ℓ, γ, and A_1.

(ii) *If $\ell \geq 1$, then*

$$\Big\| \frac{\partial^\ell \mathbb{P}_{\gamma\lambda}\psi}{\partial\lambda^\sigma dx^\phi \partial t^\theta} \Big\|_\infty \leq C(\ell, \gamma, A_1)\lambda^{1-\ell-\theta}.$$

Given u defined on a basic parabolic Lipschitz domain $\Omega(\psi)$ we consider the *pull-back function* $u_1 = u \circ \rho$ defined on $\widetilde{\mathbb{R}}^{n+1}_+$. A fundamental observation in [18] (see also [16, 17]) is that if H denotes the heat operator $\Delta - \partial/\partial t$, and u is a solution to $Hu = 0$, then the pull-back u_1 of u is a weak solution to $H_1 \widetilde{u} = 0$, where the operator H_1 is defined by

$$H_1 u_1 \equiv \operatorname{div} A_1 \nabla u_1 - \Big(1 + \frac{\partial}{\partial\lambda}\mathbb{P}_{\gamma\lambda}\psi\Big)\frac{\partial v}{\partial t} + \Big(\frac{\partial}{\partial t}\mathbb{P}_{\gamma\lambda}\psi\Big)\frac{\partial}{\partial\lambda}u_1,$$

and the $(n \times n)$ matrix $A_1(X,t) = (\widetilde{a}_{ij}(X,t))$ is given by

$$\widetilde{a}_{j,j} = 1 + \frac{\partial}{\partial\lambda}\mathbb{P}_{\gamma\lambda}\psi, \qquad \widetilde{a}_{0,j} = \widetilde{a}_{j,0} = -\frac{\partial}{\partial x_j}, P_{\gamma\lambda}\psi, \quad 1 \le j \le n-1$$

$$\widetilde{a}_{0,0} = \frac{1 + |\nabla_x \mathbb{P}_{\gamma\lambda}\psi|^2}{1 + \frac{\partial}{\partial\lambda}\mathbb{P}_{\gamma\lambda}\psi}, \qquad \widetilde{a}_{i,j} = 0 \quad 1 \le i, j \le n-1.$$

By this observation, a weak solution to $H_1 u_1 = 0$ is also a weak solution to $\widetilde{H} u_1 = 0$, where

$$\widetilde{H} v \equiv \operatorname{div} \widetilde{A} \nabla v + \vec{B} \cdot \nabla v - \frac{\partial v}{\partial t}, \tag{4.3}$$

and where $\widetilde{A}(X,t)$ is a new matrix of coefficients, and $\vec{B}(X,t)$ is an n-dimensional vector of functions.

With (4.2) at hand, and using Lemma 4.1, the following result can be established (see *e.g.*, [18, pp. 10–11]).

Proposition 4.2 *If H denotes the heat operator $\Delta - \partial/\partial t$ and u is a solution to $Hu = 0$ on a basic parabolic Lipschitz domain $\Omega(\psi)$, ψ compactly supported, then the pull-back u_1 of u is a weak solution to $\widetilde{H} u_1 = 0$, where the operator $\widetilde{H}$ has the form* (4.3)*. Moreover, for the coefficients $\widetilde{A}(X,t)$ and $\vec{B}(X,t)$ of $\widetilde{H}$ we have:*

- *$\widetilde{A}(X,t)$ satisfies the ellipticity condition* (1.2)*,*
- *$\widetilde{A}(X,t)$ and $\vec{B}(X,t)$ satisfy the following estimates:*

$$\int_0^r \int_{\Delta_r} \Big(\big| \nabla_{\lambda,x} \widetilde{A}(\lambda,x,t) \big|^2 + \big| \vec{B}(\lambda,x,t) \big|^2 \Big) \lambda \, dxdt \, d\lambda \le C_1 |\Delta_r|, \tag{4.4}$$

$$\int_0^r \int_{\Delta_r} \Big| \frac{\partial \widetilde{A}}{\partial t}(\lambda,x,t) \Big|^2 \lambda^3 dxdt \, d\lambda \le C_1 |\Delta_r|. \tag{4.5}$$

Here and in the future we adopt the notation of $|\nabla_{\lambda,x} A(\lambda,x,t)|$ for the euclidian magnitude of the vector containing the derivatives (with respect to λ and x) of all the entries of A. Similarly an integral or a derivative of the matrix $A(\lambda,x,t)$ is the matrix whose entries are respectively the derivative or integral of the corresponding entry of A.

Let $\mathcal{L}$ denote the class of operators L as in (1.1), and such that the coefficients of corresponding pull-back operator $\widetilde{L}$ as in (4.3) satisfy (4.4)–(4.5). By Proposition 4.2 this class is nonempty, as the heat operator belongs to $\mathcal{L}$.

One of our goals in [25] was to prove that for $L \in \mathcal{L}$ one can solve an initial L^p Dirichlet problem associated with $Lu = 0$ on a basic parabolic Lipschitz domain $\Omega(\psi)$. In [25, Theorem 1.3] we proved the following theorem.

Theorem 4.3 ([25, Theorem 1.3]) *Let $\psi\colon \mathbb{R}^n \to \mathbb{R}$ be a nonnegative compactly supported parabolic Lipschitz function with small constant A_1 such that (4.2) holds. Then for any operator $L \in \mathcal{L}$ the L^p Dirichlet problem associated with $Lu = 0$ over $\Omega(\psi)$ is solvable for certain $p > 1$.*

A result like Theorem 4.3 was developed for operators of elliptic type in [21], where a condition related to (4.4) and (4.5) was considered. A parabolic adaptation of the statement of [21, Theorem 2.6] is precisely Theorem 3.2.

5 Proof of Theorem 3.1

In this section we include an adaptation of an idea from [9], which, along with estimates and results from [24], yield Theorem 3.1. Recall that we are given L_0 and L_1 two operators as in (1.1) with associated matrices $A_0(X,t)$ and $A_1(X,t)$ respectively. For $(X,t) \in \Omega$ we have defined $C(X,t) = C_{\delta(X,t)/4}(X,t)$, and we have set $\mathcal{E}(X,t) = A_0(X,t) - A_1(X,t)$ and

$$\mathcal{A}(X,t) = \sup\Big\{ \big| A_0(Y,s) - A_1(Y,s)\big|^2 : (Y,s) \in C(X,t)\Big\}. \tag{5.1}$$

By assumption we know that

$$\lim_{\operatorname{diam}\Delta\to 0}\Big[\frac{1}{\sigma(\Delta)}\int_{\Psi(\Delta)}\frac{\mathcal{A}(X,t)}{\delta(X,t)}\,dXdt\Big] = 0, \tag{5.2}$$

where $\Delta \subset \partial\Omega$ denotes any surface ball on S_T.

We will describe the proof working locally in $\widetilde{\mathbb{R}}^{n+1}_+$, where the next constructions are easier to describe. Then the adaptations to any basic parabolic Lipschitz domain can be implemented as in [24, Section 6]. In fact, by using estimates proven on basic domains above a Lip(1, 1/2) graph in [24] (see also [13]) we avoid losing generality in our reasonings.

