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DEGENERATIONS OF ORBIFOLD CURVES AS NONCOMMUTATIVE VARIETIES

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Abstract. Boundary points on the moduli space of pointed curves corresponding to collisions of marked points have modular interpretations as degenerate curves. In this paper, we study degenerations of orbifold projective curves corresponding to collisions of stacky points from the point of view of noncommutative algebraic geometry.

§1. Introduction

A noncommutative projective variety in the sense of Artin–Zhang [5] is the Grothendieck category $\operatorname{Qgr} A$ obtained as the quotient of the category $\operatorname{Gr} A$ of graded right modules over a graded associative algebra A satisfying the condition χ_1 by the full subcategory $\operatorname{Tor} A$ consisting of torsion modules.

In [2], a class of AS-regular algebras determined by the quiver in Figure 1 describing noncommutative cubic surfaces are introduced. The moduli stack of this class of algebras is an open substack of the quotient of an affine space by a linear action of a torus. A choice of a stability condition gives an 8-dimensional smooth proper toric stack, which contains the moduli stack of smooth marked cubic surfaces as a locally closed substack. An interesting feature of this moduli stack is that Qgr of the algebra associated with the point on the boundary corresponding to a degenerate cubic surface with a rational double point is the category of quasi-coherent sheaves not on the singular cubic surface but on a noncommutative crepant resolution (which is derived-equivalent to the minimal resolution by the McKay correspondence), and hence is of finite homological dimension.

In this paper, we explore similar phenomena in dimension one. As a direct one-dimensional analog of (not necessarily commutative) del Pezzo surfaces, we consider orbifold projective lines studied in depth in [9]. Given a positive integer n, a sequence $\boldsymbol{r} = (r_1, \dots, r_n)$ of integers greater than 1, and a sequence

$$\boldsymbol{\lambda} = (\lambda_1 = \infty, \lambda_2 = 0, \lambda_3 = 1, \lambda_4, \dots, \lambda_n) \in \left(\left(\mathbb{P}^1\right)^n \setminus \Delta\right) \Big/ \mathrm{PGL}(2) \tag{1.1}$$

of pairwise distinct points on $\mathbb{P}^1$ where

$$\Delta := \left\{ (\lambda_i)_{i=1}^n \in \left(\mathbb{P}^1\right)^n \,\middle|\, \lambda_i = \lambda_j \text{ for some } i \neq j \right\} \tag{1.2}$$

is the big diagonal, an *orbifold projective line* is defined by

$$\mathbf{X} = \mathbf{X}_{\boldsymbol{r},\boldsymbol{\lambda}} := [(\operatorname{Spec} S \setminus \mathbf{0})/L^\vee], \tag{1.3}$$

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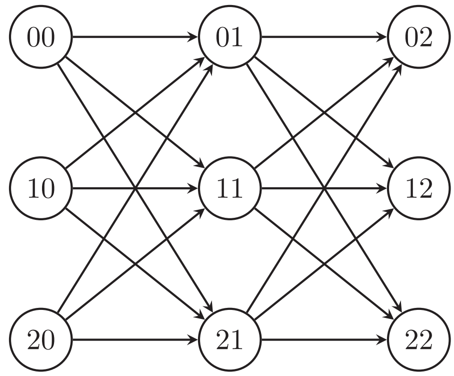

Figure 1.
The quiver describing noncommutative cubic surfaces.

where

$$S = S_{\boldsymbol{r},\boldsymbol{\lambda}} := \mathbf{k}[x_1,\dots,x_n]\big/(x_i^{r_i} - x_2^{r_2} + \lambda_i x_1^{r_1})_{i=3}^n \tag{1.4}$$

is the homogeneous coordinate ring of $\mathbf{X}$ and $L^\vee := \operatorname{Spec}\mathbf{k}[L]$ is the Cartier dual of the abelian group

$$L = L_{\boldsymbol{r}} := \mathbb{Z}\vec{c} \oplus \bigoplus_{i=1}^n \mathbb{Z}\vec{x}_i \Big/ \langle \vec{c} - r_i\vec{x}_i\rangle_{i=1}^n \tag{1.5}$$

of rank one. The coaction $S \to \mathbf{k}[L] \otimes S$ dual to the action $L^\vee \times \operatorname{Spec} S \to \operatorname{Spec} S$ is given by $x_i \mapsto \vec{x}_i \otimes x_i$ for any $i \in \{1,\dots,n\}$. The morphism $c\colon \mathbf{X} \to \mathbb{P}^1$ to the coarse moduli scheme is the r_ith root construction at λ_i in a neighborhood of $\lambda_i \in \mathbb{P}^1$ for each i, and the Picard group $\operatorname{Pic}\mathbf{X}$ can be identified with L in such a way that $\mathcal{O}_{\mathbf{X}}(\vec{x}_i)$ is the universal bundle associated with the root construction, so that $\mathcal{O}_{\mathbf{X}}(\vec{c}) = \mathcal{O}_{\mathbf{X}}(r_i\vec{x}_i) \simeq \mathcal{O}_{\mathbf{X}}(1) := c^*\mathcal{O}_{\mathbb{P}^1}(1)$. The dualizing element

$$\vec{\omega} := (n-2)\vec{c} - \vec{x}_1 - \cdots - \vec{x}_n \tag{1.6}$$

corresponds to the canonical sheaf of $\mathbf{X}$. We equip L with a structure of an ordered set so that $\vec{a} \preceq \vec{b}$ if

$$\vec{b} - \vec{a} \in L^{\geq 0} := \left\{ \sum_{i=1}^n a_i\vec{x}_i \,\middle|\, a_i \in \mathbb{Z}^{\geq 0} \right\}. \tag{1.7}$$

Note that one has $\vec{a} \preceq \vec{b}$ if and only if

$$\operatorname{Hom}\left(\mathcal{O}_{\mathbf{X}}(\vec{a}), \mathcal{O}_{\mathbf{X}}\left(\vec{b}\right)\right) \neq 0. \tag{1.8}$$

The endomorphism algebra of the full strong exceptional collection

$$(\mathcal{O}_{\mathbf{X}}(\vec{a}))_{0 \preceq \vec{a} \preceq \vec{c}} \tag{1.9}$$

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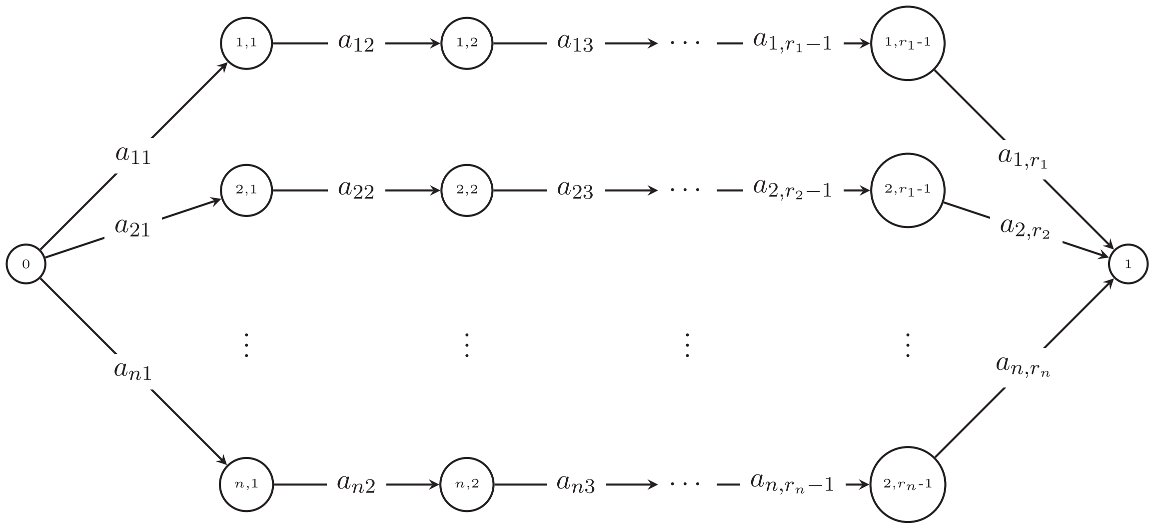

Figure 2.
The quiver describing orbifold projective lines.

of line bundles given in [9, Proposition 4.1] is described by the quiver $Q = Q_{\boldsymbol{r}}$ in Figure 2 with relations

$$\mathcal{I} = \mathcal{I}_{\boldsymbol{r},\boldsymbol{\lambda}} = \operatorname{span}\{a_{ir_i} \cdots a_{i1} - a_{1r_1} \cdots a_{11} + \lambda_i a_{2r_2} \cdots a_{21} \mid 3 \le i \le n\}, \tag{1.10}$$

which is a two-sided ideal of the path algebra $\mathbf{k}Q$ contained in $\mathbf{e}_1\mathbf{k}Q\mathbf{e}_0$ as a vector subspace.

From the ideal $\mathcal{I}$ of relations of the quiver Q, one can recover the ideal of relations of the homogeneous coordinate ring (1.4) as the ideal generated by the image of $\mathcal{I}$ by the linear map

$$\mathbf{e}_1\mathbf{Q}\mathbf{e}_0 \to \mathbf{k}[x_1, \ldots, x_n] \qquad a_{ir_i} \cdots a_{i1} \mapsto x_i^{r_i}, \quad i \in \{1, \ldots, n\}. \tag{1.11}$$

This allows the construction of $\mathbf{X}$ as the fine moduli stack of *refined* representations of the quiver

$$\Gamma = (Q, \mathcal{I}) \tag{1.12}$$

with relations [4].