In this new setting of $\widetilde{\mathbb{R}}^{n+1}_+$, for $R_0 > 0$ and $(x_0,t_0) \in \mathbb{R}^{n-1}\times\mathbb{R}$ fixed, we let

$$\widetilde{\Delta} \equiv \widetilde{\Delta}_{R_0}(x_0,t_0) = C_{R_0}(0,x_0,t_0)\cap\{\lambda = 0\} \quad\text{and}\quad \widetilde{\Psi} \equiv C_{R_0}(0,x_0,t_0)\cap\widetilde{\mathbb{R}}^{n+1}_+.$$

Also let $S_{\widetilde{\Psi}} = \partial\widetilde{\mathbb{R}}^{n+1}_+ \cap \partial\widetilde{\Psi}$.

We will simply use the notation $\mathrm{RH}_q(d\sigma)$, $\mathrm{VMO}(d\sigma)$ or $A_\infty(d\sigma)$ to refer to weight classes in this new environment.

For $0 < \theta \le 1$ the matrix $A^\theta = (a^\theta_{i,j})$ is defined by $a^\theta_{i,j} = (1-\theta)a^0_{i,j} + \theta a^1_{i,j}$. Note that the coefficients A^θ satisfy (1.2), with new ellipticity constants that depend on λ_1, λ_2, and the smooth approximate identity $\mathbb{P}$.

Assuming that the coefficients in A are smooth, define

$$L_\theta u = \operatorname{div} A^\theta \nabla u - \partial_t u,$$

and let ω_θ denote the parabolic measure associated with L_θ on $\widetilde{\Psi}$ with pole at $\widetilde{\mathcal{A}}_0 = (R_0/2, x_0, t_0 + R_0^2)$.

Given $(\lambda, x, t) \in \widetilde{\Psi}$ denote by $C(\lambda, x, t) = C_{\lambda/2}(\lambda, x, t)$. In this new setting (5.1) is translated into

$$\mathcal{A}(\lambda, x, t) = \sup\Big\{ \big| A_0(\zeta, y, s) - A_1(\zeta, y, s) \big|^2 : (\zeta, y, s) \in C(\lambda, x, t) \Big\},$$

with similar adjustment to define $\mathcal{E}(\lambda, x, t)$. Hypothesis (5.2) now becomes

$$\lim_{\operatorname{diam}\Delta \to 0} \Big[\frac{1}{\sigma(\Delta)} \int_{\Psi(\Delta)} \frac{\mathcal{A}(\lambda, x, t)}{\lambda}\, d\lambda dx dt \Big] = 0. \tag{5.3}$$

Since $\log(d\omega_0/d\sigma) \in \mathrm{VMO}(d\sigma)$, $\omega_0 \in \mathrm{RH}_2(d\sigma)$. Using [24, Theorem 6.5] we conclude that $\omega_\theta \in \mathrm{RH}_2(d\sigma)$ for $0 \le \theta \le 1$ with uniform constants.

Now for $0 < \theta < 1$ consider the auxiliary functions v_θ defined as the solutions of the following initial Dirichlet problem associated with L_θ^*, the formal adjoint of L_θ (see (1.4)):

$$\begin{cases} L_\theta^* v_\theta = -\varphi & \text{on } \widetilde{\Psi}, \\ v_\theta = 0 & \text{on } \partial_p \widetilde{\Psi}, \end{cases}$$

with $\varphi \in C_0^\infty(\widetilde{\Psi})$ satisfying $\varphi = 1$ on $\widetilde{\Psi}(\delta_0/2)$, $\varphi = 0$ on $\widetilde{\Psi} \setminus \widetilde{\Psi}(\delta_0/4)$, where

$$\widetilde{\Psi}(\tau) = \big\{ \mathbf{Y} \in \widetilde{\Psi} : d(\mathbf{Y}; \partial_p \widetilde{\Psi}) > \tau \big\}$$

for $\tau > 0$. The $\delta_0 > 0$ is chosen as a fixed small fraction of R_0, say $\delta_0 = R_0/100$.

Denoting by $g_\theta(\mathbf{X}, \mathbf{Y})$ the Green's function of L_θ on $\widetilde{\Psi}$ with pole at $\mathbf{X}$, we have

$$v_\theta(\mathbf{Y}) = \int_{\widetilde{\Psi}} g_\theta(\mathbf{X}, \mathbf{Y}) \varphi(\mathbf{X}) d\mathbf{X}.$$

Set the notation $K_\theta(\mathbf{Q}) = dv_\theta/dN(\mathbf{Q})$ for the *co-normal derivative* of v_θ at $\mathbf{Q} = (0, q, s)$, namely

$$K_\theta(\mathbf{Q}) = \big\langle A(\mathbf{Q}) \nabla v_\theta(\mathbf{Q}), N(\mathbf{Q}) \big\rangle,$$

which is well defined for $\mathbf{Q} \in S_{\widetilde{\Psi}}$.

Recall that

$$\frac{d\omega_\theta}{d\sigma}(\mathbf{Q}) = \frac{\partial g_\theta(\widetilde{\mathcal{A}}_0, \mathbf{Q})}{\partial N} \quad \text{for } \sigma\text{-almost every } \mathbf{Q} \in S_{\widetilde{\Psi}}, \tag{5.4}$$

where this time the *co-normal derivative* of the Green's function is defined as

$$\frac{\partial g_\theta(\widetilde{\mathcal{A}}_0, \mathbf{Q})}{\partial N} = \big\langle A(\mathbf{Q}) \nabla g_\theta(\widetilde{\mathcal{A}}_0, \mathbf{Q}), N(\mathbf{Q}) \big\rangle.$$

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Next we note that the following estimates also hold:

$$v_\theta(\mathbf{X}) \approx g_\theta(\widetilde{\mathcal{A}}_0, \mathbf{X}) \qquad \text{for every } \mathbf{X} \in \widetilde{\Psi} \setminus \widetilde{\Psi}(\delta_0), \tag{5.5}$$

$$\frac{\partial v_\theta}{\partial N}(\mathbf{Q}) \approx \frac{\partial g_\theta(\widetilde{\mathcal{A}}_0, \mathbf{Q})}{\partial N} \qquad \text{for } \sigma\text{-almost every } \mathbf{Q} \in S_{\widetilde{\Psi}}. \tag{5.6}$$

The estimates in (5.5) follow from the comparison principle in [24, Lemma 2.10] and an elliptic type Harnack principle (see *e.g.*, [13, Theorem 3], [24, Corollary 3.1]). Since the inner normal vector in $\widetilde{\mathbb{R}}^{n+1}_+$ coincides with the λ direction, we can obtain (5.6) again by the comparison principle and an elliptic type Harnack principle.

Now the key step to prove Theorem 3.1 is the following proposition.

Proposition 5.1 *There exists a real-valued function $\Phi(\rho)$ defined for $\rho > 0$ satisfying $\lim_{\rho\to 0} \Phi(\rho) = 0$ and such that for any surface cube $\Delta_\rho \subset \widetilde{\Delta}$ with $\Delta_{8\rho} \Subset \widetilde{\Delta}$,*

$$\Big(\frac{1}{\sigma(\Delta_\rho)} \int_{\Delta_\rho} K_1^2 d\sigma\Big)^{1/2} \lesssim \big(1 + \Phi(\rho)\big) \frac{1}{\sigma(\Delta_\rho)} \int_{\Delta_\rho} K_1 d\sigma.$$

Armed with this proposition, Theorem 3.1 follows by the arguments in [9, p. 358] (see also [8, p. 190]). These arguments are adaptable, as they rely on a measure theoretic lemma of [28] and an adequate version of the John–Nirenberg Lemma, and both results hold in our parabolic setting.

To prove Proposition 5.1 we need more definitions and another auxiliary result. Fix the surface ball $\Delta_\rho \subset \widetilde{\Delta}$, $\rho > 0$, with $\Delta_{8\rho} \Subset \widetilde{\Delta}$. For $f \in L^2(S_{\widetilde{\Psi}})$, $f \geq 0$, supported in Δ_ρ with

$$\frac{1}{\sigma(\Delta_\rho)} \int_{\Delta_\rho} f^2 d\sigma \leq 1,$$

and for $0 \leq \theta \leq 1$ set $K_\theta(E) = \int_E K_\theta d\sigma$, where $E \subset S_{\widetilde{\Psi}}$ is a Borel set. Define

$$\Upsilon(\theta) = \frac{1}{K_\theta(\Delta_\rho)} \int_{\Delta_\rho} f K_\theta d\sigma.$$

Proposition 5.2 *There exist constants $0 < \beta < 1$ and $0 < \widetilde{\alpha} < 1$ such that*

$$\Big|\frac{d\Upsilon}{d\theta}\Big| \lesssim \Big[\rho^{\widetilde{\alpha}} + \sup_{\substack{\mathbf{Q}\in S_{\widetilde{\Psi}}\\ s<\rho^\beta}} \Big(\frac{1}{\sigma(\Delta_s(\mathbf{Q}))} \int_{\Psi_s(\mathbf{Q})} \mathcal{A}(\mathbf{X}) \frac{d\mathbf{X}}{\delta(\mathbf{X})}\Big)^{1/2}\Big]$$

The proof of this proposition is given below. Proposition 5.2 implies Proposition 5.1 by the following argument. Using the fundamental theorem of calculus we first obtain the estimate

$$\frac{1}{K_1(\Delta_\rho)} \int_{\Delta_\rho} f K_1 d\sigma - \frac{1}{K_0(\Delta_\rho)} \int_{\Delta_\rho} f K_0 d\sigma \leq \rho^{\widetilde{\alpha}} + \sup_{\substack{\mathbf{Q}\in S_{\widetilde{\Psi}}\\ s<\rho^\beta}} \Big(\frac{1}{\sigma(\Delta_s(\mathbf{Q}))} \int_{\Psi_s(\mathbf{Q})} \mathcal{A}(\mathbf{X}) \frac{d\mathbf{X}}{\delta(\mathbf{X})}\Big)^{1/2}.$$

Now duality and the fact that $\log(\omega_0/d\sigma) \in \mathrm{VMO}(S_{\widetilde{\Psi}})$ implies

$$\frac{1}{\sigma(\Delta_\rho)} \int_{\Delta_\rho} \Big(\frac{d\omega_0}{d\sigma}\Big)^2 d\sigma \lesssim \big(1+\Phi(\rho)\big) \frac{1}{\sigma(\Delta_\rho)} \int_{\Delta_\rho} \frac{d\omega_0}{d\sigma} d\sigma$$

(see [9, p. 357]). This and (5.3) together imply Proposition 5.1.

It only remains to prove Proposition 5.2. After proving it we will have finished the proof of Theorem 3.1. To proceed with the proof of Proposition 5.2 we define

$$f_{\Delta_\rho,K_\theta} = \frac{1}{K_\theta(\Delta_\rho)} \int_{\Delta_\rho} f K_\theta d\sigma \quad \text{and} \quad h_\theta = \frac{(f - f_{\Delta_\rho,K_\theta})\chi_{\Delta_\rho}}{K_\theta(\Delta_\rho)},$$

and denote by u_θ the solution of

$$\begin{cases} L_\theta u_\theta = 0 & \text{on } \widetilde{\Psi}, \\ u_\theta = h_\theta & \text{on } \partial_p \widetilde{\Psi}. \end{cases}$$

Lemma 5.3 *There exists $\beta > 0$ such that for $\mathbf{Q}_0 \in S_{\widetilde{\Psi}}$ and $\mathbf{X} \in \widetilde{\Psi}$ such that $\|\mathbf{X}-\mathbf{Q}_0\| > 2r$*

$$(5.7) \qquad |u_\theta(\mathbf{X})| \lesssim \Big(\frac{r}{\|\mathbf{X}-\mathbf{Q}_0\|}\Big)^\beta \frac{1}{\omega_\theta(\Delta_{d(\mathbf{X},\mathbf{Q}_0)})}.$$

The proof of Lemma 5.3 follows along the lines of [9], and we include some details of this adaptation.

Proof Define for $\mathbf{X} \in \widetilde{\Psi}$ and $\mathbf{Q} \in S_{\widetilde{\Psi}}$,

$$\mathcal{G}_\theta(\mathbf{X},\mathbf{Q}) = \lim_{\mathbf{Q}'\to\mathbf{Q}} \frac{g_\theta(\mathbf{X};\mathbf{Q}')}{\nu_\theta(\mathbf{Q}')}, \quad 0<\theta<1.$$

This limit exists by [13, Theorem 6]. Moreover the following result holds.

Lemma 5.4 *The function u_θ can be represented by*

$$(5.8) \qquad u_\theta(\mathbf{X}) = \int \mathcal{G}_\theta(\mathbf{X},\mathbf{Q}) h_\theta(\mathbf{Q}) K_\theta(\mathbf{Q}) d\sigma.$$

Also, if $\mathbf{Q}_0 \in S_T$, then for $\mathbf{X} \in \widetilde{\Psi}$ and $r > 0$,

$$(5.9) \qquad |\mathcal{G}_\theta(\mathbf{X},\mathbf{Q}) - \mathcal{G}_\theta(\mathbf{X},\mathbf{Q}_0)| \lesssim \Big(\frac{r}{\|\mathbf{X}-\mathbf{Q}_0\|}\Big)^\alpha \frac{1}{\omega_\theta(\Delta_{d(\mathbf{X},\mathbf{Q}_0)})}$$

whenever $\|\mathbf{X}-\mathbf{Q}_0\| > 2r$, and $\mathbf{Q} \in \Delta_r(\mathbf{Q}_0)$.

Before proving this result we explain how to finish the proof of Lemma 5.3 using Lemma 5.4. Since h_θ has zero average with respect to K_θ we use (5.8) to write

$$u_\theta(\mathbf{X}) = \int_{\widetilde{\Delta}} \big[\, \mathcal{G}_\theta(\mathbf{X},\mathbf{Q}) - \mathcal{G}_\theta(\mathbf{X},\mathbf{Q}_0)\big]\, h_\theta(\mathbf{Q}) K_\theta(\mathbf{Q}) d\sigma.$$

Now since h_θ is supported in Δ_ρ, we can apply Hölder's inequality and (5.9) to obtain

$$|u_\theta(\mathbf{X})| \le \Big(\frac{r}{\|\mathbf{X}-\mathbf{Q}_0\|}\Big)^{\alpha} \frac{1}{\omega_\theta(\Delta_{d(\mathbf{X},\mathbf{Q}_0)})} \frac{\sigma(\Delta_\rho)}{K_\theta(\Delta_\rho)} \Big(\frac{1}{\sigma(\Delta_\rho)} \int_{\Delta_\rho} K_\theta^2 d\sigma\Big)^{1/2}.$$

As observed above, [24, Theorem 6.5] implies that $k_\theta = d\omega_\theta/d\sigma \in \mathrm{RH}_2(d\sigma)$ with uniform weight constants, and by (5.4) we conclude the proof of Lemma 5.3. ■

Proof of Lemma 5.4 To prove (5.8) we start by using the representation formula

$$u_\theta(\mathbf{X}) = \int h_\theta(\mathbf{Q}) d\omega^{\mathbf{X}}(\mathbf{Q}) = \int h_\theta(\mathbf{Q}) \frac{d\omega^{\mathbf{X}}}{d\sigma}(\mathbf{Q})\, d\sigma(\mathbf{Q}) \approx \int h_\theta(\mathbf{Q}) \frac{dg_\theta(\mathbf{X},\mathbf{Q})}{dN}\, d\sigma(\mathbf{Q}),$$

where in the last estimates we used (5.6).