The *moduli stack of relations*, discussed more generally in [3], is defined as the stack quotient

$$\mathscr{R} := \left[\operatorname{Gr}_{n-2}(\mathbf{e}_1\mathbf{k}Q\mathbf{e}_0) / \mathbb{G}_{\mathrm{m}}^{Q_1}\right] \tag{1.13}$$

of the Grassmannian of $(n-2)$-dimensional subspaces of the n-dimensional vector space $\mathbf{e}_1\mathbf{k}Q\mathbf{e}_0$ by the action of the torus $\mathbb{G}_{\mathrm{m}}^{Q_1}$ coming from the change of basis of $\mathbf{k}a$ for each arrow $a \in Q_1$.

REMARK 1.1. The stack $\mathscr{R}$ has a non-trivial group of generic automorphisms, which can be rigidified by taking the quotient by $(\mathbb{G}_{\mathrm{m}})^{n-1}$ instead of $\mathbb{G}_{\mathrm{m}}^{Q_1}$.

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If we write

$$\boldsymbol{r} = \Bigl(\overbrace{r_1,\dots,r_1}^{d_1},\overbrace{r_2,\dots,r_2}^{d_2},\dots,\overbrace{r_\ell,\dots,r_\ell}^{d_\ell}\Bigr) \tag{1.14}$$

for pairwise distinct integers $r_1,\dots,r_\ell$, then the automorphism group of the quiver Q is given by

$$\operatorname{Aut} Q \simeq \mathfrak{S}_{\boldsymbol{d}} \coloneqq \prod_{i=1}^{\ell} \mathfrak{S}_{d_i}. \tag{1.15}$$

The isomorphism class of an orbifold projective line $\mathbf{X}$ determines the full strong exceptional collection (1.9), which in turn determines the isomorphism class of the quiver $(Q,\mathcal{I})$ with relations. Therefore, the subspace $\mathcal{I} \subset \mathbf{e}_1 \mathbf{k} Q \mathbf{e}_0$ is determined by the isomorphism class of $\mathbf{X}$ up to the action of $\mathbb{G}_{\mathrm{m}}^{Q_1} \rtimes \operatorname{Aut} Q$.

In order to find a well-behaved open substack of the non-separated stack $\mathscr{R}$, replace $\operatorname{Gr}(n-2,n)$ with $\operatorname{Gr}(2,n)$ and use the Gelfand–MacPherson correspondence [10]

$$\operatorname{Gr}(2,n)/\!/(\mathbb{G}_{\mathrm{m}})^{n-1} \simeq (\mathrm{GL}_2 \backslash\!\backslash \operatorname{Mat}(2,n))/\!/(\mathbb{G}_{\mathrm{m}})^{n-1} \tag{1.16}$$
$$\simeq \mathrm{PGL}_2 \backslash\!\backslash (\operatorname{Mat}(2,n)/\!/(\mathbb{G}_{\mathrm{m}})^{n}) \tag{1.17}$$
$$\simeq \mathrm{PGL}_2 \backslash\!\backslash \left(\mathbb{P}^1\right)^n. \tag{1.18}$$

A stability condition in the sense of geometric invariant theory [11] is determined by a *stability parameter* $\chi = (\chi_i)_{i=1}^n \in \mathbb{Z}^n \cong \operatorname{Pic}\left(\mathbb{P}^1\right)^n$ in such a way that a point $\boldsymbol{\lambda} = (\lambda_i)_{i=1}^n \in \left(\mathbb{P}^1\right)^n$ is χ-*semistable* if for any $I \subset \{1,\dots,n\}$ such that $\lambda_i = \lambda_j$ for $i,j \in I$, one has

$$\sum_{i \in I} \chi_i \le \frac{1}{2} \sum_{i=1}^{n} \chi_i. \tag{1.19}$$

The point $\boldsymbol{\lambda}$ is χ-*stable* if the above inequality is strict. A stability parameter χ is *generic* if semistability implies stability. For a generic stability parameter, the open substack $\mathscr{R}^{\mathrm{s}} \subset \mathscr{R}$ consisting of stable points is a principal $B(\mathbb{G}_{\mathrm{m}})^{|Q_1|-n+1}$-bundle over a smooth projective scheme.

We now ask if the points on the boundary of $\mathscr{R}^{\mathrm{s}}$ (i.e., those which do not come from $\left(\mathbb{P}^1\right)^n \setminus \Delta$) have "geometric" interpretations.

To give a "commutative" answer to this problem, first note that the definitions of $S_{\boldsymbol{r},\boldsymbol{\lambda}}$ and $\mathbf{X}_{\boldsymbol{r},\boldsymbol{\lambda}}$ in (1.4) and (1.3) make sense for any $(\lambda_4,\dots,\lambda_n) \in \mathbb{A}^{n-3}$, and can be interpreted as the fine moduli stack of refined representations of Γ in (1.12). The resulting stack is described as follows:

THEOREM 1.2. *For any $(\lambda_4,\dots,\lambda_n) \in \mathbb{A}^{n-3}$, one has an isomorphism*

$$\mathbf{X}_{\boldsymbol{r},\boldsymbol{\lambda}} \simeq \mathbf{X}_{r_1,\lambda_1} \times_{\mathbb{P}^1} \mathbf{X}_{r_2,\lambda_2} \times_{\mathbb{P}^1} \cdots \times_{\mathbb{P}^1} \mathbf{X}_{r_n,\lambda_n} \tag{1.20}$$

of stacks.

To search for a "noncommutative" answer to the same problem, note that a stability condition in geometric invariant theory comes from a choice of a ray in the group of characters. In the world of stacks, we must specify not only a ray but a submonoid, since taking the Veronese subring changes Qgr in general. In the noncommutative world, we can

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choose a subset of the group of characters which may not be closed under addition (so that the resulting "noncommutative scheme" may not be a quotient by a group action).

Consider the $\mathbf{k}$-linear category $\mathsf{S} = \mathsf{S}_{\boldsymbol{r},\boldsymbol{\lambda}}$ whose set of objects is L and whose space of morphisms from $\vec{a}$ to $\vec{b}$ is the space $S_{\vec{b}-\vec{a}}$ of homogeneous elements of degree $\vec{b}-\vec{a}$. For a subset $K \subset L$ containing $\mathbb{Z}\vec{c}$ and invariant under translation by $\vec{c}$, we set

$$\mathsf{S}_K := (\text{the full subcategory of } \mathsf{S} \text{ consisting of } K). \tag{1.21}$$

A right S_K-module is a contravariant functor from S_K to $\operatorname{Mod}\mathbf{k}$. An S_K-module M is said to be *torsion* if for any $m \in M$, there exists $\vec{a} \in K$ such that $ms = 0$ for any $\vec{b} \in K$ satisfying $\vec{b} \succeq \vec{a}$ and any $s \in S_{\vec{b}}$. The quotient of the Grothendieck category $\operatorname{Gr}\mathsf{S}_K$ of right S_K-modules by the localizing subcategory $\operatorname{Tor}\mathsf{S}_K$ consisting of torsion modules will be denoted by $\operatorname{Qgr}\mathsf{S}_K$. As is common in noncommutative algebraic geometry (cf. e.g., [14, Section 3.5]), we write $\operatorname{Qcoh}\mathbf{X}_K := \operatorname{Qgr}\mathsf{S}_K$ and talk about quasi-coherent sheaves on $\mathbf{X}_K$, although the symbol $\mathbf{X}_K$ alone does not make any sense. We write the image in $\operatorname{Qcoh}\mathbf{X}_K$ of the projective module $\mathbf{e}_{\vec{a}}\mathsf{S}_K$ represented by the object $\vec{a} \in L$ as $\mathcal{O}_{\mathbf{X}_K}(\vec{a})$. For $K = L$, one has

$$\mathbf{X}_L \simeq \mathbf{X}, \tag{1.22}$$

and for $K = \mathbb{Z}\vec{c}$, one has

$$\mathbf{X}_{\mathbb{Z}\vec{c}} \simeq \mathbb{P}^1. \tag{1.23}$$

A flag $K' \subset K \subset L$ of subsets gives a full embedding $\mathsf{S}_{K'} \to \mathsf{S}_K$, the restriction $\operatorname{Gr}\mathsf{S}_K \to \operatorname{Gr}\mathsf{S}_{K'}$ along which induces a functor $\operatorname{Qgr}\mathsf{S}_K \to \operatorname{Qgr}\mathsf{S}_{K'}$ since a torsion module restricts to a torsion module. This functor is exact and can be regarded as the push-forward along the "noncommutative contraction" $\mathbf{X}_K \to \mathbf{X}_{K'}$.

Let $\varpi\colon \mathbf{X} \to \mathbb{P}^1$ be the noncommutative contraction corresponding to the inclusion $\mathbb{Z}\vec{c} \subset L$, and

$$\mathcal{A}_{\mathbf{X}_K} := \varpi_* \mathcal{E}nd\left(\bigoplus_{\vec{a}\in K_0} \mathcal{O}_{\mathbf{X}}(\vec{a})\right), \tag{1.24}$$

where

$$K_0 := \{\vec{a} \in K \mid 0 \preceq \vec{a} \prec \vec{c}\}. \tag{1.25}$$

We say that K_0 is a fundamental domain of the translation by $\vec{c}$ if the map

$$K_0 \times \mathbb{Z} \to K, \qquad (\vec{a}, m) \mapsto \vec{a} + m\vec{c} \tag{1.26}$$

is a bijection.

THEOREM 1.3. *If K_0 is a fundamental domain of the translation by $\vec{c}$, then one has an equivalence*

$$\operatorname{Qcoh}\mathbf{X}_K \simeq \operatorname{Qcoh}\mathcal{A}_{\mathbf{X}_K} \tag{1.27}$$

of categories.

Let us summarize what we have done so far. For any subset $K \subset L$ containing $\mathbb{Z}\vec{c}$ and invariant under translation by $\vec{c}$, we have constructed a noncommutative variety $\mathbf{X}_K$ and a sheaf of $\mathcal{O}_{\mathbb{P}^1}$-algebras $\mathcal{A}_{\mathbf{X}_K}$ on $\mathbb{P}^1$ satisfying (1.27). On the other hand, if $\lambda_i \neq \lambda_j$ for

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all $i \neq j$, then one has $D^b \operatorname{coh} X_{\boldsymbol{r},\boldsymbol{\lambda}}$ is equivalent to $D^b \operatorname{mod} \Gamma$ since the path algebra of Γ is isomorphic to the endomorphism algebra of the full strong exceptional collection (1.9). Now we ask if there is a choice of K such that the derived equivalence holds for any $\boldsymbol{\lambda}$. The answer is affirmative. Take

$$I = \{m\vec{x}_i \mid m \in \mathbb{Z} \text{ and } i \in \{1,\dots,n\}\} \tag{1.28}$$

and write

$$\mathbf{Y} := \mathbf{X}_I \tag{1.29}$$

(or $\mathbf{Y}_{\boldsymbol{r},\boldsymbol{\lambda}} := (\mathbf{X}_{\boldsymbol{r},\boldsymbol{\lambda}})_I$ when we want to make the dependence on $\boldsymbol{r}$ and $\boldsymbol{\lambda}$ explicit). This choice is motivated by the fact that for any $k \in \mathbb{Z}$, the set $\{\vec{a} \in I \mid k\vec{c} \preceq \vec{a} \preceq (k+1)\vec{c}\}$ can naturally be identified with the set of vertices of the quiver in Figure 2, and we have the following theorem.

THEOREM 1.4. *For any $\mathcal{I} \in \operatorname{Gr}_{n-2}(\mathbf{e}_1 \mathbf{k} Q \mathbf{e}_0)$, one has a quasi-equivalence*

$$D^b \operatorname{coh} \mathbf{Y} \simeq D^b \operatorname{mod} \Gamma \tag{1.30}$$

of dg categories, where Γ is defined in (1.12).

Therefore, there are two "geometric" interpretations of the points on the boundary of the compact moduli space of orbifold projective lines. One is "commutative" and given by the stack $\mathbf{X}_{\boldsymbol{r},\boldsymbol{\lambda}}$, and the other is "noncommutative" and given by the noncommutative variety $\mathbf{Y}_{\boldsymbol{r},\boldsymbol{\lambda}}$. Both are closely related to the quiver Γ with relations. The former is the fine moduli stack of refined representations of Γ, and the latter is derived equivalent to Γ. Moreover, any choice of a subset K of the Picard group L containing $\mathbb{Z}\vec{c}$ and invariant under translation by $\vec{c}$ produces a noncommutative variety $(\mathbf{X}_{\boldsymbol{r},\boldsymbol{\lambda}})_K$ with an equivalence (1.27), and $\mathbf{X}_{\boldsymbol{r},\boldsymbol{\lambda}}$ and $\mathbf{Y}_{\boldsymbol{r},\boldsymbol{\lambda}}$ are given by $K = L$ and $K = I$, respectively.

Now we ask if the constructions of $\mathbf{X}_{\boldsymbol{r},\boldsymbol{\lambda}}$ and $\mathbf{Y}_{\boldsymbol{r},\boldsymbol{\lambda}}$ can be generalized to curves which are not necessarily rational. The right-hand side of (1.20) is the fiber product in the category of stacks, and makes sense in much greater generality. With the fact that the orbifold projective line $\mathbf{X}_{\boldsymbol{r},\boldsymbol{\lambda}}$ is obtained from $\mathbb{P}^1$ by the root construction of orders $\boldsymbol{r}$ at the points $\boldsymbol{\lambda}$ in mind, let C be a smooth curve and $\mathscr{X}_r$ be the stack obtained from $C \times C$ by the rth root construction along the diagonal. For $i \in \{1,2\}$, let $\pi_i \colon \mathscr{X}_r \to C$ be the structure morphism $\mathscr{X}_r \to C \times C$ of the root construction composed with the ith projection. Now

$$\pi_{\boldsymbol{r}} \colon \mathscr{X}_{\boldsymbol{r}} := \mathscr{X}_{r_1} {}_{\pi_1}\!\times_{\pi_1} \mathscr{X}_{r_2} {}_{\pi_1}\!\times_{\pi_1} \cdots {}_{\pi_1}\!\times_{\pi_1} \mathscr{X}_{r_n} \xrightarrow{\pi_2 \times \cdots \times \pi_2} C^n \tag{1.31}$$

gives a flat (but non-smooth) family of proper orbifold curves, which restricts to a smooth family

$$\pi_{\boldsymbol{r}}|_{\pi_{\boldsymbol{r}}^{-1}(C^n \setminus \Delta)} \colon \pi_{\boldsymbol{r}}^{-1}(C^n \setminus \Delta) \to C^n \setminus \Delta \tag{1.32}$$

over the complement of the big diagonal. The fiber of $\pi_{\boldsymbol{r}}$ over $\boldsymbol{\lambda} \in C^n$ will be denoted by $\mathbf{X}_{\boldsymbol{r},\boldsymbol{\lambda}}$, generalizing the case where $C = \mathbb{P}^1$.

The construction of $\mathcal{A}_{\mathbf{Y}}$ also makes sense in this generality. Set

$$\mathcal{A}_{\mathscr{Y}_{\boldsymbol{r}}} := \varpi_* \mathcal{E}nd\left(\bigoplus_{\boldsymbol{a} \in I_0} \mathcal{O}_{\mathscr{X}_{\boldsymbol{r}}}(\boldsymbol{a})\right), \tag{1.33}$$

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where $\varpi\colon \mathscr{X}_{\boldsymbol{r}} \to C \times C^n$ is the structure morphism to the coarse moduli scheme,

$$I_0 \coloneqq \bigcup_{k=1}^{n} \{(a_i)_{i=1}^n \in \mathbb{Z}^n \mid a_i = 0 \text{ for } i \neq k \text{ and } 0 \leq a_k < r_k\}, \tag{1.34}$$

and

$$\mathcal{O}_{\mathscr{X}_{\boldsymbol{r}}}(\boldsymbol{a}) \coloneqq \mathcal{O}_{\mathscr{X}_{r_1}}(a_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathscr{X}_{r_n}}(a_n) \tag{1.35}$$

is the exterior tensor product of the a_ith tensor powers of the universal bundles on the root stacks $\mathscr{X}_{r_i}$. The symbol $\mathscr{Y}_{\boldsymbol{r}}$ alone does not make sense, and is meant to denote the family of noncommutative algebraic varieties over C^n such that $\operatorname{Qcoh}\mathscr{Y}_{\boldsymbol{r}} = \operatorname{Qcoh}\mathcal{A}_{\mathscr{Y}_{\boldsymbol{r}}}$. For $\boldsymbol{\lambda} \in C^n$, we write the restriction of $\mathcal{A}_{\mathscr{Y}_{\boldsymbol{r}}}$ to the fiber $C \times \boldsymbol{\lambda}$ over $\boldsymbol{\lambda}$ as $\mathcal{A}_{\mathbf{Y}_{\boldsymbol{r},\boldsymbol{\lambda}}}$. It is clear from (1.24) and (1.33) that this $\mathcal{A}_{\mathbf{Y}_{\boldsymbol{r},\boldsymbol{\lambda}}}$ is a generalization of that in (1.29) to the case where C may not be $\mathbb{P}^1$.

THEOREM 1.5. *For any smooth curve C, any positive integer n, and any $\boldsymbol{r} \in \left(\mathbb{Z}^{>1}\right)^n$, one has the following:*

1. *The sheaf $\mathcal{A}_{\mathscr{Y}_{\boldsymbol{r}}}$ of $\mathcal{O}_{C \times C^n}$-algebras is flat over C^n.*
2. *For any $\boldsymbol{\lambda} \in C^n \setminus \Delta$, one has an equivalence $\operatorname{Qcoh}\mathcal{A}_{\mathbf{Y}_{\boldsymbol{r},\boldsymbol{\lambda}}} \simeq \operatorname{Qcoh}\mathbf{X}_{\boldsymbol{r},\boldsymbol{\lambda}}$.*
3. *For any $\boldsymbol{\lambda} \in C^n$, the category $\operatorname{Qcoh}\mathcal{A}_{\mathbf{Y}_{\boldsymbol{r},\boldsymbol{\lambda}}}$ has finite homological dimension.*

Theorem 1.5 can be compared with the existence of a "degenerate" noncommutative cubic surface of finite homological dimension whose commutative counterpart is a singular cubic surface. Unlike a noncommutative crepant resolution of a quotient singularity, the stack $\mathbf{X}_{\boldsymbol{r},\boldsymbol{\lambda}}$ is singular whereas the "noncommutative contraction" $\mathbf{Y}_{\boldsymbol{r},\boldsymbol{\lambda}}$, which is "below" $\mathbf{X}_{\boldsymbol{r},\boldsymbol{\lambda}}$, is smooth.