We now use results in [13, Section 3], on Hölder continuity of quotients of solutions vanishing on a portion of the boundary. More precisely, for $\mathbf{X} \in \widetilde{\Psi}$, $\mathbf{Q}, \mathbf{Q}_0 \in S_{\widetilde{\Psi}}$ satisfying $\|\mathbf{Q}-\mathbf{Q}_0\| \le r/2$, $2r \le \|\mathbf{X}-\mathbf{Q}_0\|$, take $\mathbf{Q}', \mathbf{Q}'' \in \widetilde{\Psi}$, with $\mathbf{Q}'$ approaching $\mathbf{Q}_0$, and $\mathbf{Q}''$ approaching $\mathbf{Q}$

$$\Big| \frac{g_\theta(\mathbf{X},\mathbf{Q}'')}{v_\theta(\mathbf{Q}'')} - \frac{g_\theta(\mathbf{X},\mathbf{Q}')}{v_\theta(\mathbf{Q}')} \Big| \lesssim \Big(\frac{\|\mathbf{Q}'-\mathbf{Q}''\|}{\|\mathbf{X}-\mathbf{Q}_0\|}\Big)^{\alpha} \frac{g_\theta(\mathbf{X}, \overline{\mathcal{A}}_{\delta(\mathbf{X};\mathbf{Q}_0)}(\mathbf{Q}_0))}{v_\theta(\underline{\mathcal{A}}_{\delta(\mathbf{X},\mathbf{Q}_0)}(\mathbf{Q}_0))}.$$

Taking limits and using (5.5) and the comparison between Green's function and parabolic measure (see *e.g.*, [24, Lemmata 2.8 and 2.9]) we conclude (5.9), after applying a Backward Harnack inequality near the boundary [13, Theorem 4]. ■

Proof of Proposition 5.2 Given a real valued function F let $dF/d\theta = \dot{F}$ denote the derivative with respect to θ, whenever this derivatives makes sense. Also, for the v_θ already defined above, we set $\dot{v}_\theta$ to be the weak limit in $V_0(\widetilde{\Psi})$ (see page 430) of the ratio $(v_\theta - v_{\theta+\eta})/\eta$ as $\eta \to 0$. Similarly, $\dot{K}_\theta$ is the weak limit in $L^2(S_{\widetilde{\Psi}})$ of $(K_\theta - K_{\theta+\eta})/\eta$ as $\eta \to 0$. Incidentally, observe that actually $\|K_\theta\|_{L^2(S_{\widetilde{\Psi}})} \lesssim 1$ uniformly on $0 < \theta < 1$.

In order to estimate $|\dot{\Upsilon}(\theta)|$, first we note that for $f \in L^2(\partial_p \widetilde{\Psi})$ supported in Δ_ρ we have

$$\dot{\Upsilon}(\theta) = \frac{1}{K_\theta(\Delta_\rho)} \int_{\Delta_\rho} \dot{K}_\theta (f - f_{\Delta_\rho, K_\theta}) d\sigma = \int_{\Delta_\rho} \dot{K}_\theta h_\theta \, d\sigma. \tag{5.10}$$

On the other hand

$$\dot{K}_\theta = \frac{d}{d\theta} \langle A^\theta \nabla v_\theta, \vec{N}\rangle = \langle A \nabla \dot{v}_\theta, \vec{N}\rangle = \frac{d\dot{v}_\theta}{dN} \quad \sigma\text{-almost everywhere in } S_{\widetilde{\Psi}}, \tag{5.11}$$

since the coefficients A^θ coincide in $S_{\widetilde{\Psi}}$, for $0<\theta<1$.

Next observe that

$$L^\theta \dot{v}_\theta + \operatorname{div}(\mathcal{E}\nabla v_\theta) = 0, \quad \dot{v}_\theta = 0 \text{ on } \partial_p\widetilde{\Psi}. \tag{5.12}$$

Recalling that h_θ is supported in Δ_ρ, integration by parts and (5.10)–(5.12) imply

$$\left|\dot{\Upsilon}(\theta)\right| \lesssim \int_{\widetilde{\Psi}} |\mathcal{E}(\mathbf{X})||\nabla v_\theta(\mathbf{X})||\nabla u_\theta(\mathbf{X})|d\mathbf{X}. \tag{5.13}$$

The proof now follows the lines of the one in [9, pp. 362–365]. For completeness we include some details of the adaptation of that argument.

We handle the integral in (5.13) by considering the part close to $\mathbf{Q}_0 \in S_{\widetilde{\Psi}}$ and its complement:

$$\begin{aligned}\int_{\widetilde{\Psi}} |\mathcal{E}(\mathbf{X})||\nabla v_\theta(\mathbf{X})||\nabla u_\theta(\mathbf{X})|d\mathbf{X} &= \int_{C(\rho^{\beta'})} |\mathcal{E}(\mathbf{X})||\nabla v_\theta(\mathbf{X})||\nabla u_\theta(\mathbf{X})|d\mathbf{X} \\ &\quad + \int_{\widetilde{\Psi}_0\setminus C(\rho^{\beta'})} |\mathcal{E}(\mathbf{X})||\nabla v_\theta(\mathbf{X})||\nabla u_\theta(\mathbf{X})|d\mathbf{X},\end{aligned} \tag{5.14}$$

where β' is a number to be determined, and for any $M>0$ we have adopted the notation $C(M) \equiv C_M(\mathbf{Q}_0)\cap\widetilde{\Psi}_0$.

Fix $k_0>1$ such that $\rho^{\beta'} \approx 2^{k_0+1}$ and write the first integral as

$$\begin{aligned}&\int_{C(\rho^{\beta'})} |\mathcal{E}(\mathbf{X})||\nabla u_\theta(\mathbf{X})||\nabla v_\theta(\mathbf{X})|d\mathbf{X} \\ &\quad\le \int_{C(8\rho)} |\mathcal{E}(\mathbf{X})||\nabla u_\theta(\mathbf{X})||\nabla v_\theta(\mathbf{X})|d\mathbf{X} \\ &\quad\quad + \sum_j^{k_0} \int_{C(2^{j+1}\rho)\setminus C(2^j\rho)} |\mathcal{E}(\mathbf{X})||\nabla u_\theta(\mathbf{X})||\nabla v_\theta(\mathbf{X})|d\mathbf{X} \\ &\quad\equiv I + \sum_j I_j.\end{aligned} \tag{5.15}$$

Let Λ be a *parabolic dyadic decomposition* (see *e.g.*, [26]) of $\Delta_{8\rho}$. That is, $\Delta_{8\rho}$ is decomposed in parabolic cubes in $\mathbb{R}^n$ of diameters that are dyadic fractions of 8ρ. Let $R(J)$ be the rectangle associated with $J\in\Lambda$ defined as

$$R(J) = \left\{\mathbf{X} = (\lambda,x,t)\in\widetilde{\Psi} : (x,t)\in J,\ \operatorname{diam}(J)/2 \le \lambda \le \operatorname{diam}(J)\right\}.$$

In particular $R(J)$ has a volume comparable to $(\operatorname{diam} J)^{n+2} \approx \delta(\mathbf{X})^{n+2}$, $\sigma(J) \approx (\operatorname{diam} J)^{n+1} \approx \delta(\mathbf{X})^{n+1}$, and $\delta(\mathbf{X}) \approx \operatorname{diam} J \approx \operatorname{diam} R(J)$ for $\mathbf{X}\in R(J)$. In all the previous definitions the diameter is defined using the parabolic metric, and cubes are parabolic cubes.