The existence of distinct flat extensions $\operatorname{Qcoh}\mathscr{X}_{\boldsymbol{r}}$ and $\operatorname{Qcoh}\mathscr{Y}_{\boldsymbol{r}}$ of the family (1.32) of abelian categories over $C^n \setminus \Delta$ illustrates the non-separatedness of the "moduli space of abelian categories." When the coarse moduli space of $\mathbf{X}$ is $\mathbb{P}^1$, these extensions come from two subsets of $L \simeq \operatorname{Pic}\mathbf{X}$. While these subsets make sense only when the coarse moduli space of $\mathbf{X}$ is $\mathbb{P}^1$, there is a third choice which makes sense for any curve, namely, $\mathbb{Z}\vec{\omega} \subset L$ for the dualizing element $\vec{\omega}$. This choice is canonical, and leads to a modular compactification of the moduli stack of orbifold curves. This is a "purely commutative" story, which is discussed in a separate paper [1].

This paper is organized as follows: In Section 2, we prove Lemma 2.1, of which Theorem 1.2 is an immediate consequence. In Section 3, we discuss the sheaves of algebras $\mathcal{A}_{\mathbf{X}}$ and $\mathcal{A}_{\mathbf{Y}}$ in the case when C is an affine line to illustrate their definitions and to motivate the introduction of the path algebra of a quiver labeled by effective divisors in Section 4, which can describe $\mathcal{A}_{\mathbf{Y}}$ as shown in Section 5. Theorems 1.5, 1.3, and 1.4 are shown in Sections 6, 7, and 8, respectively.

1.1 Notations and conventions

We will work over an algebraically closed field $\mathbf{k}$ of characteristic zero throughout the paper. In particular, all schemes, stacks, and their isomorphisms are defined over $\mathbf{k}$, and all (dg) categories and functors are linear over $\mathbf{k}$.

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§2. Fiber products of global quotients

Lemma 2.1. *Let $\mathbb{Y}_i = [X_i/G_i]$ for $i = 0, \ldots, n$ be the stack quotients of schemes X_i by actions of algebraic groups G_i. Let further $X_i \to X_0$ for $i = 1, \ldots, n$ be morphisms of schemes that are intertwined by group homomorphisms $G_i \to G_0$ yielding morphisms of stacks $\mathbb{Y}_i \to \mathbb{Y}_0$. Then one has an isomorphism*

$$\mathbb{Y}_1 \times_{\mathbb{Y}_0} \cdots \times_{\mathbb{Y}_0} \mathbb{Y}_n \xrightarrow{\sim} \mathbb{Y} := [X/G] \tag{2.1}$$

of stacks where $X := X_1 \times_{X_0} \cdots \times_{X_0} X_n$ and $G := G_1 \times_{G_0} \cdots \times_{G_0} G_n$.

See, e.g., [13, Tag 003O] for the definition of fiber products.

Proof. We begin with defining the morphism (2.1) as a functor between categories fibered in groupoids over the category of $\mathbf{k}$-schemes.

To define the action of the functor on objects, fix a test scheme S. An object of the left-hand side of (2.1) over S is an n-tuple whose ith entry is as follows:

- For $i = 1, \ldots, n$ a principal G_i-bundle
$$P_i \to S. \tag{2.2}$$
- For $i = 1, \ldots, n$ a G_i-equivariant morphism
$$f_i \colon P_i \to X_i. \tag{2.3}$$
- An isomorphism of principal G_0-bundles
$$\varphi_{ij} \colon P_j \times^{G_j} G_0 \xrightarrow{\sim} P_i \times^{G_i} G_0 \tag{2.4}$$
for $i, j = 1, \ldots, n$ satisfying the following conditions.
$$\varphi_{ii} = \mathrm{id} \tag{2.5}$$
$$\varphi_{ij}\varphi_{jk}\varphi_{ki} = \mathrm{id} \tag{2.6}$$
$$f_i\varphi_{ij} = f_j. \tag{2.7}$$

Let $e_0 \in G_0(\mathbf{k})$ be the identity. Take P to be the limit of the diagram consisting of φ_{ij} for $i, j = 1, \ldots, n$ and $P_i \xrightarrow{\mathrm{id}_{P_i} \times e_0} P_i \times^{G_i} G_0$ for $i = 1, \ldots, n$. Then P admits a standard structure of a principal G-bundle over S together with a G-equivariant morphism

$$f \colon P \to X. \tag{2.8}$$

Thus we have obtained an object on the right-hand side of (2.1) over S.

We now move on to morphisms. A morphism

$$(P_i, f_i, \varphi_{ij}) \to \left(P_i', f_i', \varphi_{ij}'\right) \tag{2.9}$$

in the fiber category of the left-hand side of (2.1) over S is an n-tuple of G_i-bundle isomorphisms $(\psi_i \colon P_i \to P_i')_{i=1,\ldots,n}$ such that $f_i'\psi_i = f_i$.

If we let $f \colon P \to X$ and $f' \colon P' \to X$ be the images of the source and the target of the morphism (2.9) under (2.1), then the morphisms ψ_i induce an isomorphism of principal G-bundles $\psi \colon P \to P'$ satisfying $f'\psi = f$.

We now define a morphism which we will show is the inverse of (2.1), based on that the left-hand side of (2.1) is the limit of a diagram of stacks.

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In order to define the morphism $p_i\colon \mathbb{Y} \to \mathbb{Y}_i$ for $i = 1, \dots, n$, take an object of $\mathbb{Y}$ over S; i.e., a G-bundle $P \to S$ together with an G-equivariant morphism $f\colon P \to X$. To such data, via the projection $G \to G_i$, we associate the G_i-bundle $P_i := P \times^G G_i$ and the G_i-equivariant morphism defined as follows:

$$f_i\colon P_i \xrightarrow{f \times \mathrm{id}_{G_i}} X \times^G G_i \to X_i. \tag{2.10}$$

Thus we have obtained the map $\mathbb{Y}(S) \to \mathbb{Y}_i(S)$. It is straightforward to confirm that morphisms of the category $\mathbb{Y}$ naturally induce those of $\mathbb{Y}_i$, so we omit the details.

We also have to construct for $1 \le i \ne j \le n$ a natural isomorphism $q_i p_i \Rightarrow q_j p_j$. This is obtained from the canonical isomorphism

$$P_i \times^{G_i} G_0 \xrightarrow{\sim} P_j \times^{G_j} G_0. \tag{2.11}$$

It remains to show that the morphism of stacks we have just constructed is inverse to (2.1), which is straightforward and hence left to the reader. ∎

Proof of Theorem 1.2. For $i \in \{1, \dots, n\}$ one has $\mathbf{X}_{r_i,\lambda_i} \simeq \mathbb{P}(1, r_i) \simeq \left[\left(\mathbb{A}^2 \setminus \mathbf{0}\right)/\mathbb{G}_\mathrm{m}\right]$ where $\mathbb{G}_\mathrm{m}$ acts on $\mathbb{A}^2$ by $\mathbb{G}_\mathrm{m} \ni \alpha\colon (x,y) \mapsto (\alpha x, \alpha^{r_i} y)$. The morphism $\mathbf{X}_{r_i,\lambda_i} \to \mathbb{P}^1$ is given by $(x,y) \mapsto (y, x^{r_1})$ if $i = 1$ and $(x,y) \mapsto (x^{r_i} + \lambda_i y, y)$ if $i \in \{2, \dots, n\}$. Now we apply Lemma 2.1 to $X_i = \mathbb{A}^2 \setminus \mathbf{0}$ and $G_i = \mathbb{G}_\mathrm{m}$ for $i \in \{0, \dots, n\}$. The group $L^\vee$ is isomorphic to the fiber product of the morphisms $\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$, $\alpha \mapsto \alpha^{r_i}$ for $i \in \{1, \dots, n\}$. It remains to show

$$\left(\mathbb{A}^2 \setminus \mathbf{0}\right) \times_{\mathbb{A}^2 \setminus \mathbf{0}} \cdots \times_{\mathbb{A}^2 \setminus \mathbf{0}} \left(\mathbb{A}^2 \setminus \mathbf{0}\right) \simeq \operatorname{Spec} S \setminus \mathbf{0}, \tag{2.12}$$

where S is defined in (1.4). (2.12) follows from

$$\mathbb{A}^2 \times_{\mathbb{A}^2} \cdots \times_{\mathbb{A}^2} \mathbb{A}^2 \simeq \operatorname{Spec} S. \tag{2.13}$$

The left-hand side of (2.13) is the spectrum of

$$\mathbf{k}[x_1, y_1] \otimes_{\mathbf{k}[x_0,y_0]} \cdots \otimes_{\mathbf{k}[x_0,y_0]} \mathbf{k}[x_n, y_n], \tag{2.14}$$

which is isomorphic to the quotient of $\mathbf{k}[x_1, y_1, \dots, x_n, y_n]$ by the ideal

$$(x_1^{r_1} - y_2) + (y_1 - x_2^{r_2}) + (y_1 - (x_i^{r_i} + \lambda_i y_i), x_1^{r_1} - y_i)_{i=3}^n. \tag{2.15}$$

Note that this ideal also contains $y_i - x_2^{r_2}$ for $i \in \{3, \dots, n\}$. By eliminating all the y_i's, we see that this ring is isomorphic to S, and Theorem 1.2 is proved. ∎

§3. Sheaves of algebras on the affine line

In this section, we describe the sheaves of algebras $\mathcal{A}_\mathbf{X}$ and $\mathcal{A}_\mathbf{Y}$ for the toy model

$$C = \mathbb{A}^1_z := \operatorname{Spec} \mathbf{k}[z] \tag{3.1}$$

to illustrate their definitions given in Section 1 and to motivate the construction in Section 4. Choose $r \in \mathbb{Z}^{>1}$ and $\lambda \in \mathbf{k}$. Consider the group action $\boldsymbol{\mu}_r \curvearrowright \mathbb{A}^1_x$ defined by

$$\boldsymbol{\mu}_r \ni \zeta\colon x \mapsto \zeta(x - \lambda) + \lambda \tag{3.2}$$

and let

$$\mathbf{X}_{r,\lambda} = [\operatorname{Spec} \mathbf{k}[x]/\boldsymbol{\mu}_r], \tag{3.3}$$

$$\mathcal{A}_\mathbf{X} = \mathcal{A}_\mathbf{Y} = \mathbf{k}[x] \rtimes \boldsymbol{\mu}_r. \tag{3.4}$$

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The right-hand side of (3.4) is the skew group ring with respect to the action (3.2). The coarse moduli of $\mathbf{X}_{r,\lambda}$ is as follows, which at the same time depicts $\mathbf{X}_{r,\lambda}$ as the root stack of $\operatorname{Spec}\mathbf{k}[z]$ ramified of order r at the origin.

$$\varpi\colon \mathbf{X}_{r,\lambda} \to \operatorname{Spec}\mathbf{k}[z], \quad z \mapsto (x-\lambda)^r. \tag{3.5}$$

Our assumption that $\mathbf{k}$ is an algebraically closed field of characteristic zero implies the isomorphism $\mathbf{k}[\boldsymbol{\mu}_r] \simeq \mathbf{k}^{\times r}$ as $\mathbf{k}$-algebras and

$$\mathbf{k}[x] \rtimes \boldsymbol{\mu}_r \simeq \mathbf{k}[x]^{\oplus r} \tag{3.6}$$

as $\mathbf{k}[x]$-modules.

One has $\mathbf{k}[x] \simeq \mathbf{k}[z]^{\oplus r}$ as $\mathbf{k}[z]$-modules, so that

$$\mathbf{k}[x] \rtimes \boldsymbol{\mu}_r \simeq \mathbf{k}[z]^{\oplus r^2} \tag{3.7}$$

as $\mathbf{k}[z]$-modules. If we write the universal bundle on $\mathbf{X}_{r,\lambda}$ as the root stack as $\mathcal{O}(1)$, then one has

$$\mathbf{k}[x] \rtimes \boldsymbol{\mu}_r \simeq \varpi_* \mathcal{E}nd\left(\bigoplus_{a=0}^{r-1} \mathcal{O}(a)\right). \tag{3.8}$$

The $\mathbf{k}$-algebra $\mathbf{k}[x] \rtimes \boldsymbol{\mu}_r$ is described by a quiver such that vertices are the idempotents $\mathbf{e}_i \in \mathbf{k}[\boldsymbol{\mu}_r]$ for $i \in \operatorname{Hom}(\boldsymbol{\mu}_r, \mathbb{G}_\mathrm{m}) \simeq \mathbb{Z}/r\mathbb{Z}$ and there is one arrow from $\mathbf{e}_i$ to $\mathbf{e}_{i+1}$ where $1 := \deg(x-\lambda) \in \operatorname{Hom}(\boldsymbol{\mu}_r, \mathbb{G}_\mathrm{m})$.

For $n \geq 1$ the $\mathbf{k}$-algebra

$$\mathcal{A}_\mathbf{X} \simeq \mathcal{A}_{r_1,\lambda_1} \otimes_{\mathbf{k}[z]} \cdots \otimes_{\mathbf{k}[z]} \mathcal{A}_{r_n,\lambda_n} \tag{3.9}$$

is described by a quiver with vertices $\mathbf{e}_{\boldsymbol{i}}$ for $\boldsymbol{i} \in \prod_{k=1}^n \mathbb{Z}/r_k\mathbb{Z}$ and arrows $a_{\boldsymbol{i},j}$ for $(\boldsymbol{i},j) \in \prod_{k=1}^n \mathbb{Z}/r_k\mathbb{Z} \times \{1,\dots,n\}$, equipped with relations. The arrow $a_{\boldsymbol{i},j}$ goes from $\mathbf{e}_{\boldsymbol{i}}$ to $\mathbf{e}_{\boldsymbol{i}'}$ where $\boldsymbol{i}'$ is obtained from $\boldsymbol{i}$ by increasing the jth component by 1. The algebra $\mathcal{A}_\mathbf{Y}$ is described by the full subquiver consisting of vertices $\mathbf{e}_{\boldsymbol{i}}$ where $\boldsymbol{i} = (i_1,\dots,i_n)$ runs over the subset of $\prod_{k=1}^n \mathbb{Z}/r_k\mathbb{Z}$ such that i_k is non-zero for at most one $k \in \{1,\dots,n\}$. In order to describe $\mathcal{A}_\mathbf{Y}$ not as a $\mathbf{k}$-algebra but as a $\mathbf{k}[z]$-algebra (or a sheaf of $\mathcal{O}_C$-modules), we introduce the notion of the *path algebra of a* $\operatorname{Div}_{\text{eff}} C$*-labeled quiver* in Section 4 below.

§4. Path algebras of quivers with arrows labeled by effective divisors

A *quiver* $Q = (Q_0, Q_1, s, t)$ consists of a set Q_0 of *vertices*, a set Q_1 of *arrows*, and two maps $s, t\colon Q_1 \to Q_0$ called the *source* and the *target*. The quiver Q is said to be *finite* if the sets Q_0 and Q_1 are finite. The *path category* of the quiver is the category with objects Q_0 freely generated by Q_1. A *path* is a morphism of the path category. The set of paths is denoted by $\mathcal{P}$.

We consider only finite quivers unless otherwise specified.

Definition 4.1. A *labeling* of a quiver $Q = (Q_0, Q_1, s, t)$ by a set S is a map

$$D_\bullet\colon Q_1 \to S, \qquad \rho \mapsto D_\rho \tag{4.1}$$

of sets.

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If S is a monoid, then an S-labeling $D_\bullet$ induces a functor from the path category of Q to the category with one object and the set S of morphisms. The resulting map $\mathcal{P} \to S$ of sets will be denoted by $D_\bullet$ again by abuse of notation.

We henceforth restrict ourselves to labeling by the monoid $G = \operatorname{Div}_{\mathrm{eff}} C$ of effective Cartier divisors on a scheme C.

DEFINITION 4.2. A *cycle* is a non-identity endomorphism of an object of the path category, i.e., a product $a_n \cdots a_2 a_1$ of arrows $a_i \in Q_1$ such that $s(a_{i+1}) = t(a_i)$ for $i = 1, \ldots, n-1$ and $s(a_1) = t(a_n)$. It is *simple* if $s(a_1), s(a_2), \ldots, s(a_n)$ are mutually distinct elements of Q_0.

The set $\mathcal{P}$ of paths is partitioned into the disjoint union of the set $\mathcal{P}^a$ of acyclic (cycle-free) paths and the set $\mathcal{P}^c$ of paths containing at least one cycle;

$$\mathcal{P} = \mathcal{P}^a \sqcup \mathcal{P}^c. \tag{4.2}$$

Given a simple cycle ρ, permuting the arrows cyclically naturally gives another simple cycle.

DEFINITION 4.3. The *path algebra* of a $\operatorname{Div}_{\mathrm{eff}} C$-labeled quiver $\mathcal{Q} = (Q, D_\bullet)$ is the sheaf

$$\mathcal{O}_C \mathcal{Q} := \left(\bigoplus_{\rho \in \mathcal{P}} \mathcal{O}_C(-D_\rho)\rho \right) \Big/ \mathcal{I} \tag{4.3}$$

of $\mathcal{O}_C$-algebras, where the multiplication is given by the concatenation of paths, and the ideal $\mathcal{I}$ is generated by the following relations: For any vertex $v \in Q_0$ and any simple cycle ρ at v, we identify $\mathcal{O}_C(-D_\rho)\rho$ with $\mathcal{O}_C(-D_\rho)\mathbf{e}_v \subseteq \mathcal{O}\mathbf{e}_v$, where $\mathbf{e}_v$ is the trivial path at v.

Since any simple cycle can be removed by the relation $\mathcal{I}$ in (4.3), the inclusion and the projection gives a surjection

$$\bigoplus_{\rho \in \mathcal{P}^a} \mathcal{O}_C(-D_\rho)\rho \to \mathcal{O}_C \mathcal{Q} \tag{4.4}$$

of $\mathcal{O}_C$-modules. The assumption that a quiver is finite implies that the set $\mathcal{P}^a$ and hence the sheaf $\mathcal{O}\mathcal{Q}$ of $\mathcal{O}_C$-algebras is also finite.

§5. The labeled quiver describing $\mathcal{A}_{\mathbf{Y}}$

Fix a smooth curve C, a positive integer n, a sequence $\boldsymbol{r} = (r_i)_{i=1}^n$ of positive integers, and a sequence $\boldsymbol{\lambda} = (\lambda_i)_{i=1}^n$ of points on C. Let $\mathcal{A}_{\mathbf{Y}} = \mathcal{A}_{\mathbf{Y}_{\boldsymbol{r},\boldsymbol{\lambda}}}$ be the fiber over $\boldsymbol{\lambda} \in C^n$ of the family (1.33) of sheaves of $\mathcal{O}_C$-algebras. Let further $\mathcal{Q}$ be the quiver obtained from the quiver in Figure 2 by identifying the leftmost vertex 0 with the rightmost vertex 1, equipped with the $\operatorname{Div}_{\mathrm{eff}}(C)$-labeling defined by

$$D_{a_{ij}} = \begin{cases} \lambda_i & j = r_i, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Note that for each $1 \le i \le n$ and $1 \le j \le r_i$ there is a standard isomorphism of line bundles

$$\mathcal{O}_C a_{ij} \xrightarrow{\sim} \varpi_* \mathcal{O}(\vec{x}_i) \overset{\text{canonical}}{\simeq} \varpi_* \mathcal{H}om(\mathcal{O}((j-1)\vec{x}_i), \mathcal{O}(j\vec{x}_i)) \tag{5.2}$$

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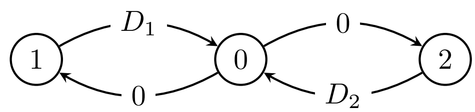
which sends a_{ij} to the tautological section of the line bundle $\mathcal{O}(\vec{x}_i)$. Lemma 5.1 below is an immediate consequence of the definition of $\mathcal{O}_C\mathcal{Q}$ in Definition 4.3.