By Cauchy's inequality, Caccioppoli's inequality, and (5.5),

(5.16)
$$I \lesssim \sum_{J\in\Lambda} \sup_{R(J)} |\mathcal{E}| \Big(\int_{R(J)} |\nabla v_\theta(\mathbf{X})|^2 \delta(\mathbf{X})^n d\mathbf{X} \Big)^{1/2} \Big(\int_{R(J)} \frac{|\nabla u_\theta(\mathbf{X})|^2}{\delta(\mathbf{X})^n} d\mathbf{X} \Big)^{1/2}$$
$$\approx \sum_{J\in\Lambda} \sup_{R(J)} |\mathcal{E}| \Big(\int_{R(J)} \delta(\mathbf{X})^{n-2} |g_\theta(\widetilde{\mathcal{A}}_0, \mathbf{X})|^2 d\mathbf{X} \Big)^{1/2} \Big(\int_{R(J)} |\nabla u_\theta(\mathbf{X})|^2 \delta(\mathbf{X})^{-n} d\mathbf{X} \Big)^{1/2}.$$

Now we use the comparison between parabolic measure and Green's function (see [24, Lemma 2.8]) and the fact that

$$\sup_{R(J)} |\mathcal{E}| \lesssim \Big(\int_{R(J)} \big[\sup_{\mathbf{Z}\in C(\mathbf{X})} |\mathcal{E}(\mathbf{Z})| \big]^2 \frac{d\mathbf{X}}{\delta(\mathbf{X})^{n+2}} \Big)^{1/2},$$

and we obtain by (5.3)

$$I \lesssim \sum_{J\in\Lambda} \Big(\int_{R(J)} \frac{|\mathcal{A}(\mathbf{X})|}{\delta(\mathbf{X})^{n+2}} |H_\theta(\mathbf{X})|^2 d\mathbf{X} \Big)^{1/2} \Big(\int_{R(J)} |\nabla u_\theta(\mathbf{X})|^2 \delta(\mathbf{X})^{-n} d\mathbf{X} \Big)^{1/2} \sigma(J),$$

where
$$H_\theta(\mathbf{X}) = \frac{\omega_\theta(\Delta_{\delta(\mathbf{X})})}{\delta^{n+1}(\mathbf{X})}$$
and where $\Delta_{\delta(\mathbf{X})}$ is a surface cube centered at $\mathbf{Q}_0$ of diameter $\delta(\mathbf{X}) \approx \operatorname{diam} R(J)$.

Let

$$a_J = \Big(\int_{R(J)} \frac{|\mathcal{A}(\mathbf{X})|}{\delta(\mathbf{X})^{n+2}} |H_\theta(\mathbf{X})|^2 d\mathbf{X} \Big)^{1/2}, \quad b_J = \Big(\int_{R(J)} |\nabla u_\theta(\mathbf{X})|^2 \delta(\mathbf{X})^{-n} d\mathbf{X} \Big)^{1/2}.$$

Consider $F(\mathbf{Q}) = a_J \chi_J(\mathbf{Q})$ and $G(\mathbf{Q}) = b_J \chi_J(\mathbf{Q})$, functions defined for $\mathbf{Q} \in \widetilde{\Delta}_0$, taking values in the space of sequences in $\ell^2(\Lambda)$. In fact one has

$$\|F(\mathbf{Q})\|_{\ell^2(\Lambda)} \lesssim \Big(\int_{\Gamma_\alpha(\mathbf{Q})} \frac{|\mathcal{A}(\mathbf{X})|}{\delta(\mathbf{X})^{n+2}} |H_\theta(\mathbf{X})|^2 d\mathbf{X} \Big)^{1/2}, \quad \|G(\mathbf{Q})\|_{\ell^2(\Lambda)} \lesssim S_\alpha u_\theta(\mathbf{Q}).$$

Here $\alpha > 0$ is an appropriately chosen aperture.

Hence, by Cauchy's inequality and Fubini's theorem, since $\sigma(J) \approx \delta(\mathbf{X})^{n+1}$, we obtain

$$I \lesssim \sum_{J\in\Lambda} a_J b_J \sigma(J) \lesssim \int F(Q) G(Q) d\sigma(Q)$$
$$\lesssim \Big(\int_{\Psi(\Delta_{2\rho})} \frac{|\mathcal{A}(\mathbf{X})|}{\delta(\mathbf{X})} |H_\theta(\mathbf{X})|^2 d\mathbf{X} \Big)^{1/2} \Big(\int_{\Delta_{2\rho}} S_\alpha^2(u_\theta) d\sigma \Big)^{1/2}.$$

Now we continue observing that the non-tangential maximal function of H_θ is bounded by the Hardy–Littlewood maximal function of $(d\omega_\theta/d\sigma)\chi_{\Delta_\rho}$ with respect to σ. This Hardy–Littlewood maximal function has L^2 norm bounded by $\omega_\theta(\Delta_{2\rho})/\sigma(\Delta_{2\rho})^{1/2}$, hence we obtain, by a property of Carleson measures [29, p. 59],

$$I \lesssim \Big(\frac{\omega_\theta(\Delta_{2\rho})}{\sigma(\Delta_{2\rho})^{1/2}} \Big) \sup_{\substack{\mathbf{Q}\in S_{\widetilde{\Psi}} \\ s<\rho^{\beta'}}} \Big(\frac{1}{\sigma(\Delta_s(\mathbf{Q}))} \int_{\Psi_s(\mathbf{Q})} \frac{|\mathcal{A}(\mathbf{X})|}{\delta(\mathbf{X})} d\mathbf{X} \Big)^{1/2} \Big(\int_{\Delta_{2\rho}} S_\alpha^2(u_\theta) d\sigma \Big)^{1/2}.$$

To estimate the factor containing the area integral, we recall that $d\omega/d\sigma \in \mathrm{RH}^2$, and so by the area integral estimates in [24, Theorem 5.1],

$$\int_{\Delta_{2\rho}} S_\alpha^2(u_\theta) d\sigma \lesssim \int_{\Delta_{2\rho}} |h_\theta|^2 d\sigma.$$

And now it is not hard to see that

$$\Big(\int_{\Delta_{2\rho}} |h_\theta|^2 d\sigma \Big)^{1/2} \leq \frac{\sigma(\Delta_{2\rho})^{1/2}}{\omega_\theta(\Delta_{2\rho})}.$$