LEMMA 5.1. *The isomorphisms (5.2) induce the following isomorphism of $\mathcal{O}_C$-algebras.*

$$\mathcal{O}_C\mathcal{Q} \xrightarrow{\sim} \mathcal{A}_{\mathbf{Y}}. \tag{5.3}$$

EXAMPLE 5.2. To illustrate the above constructions, we give an example of a family of algebras over $\mathbb{P}^1$ parametrized by the points of $\mathbb{A}^1$. Consider the labeled quiver $\mathcal{Q}$ as in (5.4), where $D_1, D_2 \in \mathrm{Div}_{\mathrm{eff}}\left(\mathbb{P}^1_{[u_0:u_1]} \times \mathbb{A}^1_\lambda\right)$ are defined by $s_1 := u_0 - \lambda u_1, s_2 := u_0 - 2\lambda u_1 \in H^0\left(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{A}^1}\right)$.



(5.4)

The path algebra $\mathcal{O}\mathcal{Q}$ can be described as the matrix algebra

$$\begin{bmatrix} \mathcal{O} & \mathcal{O}(-D_1) & \mathcal{O}(-D_2) \\ \mathcal{O} & \mathcal{O} & \mathcal{O}(-D_2) \\ \mathcal{O} & \mathcal{O}(-D_1) & \mathcal{O} \end{bmatrix}, \tag{5.5}$$

which gives a flat family of finite $\mathbb{P}^1_{[u_0:u_1]}$-algebras over $\mathbb{A}^1_\lambda$. If $\lambda \neq 0$, then D_1 and D_2 give distinct points on $\mathbb{P}^1$, so that $\mathrm{coh}\,\mathcal{O}\mathcal{Q}$ is equivalent to the category of coherent sheaves on a smooth rational orbifold projective curve with two stacky points of order 2. If $\lambda = 0$, then both D_1 and D_2 gives the point $p = [0:1] \in \mathbb{P}^1_{[u_0:u_1]}$. The resulting path algebra

$$\begin{bmatrix} \mathcal{O} & \mathcal{O}(-p) & \mathcal{O}(-p) \\ \mathcal{O} & \mathcal{O} & \mathcal{O}(-p) \\ \mathcal{O} & \mathcal{O}(-p) & \mathcal{O} \end{bmatrix} \tag{5.6}$$

has a right module $\mathbf{e}_0\mathcal{O}_p$ of homological dimension two by Proposition 6.9.

§6. Homological dimension of the path algebra

DEFINITION 6.1. A quiver is said to *have transverse cycles* if any pair of simple cycles intersect either at most one vertex or are cyclic permutations of each other.

The following result is well-known. Lacking a good reference, we provide the short proof.

PROPOSITION 6.2. *If $\mathcal{Q} = (Q, D_\bullet)$ is a $\mathrm{Div}_{\mathrm{eff}}\,C$-labeled quiver as in Definition 4.3 such that Q has transverse cycles, then the morphism (4.4) of $\mathcal{O}_C$-modules is an isomorphism.*

Proof. First we study the map at the generic point of the curve C. For this, let $\mathcal{K}$ be the function field of C. The ideal $\mathcal{I}_\mathcal{K} \triangleleft \mathcal{K}\mathcal{Q}$ of relations at the generic point is generated by elements of the form $\rho - \mathbf{e}_v$ where ρ ranges over all simple cycles and v is the starting vertex of ρ. Given any path $\rho \in \mathcal{P}^c$, we let $\bar{\rho}$ be a cycle-free path obtained from ρ by contracting all simple cycles so that $\rho \equiv \bar{\rho}$ modulo $\mathcal{I}_\mathcal{K}$. Furthermore, our assumption that Q has transverse cycles ensures that $\bar{\rho}$ is uniquely determined by ρ. In fact, to see the elements $\rho - \bar{\rho}$ form a $\mathcal{K}$-basis for $\mathcal{I}_\mathcal{K}$, we use Bergman's diamond lemma as follows (see [6, Theorem 1.2] for its statement and associated terminology. Generalization to path algebras has been settled in, say, [8, Section 2.2]). We view $\mathcal{K}\mathcal{Q}$ as an algebra over the semi-simple algebra $\mathcal{K}^{Q_0}$ freely

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generated by the bimodule $\mathcal{K}Q_1$. We partially order monomials in edges by their length. The reduction system we use replaces each monomial of edges $a_1a_2 \dots a_l$ corresponding to a simple cycle ρ beginning at v with $\mathbf{e}_v$. Furthermore, given $a \in Q_1$, it replaces $\mathbf{e}_v a$ and $a\mathbf{e}_w$ with a whenever v is the target of a and w is its source, and to 0 otherwise. We need to check the overlap $m = a_i \dots a_l a_1 \dots a_l$ which can be reduced using ρ to give $a_i \dots a_l \mathbf{e}_v = a_i \dots a_l$. If w is the source of a_i, then the monomial m can also be reduced using the cycle $a_i \dots a_l a_1 \dots a_{i-1}$ to $\mathbf{e}_w a_i \dots a_l = a_i \dots a_l$. The two reductions are the same so the overlap ambiguity is resolved. Now the transverse cycle condition ensures that the only other overlaps to check involve idempotents $\mathbf{e}_v$, and these are easily resolved. This proves Proposition 6.2 at the generic point. As the source of the map (4.4) is torsion free, this already implies the assertion. ∎

Given a simple cycle ρ on a quiver Q, we can construct a new quiver Q' by *contracting the cycle* ρ as follows. The set of vertices is defined by $Q'_0 = Q_0/\sim$, where $\sim$ is the equivalence relation which identifies all vertices of ρ but leaves all other vertices distinct. The set of arrows Q'_1 is the subset of Q_1 consisting of arrows not contained in ρ. A labeling $D_\bullet : Q_1 \to S$ restricts to a labeling $D'_\bullet := D_\bullet|_{Q'_1}$, and we write $\mathcal{Q}' = (Q', D'_\bullet)$ for $\mathcal{Q} = (Q, D_\bullet)$.

PROPOSITION 6.3. *If $\mathcal{Q}$ is a $\mathrm{Div}_{\mathrm{eff}}\, C$-labeled quiver having transverse cycles and $\mathcal{Q}'$ is the $\mathrm{Div}_{\mathrm{eff}}\, C$-labeled quiver obtained by contracting a simple cycle ρ with $D_\rho = 0$, then one has an equivalence*

$$\mathrm{Qcoh}\, \mathcal{O}_C \mathcal{Q} \simeq \mathrm{Qcoh}\, \mathcal{O}_C \mathcal{Q}' \tag{6.1}$$

of categories.

Proof. Let $v, w \in Q_0$ and consider the local projectives $P_v = \mathbf{e}_v \mathcal{O}_C \mathcal{Q}$ and $P_w = \mathbf{e}_w \mathcal{O}_C \mathcal{Q}$. If v, w are in ρ, as $D_\rho = 0$, multiplication by the "arc" of ρ from the vertex w to v induces an isomorphism $P_w \xrightarrow{\sim} P_v$. Thus the direct sum $P = \oplus_v P_v$ over all vertices of Q outside ρ and one vertex of ρ gives a local progenerator of $\mathcal{O}_C \mathcal{Q}$.

It thus suffices to show that $\mathcal{O}_C \mathcal{Q}' \simeq \mathcal{E}nd_C\, P$. From Proposition 6.2, we know that for each pair (v, w) of vertices of Q

$$\mathcal{H}om_{\mathcal{O}_C \mathcal{Q}}(P_w, P_v) = \mathbf{e}_v \mathcal{O}_C \mathcal{Q} \mathbf{e}_w = \bigoplus_\gamma \mathcal{O}_C(-D_\gamma)\gamma, \tag{6.2}$$

where γ runs over all cycle-free paths from w to v. Since Q has transverse cycles, the path of Q' which is obtained from an cycle-free path of Q in the obvious manner is again cycle-free. Moreover this yields a bijection between cycle-free paths from w to v and cycle-free paths in Q' from the image of w to the image of v. The assumption $D_\rho = 0$ ensures that this bijection respects the labeling, and Proposition 6.3 is proved. ∎

DEFINITION 6.4. A $\mathrm{Div}_{\mathrm{eff}}\, C$-labeling $D_\bullet$ of a quiver $\mathcal{Q}$ is said to be *reduced* if for every simple cycle ρ on Q, the divisor D_ρ is reduced.

Lemma 6.5 below is an immediate consequence of the definition of having transverse cycles.

LEMMA 6.5. *If Q has transverse cycles, then any two simple cycles which intersect in more than one vertex are cyclic permutations of each other.*

The proof of Lemma 6.6 below is straightforward.

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LEMMA 6.6. *Let ρ be a simple cycle on $\mathcal{Q}$ with $D_\rho = 0$ and $\mathcal{Q}'$ be the labeled quiver obtained by contracting ρ. If Q has transverse cycles, then so does Q'. If $D_\bullet$ is reduced, then so is $D'_\bullet$.*

The condition of transverse cycles has the following nice consequence.

LEMMA 6.7. *For a $\operatorname{Div}_{\mathrm{eff}} C$-labeled quiver $\mathcal{Q}$ with transverse cycles and an arrow $a \in Q_1$, the left multiplication by a map $\mathbf{e}_{s(a)}\mathcal{K}\mathcal{Q} \xrightarrow{a\cdot} \mathbf{e}_{t(a)}\mathcal{K}\mathcal{Q}$ over the function field $\mathcal{K}$ of C is injective.*

Proof. Suppose to the contrary that $a\sum_i r_i\rho_i = 0$, where $r_i \in \mathcal{K}^\times$ and ρ_i are distinct cycle-free paths ending at $s(a)$. If $a\rho_i$ are cycle-free for all i, then the assertion is obvious. For the contrary, suppose that $a\rho_i$ contains a cycle for some i. By the transversality assumption there exists a unique simple cycle $\gamma = a\gamma'$ at the vertex $t(a)$. Let ρ'_i be the unique acyclic path such that $\rho_i = \gamma'\rho'_i$. Now

$$0 = a\sum_i r_i\rho_i = \sum_{a\rho_i \in \mathcal{P}^c} r_i a\rho_i + \sum_{a\rho_i \in \mathcal{P}^a} r_i a\rho_i = \sum_{a\rho_i \in \mathcal{P}^c} r_i \mathbf{e}_{t_a}\rho'_i + \sum_{a\rho_i \in \mathcal{P}^a} r_i a\rho_i. \tag{6.3}$$

As ρ'_i for those i such that $a\rho_i \in \mathcal{P}^a$ are all distinct and do not contain the edge a, the equality (6.3) implies $r_i = 0$ for all i. ∎

Theorem 6.8 below is a consequence of Proposition 6.9 and Proposition 6.10

THEOREM 6.8. *The path algebra $\mathcal{O}_C\mathcal{Q}$ of a reduced $\operatorname{Div}_{\mathrm{eff}} C$-labeled quiver $\mathcal{Q}$ on a smooth curve C with transverse cycles has homological dimension at most two.*

PROPOSITION 6.9. *Theorem 6.8 holds if C is the spectrum of a discrete valuation ring.*

Proof. Let $\mathcal{O}$ be a discrete valuation ring with the closed point p and a uniformizing parameter t. Each vertex $v \in Q_0$ gives rise to the indecomposable projective module $P_v = \mathbf{e}_v\mathcal{O}\mathcal{Q}$ and the simple module $S_v = P_v/\operatorname{rad} P_v$. It suffices to show that $\operatorname{pd} S_v \le 2$.

Suppose first that there are no cycles through v. Then Proposition 6.2 and Lemma 6.7 imply that

$$0 \to \bigoplus_{t(a)=v} P_{s(a)} \xrightarrow{\binom{-t}{a}} \bigoplus_{t(a)=v} P_{s(a)} \oplus P_v \xrightarrow{(a\ t)} P_v \to S_v \to 0 \tag{6.4}$$

gives a projective resolution of $S_v = \mathbf{e}_v\mathcal{O}/(t)$, so that $\operatorname{pd} S_v \le 2$ in this case.

Suppose now that there exists a simple cycle ρ starting at v. If $D_\rho = 0$, then we may contract ρ to obtain a new $\operatorname{Div}_{\mathrm{eff}} C$-labeled quiver, whose path algebra is Morita equivalent to the original one by Proposition 6.3, and which still satisfies the hypotheses of Theorem 6.8 by Lemma 6.6. By repeating this operation if necessary, we may assume that there exist exactly $r+1$ simple cycles $\rho_0, \dots, \rho_r$ starting at v and that $D_{\rho_i} = p$ for all i. Lemma 6.5 implies that ρ_i and ρ_j for $i \ne j$ intersect only at the vertex v. Set

$$\rho_i = a_i\rho'_i \tag{6.5}$$

and

$$\Xi_v := \{a \in Q_1 \mid t(a) = v\} \setminus \{a_0, \dots, a_r\}. \tag{6.6}$$

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Since $D_{\rho_i} = p \neq 0$ for all i, we again have $S_v = \mathbf{e}_v\mathcal{O}/(t)$. We claim that we have a projective resolution

$$0 \to \bigoplus_{i=1}^{r} P_v \oplus \bigoplus_{a\in\Xi_v} P_{s(a)} \xrightarrow{\phi} \bigoplus_{t(a)=v} P_{s(a)} \xrightarrow{(a)} P_v \to S_v \to 0, \tag{6.7}$$

where ϕ is defined by the following block matrix (we think of the elements of the source and the target as column vectors):

$$\left[\begin{array}{ccc|c} -\rho_0' & \cdots & -\rho_0' & (\rho_0' a)_{a\in\Xi_v} \\ \hline \rho_1' & & & \\ & \cdots & & 0 \\ & & \rho_r' & \\ \hline & 0 & & (-t)_{a\in\Xi_v} \end{array}\right] : \begin{bmatrix} \bigoplus_{i=1}^r P_v \\ \bigoplus_{a\in\Xi_v} P_{s(a)} \end{bmatrix} \to \begin{bmatrix} P_{s(a_0)} \\ \bigoplus_{i=1}^r P_{s(a_i)} \\ \bigoplus_{a\in\Xi_v} P_{s(a)} \end{bmatrix}. \tag{6.8}$$

It is easy to check that (6.7) is a complex and ϕ is injective. What remains to check is the inclusion $\ker(a) \subseteq \operatorname{im}\phi$. Take an element of $\ker(a)$; namely, a column vector

$$\begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_r \\ (\sigma_a)_{a\in\Xi_v} \end{bmatrix} \in \ker(a) \tag{6.9}$$

such that

$$0 = (a)\begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_r \\ (\sigma_a)_{a\in\Xi_v} \end{bmatrix} = a_0\sigma_0 + \sum_{i=1}^{r} a_i\sigma_i + \sum_{a\in\Xi_v} a\sigma_a. \tag{6.10}$$

Without loss of generality, we assume all entries of (6.9) are linear combinations of cycle-free paths. By a direct analysis one can confirm that it can be modified by an element of $\operatorname{im}\phi$ so that all entries of (6.9) but σ_0 are zero. Exactness now follows from Lemma 6.7, which ensures that $\ker(a_0 \colon P_{s(a_0)} \to P_v) = 0$, and Proposition 6.9 is proved. ∎

PROPOSITION 6.10. *Let $\mathcal{A}$ be a coherent sheaf of algebras on a smooth curve C such that at any point $p \in C$ the homological dimension of the stalk $\mathcal{A}_p$ is at most two. Then the homological dimension of $\mathcal{A}$ is at most two.*

Proof. It suffices to show that $\operatorname{Ext}^3_{\mathcal{A}}(\mathscr{F}, -) = 0$ for every coherent $\mathcal{A}$-module $\mathscr{F}$. If $\mathscr{F}$ has dimension 0 as a sheaf on C, then this immediately follows from our assumption as the local-to-global Ext spectral sequence degenerates for the obvious reason.

Since the torsion part of $\mathscr{F}$ as an $\mathcal{O}_C$-module is automatically an $\mathcal{A}$-submodule, it remains to show the assertion under the assumption that $\mathscr{F}$ is torsion free as a sheaf on C. In this case, if we can show that locally at every closed point $\mathscr{F}$ has projective dimension at most one, then the assertion follows from the local-to-global Ext spectral sequence. To this end,

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suppose that $\mathcal{O}$ is a discrete valuation ring with a uniformizing parameter t. Consider the following part of the long exact sequence of Ext groups.

$$\mathrm{Ext}^2_{\mathcal{A}}(\mathscr{F},-) \xrightarrow{t} \mathrm{Ext}^2_{\mathcal{A}}(\mathscr{F},-) \to \mathrm{Ext}^3_{\mathcal{A}}(\mathscr{F}/t\mathscr{F},-). \tag{6.11}$$

The last term is zero by what we have already confirmed, as $\mathscr{F}/t\mathscr{F}$ is torsion. Hence the first map of (6.11) is always surjective. By the structure theorem for finitely generated modules over a discrete valuation ring or Nakayama's lemma, we see $\mathrm{Ext}^2_{\mathcal{A}}(\mathscr{F},-)=0$. Thus we conclude the proof. ∎

Proof of Theorem 1.5. Theorem 1.5.3 is a special case of Theorem 6.8. Theorem 1.5.1 is clear since $\mathcal{A}_{\mathscr{Y}_{\boldsymbol{r}}}$ is $\mathcal{O}_{C\times C^n}$-locally-free. In order to prove Theorem 1.5.2, take Zariski open subsets U_i of C such that $\lambda_i \in U_i$, $\lambda_j \notin U_i$ for $j \neq i$, and $\bigcup_{i=1}^n U_i = C$, which is possible since $\boldsymbol{\lambda} \in C^n \setminus \Delta$. Then the restriction of $\mathcal{A}_{\mathbf{Y}_{\boldsymbol{r},\boldsymbol{\lambda}}}$ to U_i is Morita equivalent to $\mathcal{O}_{U_i}\mathcal{Q}_i$ by Proposition 6.3, where $\mathcal{Q}_i$ is the labeled quiver associated with $n=1$, $\boldsymbol{r}=r_i$ and $\boldsymbol{\lambda}=\lambda_i$. Recall from Section 5 that the quiver $\mathcal{Q}_i$ is introduced in such a way that $\mathrm{Qcoh}\,\mathcal{O}_{U_i}\mathcal{Q}_i \simeq \mathrm{Qcoh}\,(\mathbf{X}_{\boldsymbol{r},\boldsymbol{\lambda}} \times_C U_i)$. These equivalences on U_i glue together to give the desired equivalence on C, and Theorem 1.5.2 is proved. ∎

§7. Graded algebras and I-algebras

Definition 7.1 [5, Conditions (4.2.1)]. Let $\mathscr{C}$ be an abelian category and $\mathcal{A}$ be an object of $\mathscr{C}$. An autoequivalence s of $\mathscr{C}$ is *ample* if

(a) for every object $\mathcal{M}$ of $\mathscr{C}$, there exist positive integers $l_1,\ldots,l_p$ and an epimorphism $\bigoplus_{i=1}^p \mathcal{A}(-l_i) \to \mathcal{M}$, and
(b) for every epimorphism $f\colon \mathcal{M} \to \mathcal{N}$, there exists an integer n_0 such that for every $n \geq n_0$, the map $H^0(\mathcal{M}(n)) \to H^0(\mathcal{N}(n))$ is surjective,

where $H^0(\mathcal{M}) := \mathrm{Hom}(\mathcal{A},\mathcal{M})$ and $\mathcal{M}(l) := s^l(\mathcal{M})$ for $l \in \mathbb{Z}$.