Therefore,

$$I \lesssim \sup_{\substack{\mathbf{Q}\in S_{\widetilde{\Psi}} \\ s<\rho^{\beta'}}} \Big(\frac{1}{\sigma(\Delta_s(\mathbf{Q}))} \int_{\Psi_s(\mathbf{Q})} \frac{|\mathcal{A}(\mathbf{X})|}{\delta(\mathbf{X})} d\mathbf{X} \Big)^{1/2}. \tag{5.17}$$

To estimate the terms I_j in (5.15), we let Λ_j denote a parabolic dyadic decomposition of $\Delta_{2^{j+1}\rho}(\mathbf{Q}_0) \setminus \Delta_{2^j\rho}(\mathbf{Q}_0)$. Since the set $\big(\Psi_{2^{j+1}\rho}(\mathbf{Q}_0) \setminus \Psi_{2^j\rho}(\mathbf{Q}_0)\big) \setminus \bigcup_{J\in\Lambda_j} R(J)$ is contained in a cylinder C_j whose diameter is proportional to its distance to the boundary and in turn proportional to $2^j\rho$,

$$\begin{aligned} I_j &\lesssim \sum_{J\in\Lambda_j} \int_{R(J)} |\mathcal{E}(\mathbf{X})||\nabla v_\theta(\mathbf{X})||\nabla u_\theta(\mathbf{X})| d\mathbf{X} + \int_{C_j} |\mathcal{E}(\mathbf{X})||\nabla v_\theta(\mathbf{X})||\nabla u_\theta(\mathbf{X})| d\mathbf{X} \\ &\equiv \sum_{J\in\Lambda_j} I_j(J) + II_j. \end{aligned}$$

We estimate each of the terms above independently.

To estimate the terms $I_j(J)$ we use the device used to handle (5.16) to obtain

$$\begin{aligned} I_j(J) &\lesssim \omega_\theta(J) \mathrm{diam}(J)^{-(2n+1)/2} \Big(\int_{R(J)} \frac{|\mathcal{E}(\mathbf{X})|^2}{\delta(\mathbf{X})} d\mathbf{X} \Big)^{1/2} \Big(\int_{R(J)} |\nabla u_\theta(\mathbf{X})|^2 d\mathbf{X} \Big)^{1/2} \\ &\lesssim \omega_\theta(J) \mathrm{diam}(J)^{-(2n+3)/2} \Big(\int_{R(J)} \frac{|\mathcal{E}(\mathbf{X})|^2}{\delta(\mathbf{X})} d\mathbf{X} \Big)^{1/2} \Big(\int_{\widetilde{R}(J)} |u_\theta(\mathbf{X})|^2 d\mathbf{X} \Big)^{1/2}, \end{aligned}$$

where $\widetilde{R}(J)$ is a small dilation of $R(J)$. By the same token

$$II_j \lesssim \omega_\theta(\Delta_{2^{j-1}\rho})\operatorname{diam}(C_j)^{-(2n+3)/2}\Big(\int_{\widetilde{C}_j} \frac{|\mathcal{E}(\mathbf{X})|^2}{\delta(\mathbf{X})} d\mathbf{X}\Big)^{1/2}\Big(\int_{\widetilde{C}_j} |u_\theta(\mathbf{X})|^2 d\mathbf{X}\Big)^{1/2}.$$

We can now apply Harnack's inequality and (5.7) in either term, obtaining for instance for $\mathbf{X} \in \widetilde{C}_j$

$$u(\mathbf{X}) \lesssim u\big(\overline{\mathcal{A}}_{2^j\rho}(\mathbf{Q}_0)\big) \lesssim \Big(\frac{\rho}{2^j\rho}\Big)^{\alpha} \frac{1}{\omega_\theta(\Delta_{2^j\rho})}.$$

Hence by the doubling property of ω

$$\begin{aligned}
II_j &\lesssim \frac{(2^j\rho)^{-(2n+3)/2}(2^j\rho)^{(n+2)/2}}{2^{\alpha j}}\Big(\int_{\widetilde{C}_j} \frac{|\mathcal{E}(\mathbf{X})|^2}{\delta(\mathbf{X})} d\mathbf{X}\Big)^{1/2} \\
&\lesssim \frac{1}{2^{\alpha j}}\Big(\frac{1}{\sigma(\pi(C_j))}\int_{\widetilde{C}_j} \frac{|\mathcal{E}(\mathbf{X})|^2}{\delta(\mathbf{X})} d\mathbf{X}\Big)^{1/2} \\
&\lesssim \frac{1}{2^{\alpha j}} \sup_{\substack{\mathbf{Q}\in S_{\widetilde{\Psi}} \\ s<\rho^{\beta'}}} \Big(\frac{1}{\sigma(\Delta_s(\mathbf{Q}))}\int_{\Psi_s(\mathbf{Q})} \frac{|\mathcal{E}(\mathbf{X})|^2}{\delta(\mathbf{X})} d\mathbf{X}\Big)^{1/2}.
\end{aligned} \tag{5.18}$$

Similarly,

$$I_j(J) \lesssim 2^{-\alpha j} \sup_{\substack{\mathbf{Q}\in S_{\widetilde{\Psi}} \\ s<\rho^{\beta'}}} \Big(\frac{1}{\sigma(\Delta_s(\mathbf{Q}))}\int_{\Psi_s(\mathbf{Q})} \frac{|\mathcal{E}(\mathbf{X})|^2}{\delta(\mathbf{X})} d\mathbf{X}\Big)^{1/2}. \tag{5.19}$$

Estimates (5.17), (5.18), and (5.19) clearly imply the desired bound for the first term in (5.14).

To handle the second term in (5.14) we first observe that $0 \le |v_\theta(\mathbf{X})| \lesssim 1$ and $\mathcal{O}(\mathbf{X}) \lesssim 1$. Hence, applying Caccioppoli's inequality, using that $u_\theta \equiv 0$ on $S_{\widetilde{\Psi}} \setminus \Delta_\rho(\mathbf{Q}_0)$, and by (5.7) we have that

$$\begin{aligned}
\Big(\int_{\widetilde{\Psi}\setminus B(\rho^\beta)} |\nabla u_\theta(\mathbf{X})|^2 d\mathbf{X}\Big)^{1/2} &\lesssim \frac{1}{\rho^\beta}\Big(\int_{C(2\rho^\beta)\setminus C(\rho^\beta/2)} |u_\theta(\mathbf{X})|^2 d\mathbf{X}\Big)^{1/2} \\
&\lesssim \rho^{-\beta(1-(n+1)/2)}\rho^{(1-\beta)\alpha}\frac{1}{\omega_r(\Delta_{\rho^\beta})}.
\end{aligned}$$

As in [9, p. 363], by a well-known property of $\mathrm{RH}_2(d\sigma)$ weights, $\omega_r(\Delta_{\rho^\beta}) \approx \rho^{\beta''}$, and we conclude that

$$\int_{\widetilde{\Psi}\setminus C(\rho^\beta)} |\mathcal{E}(\mathbf{X})||\nabla v_\theta(\mathbf{X})||\nabla u_r(\mathbf{X})| d\mathbf{X} \lesssim \rho^{\widetilde{\alpha}},$$

where $\widetilde{\alpha} > 0$ after choosing β small (see [9]). This concludes the proof of the proposition. ∎

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6 Arguments to Prove the Solvability Results

Proof of Theorem 3.2 By Theorem 4.3 it suffices to prove that any operator L as in (1.1) with coefficients satisfying (3.1) lies in the class $\mathcal{L}$.

Let $(Q,s) \in S_T$ and $r > 0$ such that $\Delta_r(Q,s) \subset S_T$. Suppose with no loss of generality that (Q,s) lies on the graph of the parabolic function ψ, supported on $S_T \cap \mathcal{Z}$, where $\mathcal{Z}$ is the local cylinder of constants A_1, r_0 corresponding to ψ.

Set $\widehat{A}(X,t) = (\widehat{a}_{i,j})_{i,j}$, where $\widehat{a}_{i,j} = a_{i,j} \circ \rho$ are the corresponding pull-back mapping of the coefficients $a_{i,j}$. We start by noting that if u is a weak solution to $Lu = 0$ on $\mathcal{Z}$, then $\widetilde{u} = u \circ \rho$ is a weak solution to

$$\widetilde{L}_1 \widetilde{u} \equiv \operatorname{div} \widetilde{A} \nabla \widetilde{u} + \vec{B} \cdot \nabla \widetilde{u} - \frac{\partial u}{\partial t} = 0 \tag{6.1}$$

where $\vec{B}(X,t)$ is the same as defined in Proposition 4.2, and $\widetilde{A} = (\widetilde{a}_{i,j})_{i,j}$ is a new matrix of coefficients given by

$$\widetilde{a}_{00} = \Big(\frac{1}{1 + \frac{\partial}{\partial\lambda} \mathbb{P}_{\gamma\lambda}\psi} \Big) \Big(\widehat{a}_{0,0} - 2 \sum_{j=1}^{n} \widehat{a}_{0,j} \frac{\partial}{\partial x_j} \mathbb{P}_{\gamma\lambda}\psi + \big\langle \widehat{A} \nabla \mathbb{P}_{\gamma\lambda}\psi, \nabla \mathbb{P}_{\gamma\lambda}\psi \big\rangle \Big), \tag{6.2}$$

$$\widetilde{a}_{jj} = \widehat{a}_{j,j} \Big(1 + \frac{\partial}{\partial\lambda} \mathbb{P}_{\gamma\lambda}\psi \Big),$$

$$\widetilde{a}_{0j} = \widetilde{a}_{j0} = \widehat{a}_{0,j} - \sum_{i=1}^{n} \widehat{a}_{i,j} \frac{\partial}{\partial x_i}, P_{\gamma\lambda}\psi, \quad 1 \le j \le n-1$$

$$\widetilde{a}_{i,j} = \widehat{a}_{i,j} \Big(1 + \frac{\partial}{\partial\lambda} \mathbb{P}_{\gamma\lambda}\psi \Big), \quad 1 \le i, j \le n-1, \tag{6.3}$$

Notice that the ellipticity condition (1.2) for $\widetilde{A}$ is actually a consequence of the corresponding one for A and of Lemma 4.1.

On the other hand, if the coefficients of $A(X,t) = (a_{i,j}(X,t))$ satisfy (3.1), then the coefficients of $\widetilde{A}$ satisfy the right estimates, as we outline now.

First note that for $(X,t) = (x_0, x, t) \in \Psi_r(Q,s)$, if we denote by $\widetilde{\delta}(X,t)$ the vertical distance from (X,t) to ∂S_T, one has for $r > 0$ sufficiently small

$$\delta(X,t) \approx \widetilde{\delta}(X,t) = |\psi(x,t) - x_0| \approx |\mathbb{P}_{\gamma\lambda}\psi(x,t), x, t) - x_0|$$

by the $\operatorname{Lip}(1, 1/2)$ property of ψ, and geometric considerations. Therefore, $\delta \circ \rho(\lambda, x, t) \approx \lambda$.

Now note that the left-hand side of (3.1) is larger than

$$\int_{\Psi_r(Q,s)} \Big[\delta(X,t) |\nabla A(X,t)|^2 + \delta(X,t)^3 \Big| \frac{\partial A}{\partial t}(X,t) \Big|^2 \Big] \, dX dt \gtrsim$$
$$\int_{\widetilde{\Psi}_r(\widetilde{Q},\widetilde{s})} \Big[\lambda |\nabla_{\lambda,x} (A \circ \rho)(\lambda, x, t)|^2 + \lambda^3 \Big| \frac{\partial (A \circ \rho)}{\partial t}(X,t) \Big|^2 \Big] \, J_\rho(\lambda, x, t) \, d\lambda dx dt.$$

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Here $(\widetilde{Q},\widetilde{s}) = \rho^{-1}(Q,s)$ and $\widetilde{\Psi}_r(\widetilde{Q},\widetilde{s}) = \widetilde{\Delta}_{\widetilde{N}r}(Q,s) \times (0,\widetilde{N}r)$, with $\widetilde{\Delta}_{\widetilde{N}r}(\widetilde{Q},\widetilde{s}) = \{(\widetilde{X},\widetilde{t}) : \|\widetilde{X} - \widetilde{Q}, \widetilde{x} - \widetilde{s}\| < \widetilde{N}r\}$, and where $0 < \widetilde{N} < 1$ depends only on dimension n and on the constant of ψ (as a parabolic Lipschitz function), and it makes possible that

$$\rho^{-1}\big(\Psi_r(Q,s)\big) \supseteq \widetilde{\Psi}_r(\widetilde{Q},\widetilde{s}).$$

Also $J_\rho = (1 + (\partial/\partial\lambda)\mathbb{P}_{\gamma\lambda})$ denotes the determinant of the Jacobian matrix of the mapping ρ.

Using (4.2) and the chain rule, one can conclude that the assumption (3.1) implies that the following estimates hold:

$$\int_{\widetilde{\Psi}_r(\widetilde{Q},\widetilde{s})} \lambda \Bigg[\Bigg| \nabla_{\lambda,x,t}\widehat{A} \cdot \begin{pmatrix} (1 + (\partial/\partial\lambda)\mathbb{P}_{\gamma\lambda}) \\ \vec{0} \\ 0 \end{pmatrix} \Bigg|^2 + \sum_{j=1}^{n} \Bigg| \nabla_{\lambda,x,t}\widehat{A} \cdot \begin{pmatrix} (\partial/\partial x_j)\mathbb{P}_{\gamma\lambda} \\ \vec{1}_j \\ 0 \end{pmatrix} \Bigg|^2 \Bigg] \, d\lambda dx dt \lesssim r^{n+1}, \tag{6.4}$$

$$\int_{\widetilde{\Psi}_r(\widetilde{Q},\widetilde{s})} \lambda^3 \Bigg[\Bigg| \nabla_{\lambda,x,t}\widehat{A} \cdot \begin{pmatrix} (\partial/\partial t)\mathbb{P}_{\gamma\lambda} \\ \vec{0} \\ 1 \end{pmatrix} \Bigg|^2 \Bigg] \, d\lambda dx dt \lesssim r^{n+1} \tag{6.5}$$

Here $\vec{0}$ denotes the $(n-1)$-dimensional zero vector, $\vec{1}_j$ denotes the $(n-1)$-dimensional vector with zeroes everywhere except in the j-th place, where there is a 1 $(1 \leq j \leq n-1)$.