Lemma 7.2 below is an immediate consequence of Definition 7.1.

Lemma 7.2. *If $\mathcal{A}$ is a coherent sheaf of algebras on a polarized scheme $(C,\mathcal{L})$, then the tensor product by $\mathcal{L}$ gives an ample autoequivalence of $(\mathrm{coh}\,\mathcal{A},\mathcal{A})$.*

Theorem 7.3 below is one of the main results of [5].

Theorem 7.3 [5, Theorem 4.5(1)]. *Let $(\mathscr{C},\mathcal{A},s)$ be a triple consisting of*

- *an abelian category $\mathscr{C}$,*
- *an object $\mathcal{A}$ of $\mathscr{C}$, and*
- *an autoequivalence s of $\mathscr{C}$*

satisfying the following three conditions:

1. *$\mathcal{A}$ is noetherian.*
2. *$A_0 := H^0(\mathcal{A})$ is a right noetherian ring and $H^0(\mathcal{M})$ is a finite A_0-module for all $\mathcal{M}$.*
3. *s is ample.*

Then the graded ring $A := \bigoplus_{i=0}^\infty H^0(\mathcal{A}(i))$ is right noetherian satisfying χ_1, and $(\mathscr{C},\mathcal{A},s)$ is isomorphic to $(\mathrm{qgr}\,A,\pi(A),(1))$.

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Given a set J, an *J-algebra* is a category whose set of objects is identified with J. A $\mathbb{Z}$-algebra is a generalization of $\mathbb{Z}$-graded algebra [7]. A $\mathbb{Z}$-algebra analog of Theorem 7.3 is given in [12, Theorem 2.4].

Let $\mathcal{A} = \mathcal{O}_C \mathcal{Q}$ be the path algebra of a quiver $\mathcal{Q} = (Q, D_\bullet)$ labeled by effective divisors on an integral polarized scheme $(C, \mathcal{L})$. The graded ring associated with the triple $(\operatorname{coh} \mathcal{A}, \mathcal{A}, \mathcal{L} \otimes (-))$ as in Theorem 7.3 will be denoted by A.

Define a J-algebra for $J := Q_0 \times \mathbb{Z}$ by

$$B_{(v,m)(w,n)} := H^0\left(\mathbf{e}_v \mathcal{A} \mathbf{e}_w \otimes_{\mathcal{O}_C} \mathcal{L}^{\otimes(n-m)}\right), \tag{7.1}$$

whose multiplication is induced from that of $\mathcal{A}$ through the morphism

$$\left(\mathbf{e}_u \mathcal{A} \mathbf{e}_v \otimes_{\mathcal{O}_C} \mathcal{L}^{\otimes(m-l)}\right) \otimes_{\mathcal{O}_C} \left(\mathbf{e}_v \mathcal{A} \mathbf{e}_w \otimes_{\mathcal{O}_C} \mathcal{L}^{\otimes(n-m)}\right) \to \mathbf{e}_u \mathcal{A} \mathbf{e}_w \otimes_{\mathcal{O}_C} \mathcal{L}^{\otimes(n-l)}. \tag{7.2}$$

One can collapse the J-structure to a $\mathbb{Z}$-structure by

$$B'_{mn} := \bigoplus_{v,w \in Q_0} B_{(v,m)(w,n)} \tag{7.3}$$

without changing the categories gr and qgr;

$$\operatorname{gr} B \simeq \operatorname{gr} B', \qquad \operatorname{qgr} B \simeq \operatorname{qgr} B'. \tag{7.4}$$

As the index set $K = I$ satisfies the assumption of Theorem 1.3, it follows that the $\mathbb{Z}$-algebra B' is related to A by $B'_{mn} = A_{n-m}$, so that

$$\operatorname{gr} A \simeq \operatorname{gr} B', \qquad \operatorname{qgr} A \simeq \operatorname{qgr} B'. \tag{7.5}$$

If $\mathcal{Q}$ is the labeled quiver introduced in the beginning of Section 5, then the resulting J-algebra B coincides with the category S_I appearing in (1.21), where the index set K is taken to be the set I defined in (1.28), so that Theorem 7.3 gives

$$\operatorname{coh} \mathcal{A}_{\mathbf{Y}} \simeq \operatorname{qgr} A \simeq \operatorname{qgr} B \simeq \operatorname{qgr} \mathsf{S}_I. \tag{7.6}$$

This proves the equivalence (1.27) for $K = I$. The equivalence for general K is proved similarly by using the K-algebra $\mathsf{S} = \mathsf{S}_K$, and Theorem 1.3 is proved.

§8. A full strong exceptional collection on $\mathbf{Y}$

In this section, we always assume $C = \mathbb{P}^1$.

THEOREM 8.1. *The sequence $(\mathcal{O}_{\mathbf{Y}}(\vec{a}))_{0 \preceq \vec{a} \preceq \vec{c}}$ of objects of $\operatorname{coh} \mathbf{Y} := \operatorname{qgr} \mathsf{S}_I$ is a full strong exceptional collection of $D^b \operatorname{coh} \mathbf{Y}$ whose total morphism algebra is the path algebra of the quiver in Figure 2 with relations (1.10).*

Proof. Under the equivalence with $\operatorname{coh} \mathcal{A}_{\mathbf{Y}}$, the object $\mathcal{O}_{\mathbf{Y}}(a\vec{x}_i)$ for $0 < a < r_i$ corresponds to the $\mathcal{A}_{\mathbf{Y}}$-module $\mathbf{e}_{i,a} \mathcal{A}_{\mathbf{Y}}$, and the objects $\mathcal{O}_{\mathbf{Y}}(0)$ and $\mathcal{O}_{\mathbf{Y}}(\vec{c})$ correspond to $\mathbf{e}_0 \mathcal{A}_{\mathbf{Y}}$ and $\mathbf{e}_0 \mathcal{A}_{\mathbf{Y}}(1)$, respectively. Since they are sheaves of projective $\mathcal{A}_{\mathbf{Y}}$-modules, Ext-groups

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between them can be computed without taking further local projective resolutions as $\mathcal{A}_{\mathbf{Y}}$-modules. Namely, we have

$$\begin{aligned}
\mathrm{Hom}^{\bullet}_{\mathrm{coh}\,\mathcal{A}_Y}\left(\mathbf{e}_{\vec{a}}\mathcal{A}_{\mathbf{Y}},\mathbf{e}_{\vec{b}}\mathcal{A}_{\mathbf{Y}}\right) &\simeq \mathbf{R}\Gamma\left(\mathbf{Y},\mathcal{H}om^{\bullet}_{\mathcal{A}_{\mathbf{Y}}}\left(\mathbf{e}_{\vec{a}}\mathcal{A}_{\mathbf{Y}},\mathbf{e}_{\vec{b}}\mathcal{A}_{\mathbf{Y}}\right)\right) && (8.1)\\
&\simeq \mathbf{R}\Gamma\left(\mathbf{Y},\mathcal{H}om_{\mathcal{A}_{\mathbf{Y}}}\left(\mathbf{e}_{\vec{a}}\mathcal{A}_{\mathbf{Y}},\mathbf{e}_{\vec{b}}\mathcal{A}_{\mathbf{Y}}\right)\right) && (8.2)\\
&\simeq \mathbf{R}\Gamma\left(\mathbf{Y},\mathcal{O}_{\mathbf{Y}}\left(\vec{b}-\vec{a}\right)\right) && (8.3)\\
&\simeq \mathbf{R}\Gamma\left(\mathbb{P}^1,\varpi_*\mathcal{O}_{\mathbf{Y}}\left(\vec{b}-\vec{a}\right)\right). && (8.4)
\end{aligned}$$

It then follows that $\left(\mathcal{O}_{\mathbf{Y}}\left(\vec{a}\right)\right)_{0\preceq\vec{a}\preceq\vec{c}}$ is a strong exceptional collection whose total morphism algebra is the path algebra of the quiver in Figure 2 with relations (1.10). In order to show that the collection is full, one can use the exact sequences

$$0\to\mathcal{O}_{\mathbf{Y}}(0)\to\mathcal{O}_{\mathbf{Y}}\left(a\vec{x}_i\right)\oplus\mathcal{O}_{\mathbf{Y}}\left(\vec{c}\right)\to\mathcal{O}_{\mathbf{Y}}\left(a\vec{x}_i+\vec{c}\right)\to 0 \qquad (8.5)$$

and their translates by $\mathbb{Z}\vec{c}$ to show that $\mathcal{A}_{\mathbf{Y}}(i)$ for any $i\in\mathbb{Z}$ is contained in the full triangulated subcategory generated by the collection. ∎

Theorem 1.4 is an immediate consequence of Theorem 8.1.

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