Hence using (6.2)–(6.3) to evaluate $|\nabla\widetilde{A}(\lambda,x,t)|^2$ and $|\partial_t\widetilde{A}(\lambda,x,t)|^2$, one may use Lemma 4.1, (4.2) and the estimates in (6.4)–(6.5), to conclude (4.4)–(4.5) holds for the coefficients of $\widetilde{L}_1$ in (6.1). This suffices to prove Theorem 3.2 by Theorem 4.3. ∎

One can actually prove a similar theorem for non-divergence operators of the form

$$Hu = \sum_{i,j=1}^{n} a_{i,j}(X,t)\frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t}, \tag{6.6}$$

where the matrix is again symmetric and satisfies an ellipticity condition as in (1.2). In this case the solutions are taken as *strong solutions*, and assuming the smoothness of the coefficients $A(X,t)$, we can only use limiting arguments to make the same conclusions for $A(X,t) \in \mathrm{VMO}$ (see details *e.g.*, in [2]).

Theorem 6.1 *Fix $T > 0$, and let Ω be an infinite starlike parabolic cylinder. Let H be any operator as described in (6.6) defined over Ω_T. Suppose that there exists a constant $C > 0$ such that for every $(Q,s) \in S_T$ and every $r > 0$ such that $\Delta_r(Q,s) \subset S_T$,*

$$\int_{\Psi_r(Q)} \Big(\sup_{(Z,\tau)\in C(X,t)} \Big[\delta(Z,\tau)|\nabla A(Z,\tau)|^2 + \delta(Z,\tau)^3 \Big| \frac{\partial A}{\partial \tau}(Z,\tau) \Big|^2 \Big] \Big) \, dX dt < C\sigma\big(\Delta_r(Q)\big).$$

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Then $k = d\omega/d\sigma \in \mathrm{RH}_p(\Omega_T)$ *for some* $p > 1$.

The key observation is that a non-divergence operator of the form $\sum a_{i,j}\partial_{i,j}u$ can be written as a divergence form operator with drift terms $\operatorname{div}A\nabla u + \vec{B}\cdot\nabla u$, with $\vec{B}$ satisfying the adequate Carleson measure condition. To prove Theorem 6.1 one uses Theorem 3.2 and the following result proved in [17, Chapter III].

Theorem 6.2 *Let* $L_1 u = \operatorname{div}A\nabla u - \partial u/\partial t$ *and* $L_2 u = \operatorname{div}A\nabla u - \partial u/\partial t + \vec{B}\cdot\nabla u$ *be two operators defined on* Ω*, where* A *satisfies the ellipticity condition* (1.2) *and suppose that there exists* $C > 0$ *such that* $\vec{B}$ *satisfies*

$$\int_{\Psi_r(Q)}\Big(\sup_{(Y,s)\in C(X,t)}\delta(Y,s)|\vec{B}(Y,s)|^2\Big)\,dXdt < C\sigma\big(\Delta_r(Q)\big) \tag{6.7}$$

for every $r > 0$ *and* $(Q,s) \in S_T$ *such that* $\Delta_r(Q,s) \subset S_T$*. Then* $\omega_{L_1} \in A_\infty(S_T,\sigma)$ *implies* $\omega_{L_2} \in A_\infty(S_T,\sigma)$.

Sketch of proof of Theorem 3.3 It has been observed before that the parabolic version of Dahlberg–Kenig–Stein mapping is a bijection between $\widetilde{\mathbb{R}}^{n+1}_+$ and a basic parabolic Lipschitz domain $\Omega(\psi)$, where ψ is a function associated with a local cylinder defining Ω. Also the Jacobian of this mapping is $J_\rho \approx 1$ for an appropriate choice of the parameter γ in its definition.

It has also been observed that an operator L as decribed in Theorem 3.3 is mapped to an operator with drift terms as in (6.1), and in particular the drift coefficients $\vec{B}$ satisfy a Carleson measure condition as in (6.7). Hence, by Theorem 6.2, in order to prove the theorem it suffices to prove it for an operator as in (6.1), defined on $\widetilde{\mathbb{R}}^{n+1}_+$, without the drift term $\vec{B}$.

We assume then that the matrix $A(\lambda,x,t)$ is defined on $\widetilde{\mathbb{R}}^{n+1}_+$. Denote by $a(\lambda,x,t)$ any of its entries and define

$$a^*(\lambda,x,t) = \int \mathcal{P}_\lambda(\lambda-\varsigma, x-z, t-\tau)a(\varsigma,z,\tau)d\varsigma\, dzd\tau$$

where $\mathcal{P} \in C_0^\infty$ supported on $\Sigma^n = \{(\lambda,x,t) : \lambda^2 + |x|^2 + t^2 < 1\}$, with

$$\int \mathcal{P}(\lambda,x,t)d\lambda dxdt = 1$$

and $P_\lambda(\varsigma,z,\tau) = \lambda^{-n-2}\mathcal{P}(\varsigma\lambda^{-1}, z\lambda^{-1}, \tau\lambda^{-2})$.

By applying the gradient to $\mathcal{P}_\lambda$ and adding, then subtracting, the constant $\operatorname{avg}((Y,s);A)$, one can check that

$$\lambda|\nabla A^*(\lambda,z,\tau)|^2 + \lambda^3\Big|\frac{\partial A^*}{\partial\tau}(\lambda,z,\tau)\Big|^2 \lesssim \frac{\mathcal{O}(\lambda,z,\tau)}{\lambda},$$

and hence by (3.2) and Theorem 3.2 the caloric measure associated with the operator as in (1.1) with matrix $A^*(X,t)$ is in $A_\infty(d\sigma)$.

Notice that for $(\xi, Y, s) \in C(\lambda, x, t)$,

$$|a(\xi, Y, s) - a^*(\lambda, x, t)| \le \int \mathcal{P}_\lambda(\lambda - \varsigma, x - z, t - \tau)|a(\varsigma, z, \tau) - a(\xi, Y, s)|d\varsigma\, dz d\tau$$

and therefore by assumption (3.2), the perturbation result [24, Theorem 6.4] can be invoked. This implies the theorem. ■

A consequence of the proof just presented is a sort of stability property of the class $\mathcal{L}$. In fact the following corollary states that a (non-empty) sub-class of $\mathcal{L}$ is stable with respect to certain perturbations.

Corollary *Define $\mathcal{L}_0$ as the class of operators as described in (1.1) such that their coefficients satisfy (3.1). Let L_0, L_1 be operators as described in (1.1). Suppose that $L_0 \in \mathcal{L}_0$ and that L_1 is a perturbation of L_0 in the sense that (5.2) holds. Then $L_1 \in \mathcal{L}_0$.*

Theorem 3.3 has another consequence for non-divergence operators of the form (6.6). In fact the proof of the following theorem follows the lines of the proof of Theorem 3.3 sketched above, only this time using Theorem 6.1 and the perturbation result [1, Theorem 1.1].

Theorem 6.3 *Fix $T > 0$, and let Ω be an infinite starlike parabolic cylinder. Let H be an operator as described in (6.6) defined over Ω. Suppose that there exists a constant $C > 0$ such that for every $(Q, s) \in S_T$ and every $r > 0$ such that $\Delta_r(Q, s) \subset S_T$,*

$$\int_{\Psi_r(Q,s)} \frac{\mathcal{O}(X,t)}{\delta(X,t)} dX dt < C\sigma\big(\Delta_r(Q,s)\big).$$

Then $k = d\omega/d\sigma \in \mathrm{RH}_p(\Omega_T)$ for some $p > 1$.

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