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SEMI r-FREE AND r-FREE INTEGERS— A UNIFIED APPROACH

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ABSTRACT. We obtain an asymptotic formula for the number of (k, r)-free integers that do not exceed x. By definition, a (k, r)-free integer is one in whose canonical representation no prime power is in the interval $[r, k-1]$ where $1<r<k$ are fixed integers. These include as special cases the r-free integers, the semi r-free integers and the k-full integers. We obtain an asymptotic formula for the number of representations of an integer as the sum of a prime and a (k, r)-free integer, and use the result to prove that every sufficiently large integer can be represented as the sum of a prime and $m = ab^k$ where a and b are both square free, $(a, b) = 1, b > 1$ and k is any fixed integer, $k \geq 3$.

§1. **Introduction and definitions.** Throughout what follows, n represents a positive integer. If $n > 1$, we assume throughout the paper that it has the canonical form

$$n = p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_j^{\alpha_j}.$$

Let r be a fixed integer >1. A well known class of integers which have been studied extensively is the set of r-free integers. An integer n is r-free if either $n = 1$, or if $n > 1$, in its canonical form given above, all the α_i's are less than r. Equivalently, n is r-free if and only if it is not divisible by the rth power of any prime. The "unitary" analogue of this concept gives rise to the set of semi r-free integers. We define an integer n to be semi r-free whenever it is not unitarily divisible by the rth power of any prime. (We recall that n is said to be unitarily divisible by an integer d if d is a divisor of n and, moreover, d and n/d are relatively prime). Thus, n is semi r-free if either $n = 1$, or if $n > 1$, in its canonical form, we have $\alpha_i \neq r$, $1 \leq i \leq j$. These integers have been studied by D. Suryanarayana [17] and others.

In this paper, we introduce a class of integers called the (k, r)-free integers— of which both the set of r-free integers as well as the set of semi r-free integers arise as special cases. Thus we unify the study of the r-free and semi r-free integers. Indeed, our (k, r)-free integers also generalize another well known class of positive integers. These are the k-full integers—that is those integers n in whose canonical form we have $\alpha_i \geq k$ for all i.

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Let r and k be fixed integers satisfying $1<r<k$. Let $Q_{(k;r)}$ be the set of all positive integers n such that either $n=1$, or if $n>1$, its canonical form satisfies the property that $\alpha_i<r$ or $\alpha_i<r$ or $\alpha_i\geq k$ for all i $(1\leq i\leq j)$. We shall call such numbers (k, r)-*free*. We also define

$$(1.1) \qquad \chi_{(k;r)}(n)=\begin{cases}0, & n\notin Q_{(k;r)}\\ 1, & n\in Q_{(k;r)}\end{cases}$$

$$(1.2) \qquad Q_{(k;r)}(x)=\sum_{n\leq x}\chi_{(k;r)}(n).$$

Notice that $Q_{(r+1;r)}$ is the set of semi r-free integers. Also, since $Q_{(k+1;r)}\subset Q_{(k;r)}$, we may define $Q_{(\infty;r)}=\bigcap_{k=r+1}^{\infty}Q_{(k;r)}$ and observe that $Q_{(\infty;r)}$ is the set of r-free integers. Similarly, the k-full integers are the integers for which $\alpha_i\geq k$, and these are given by $Q_{(k;1)}$. Thus the (k, r)-free numbers may be considered to provide a simultaneous generalization of the r-free number, the k-full numbers, as well as the semi r-free numbers.

The (k, r) free integers are also related to a class of generalized r-free integers introduced by L. Carlitz [2]. The details are given later in Section 4.

The first main result of this paper is Theorem 3.1 where we obtain an asymptotic formulae for $Q_{(k;r)}(x)$, as well as estimates for the error term, one with no assumptions, and one based on the Riemann hypothesis.

The next problem we deal with is related to the famous Hardy–Littlewood conjectures ([7], p. 609 and 611) that every sufficiently large number is the sum of a prime and a square, and every sufficiently large number is the sum of a prime and a cube. These conjectures are still open and are made on the basis of the extended Riemann hypothesis. However, it has been proved by Hooley [9] that every sufficiently large integer is the sum of a prime and two squares, assuming the extended Riemann hypothesis. This assumption was dropped in a proof of the same result by Ju. V. Linnik [11] in 1960. It has also been proved that every sufficiently large integer is the sum of two primes and a square, by G. K. Stanley ([15], 1929) with the extended Riemann hypothesis, and by T. Esterman ([4], 1937) without.

It is known (L. K. Hua [10]) that every sufficiently large number can be expressed as the sum of a prime and s kth powers of integers, if $s>s_0\sim\frac{3}{2}k$ log k. Prachar [14] proved that given a positive integer l, $\exists n>1$ and $\delta>0$ such that for N sufficiently large, at least δN of the positive integers $\leq N$ are not expressible in the form $p+m^l$ where p is a prime and m is a positive integer, $m\leq n\log N$. On the other hand, it has been shown by Davenport and Heilbronn [3] that almost all numbers n can be represented as $p+b^k$, with $k\geq 2$ fixed. Babaev [1] showed there are infinitely many n with no such representation.

It has been shown by Esterman [4] in 1931 that every sufficiently large n is the sum of a prime and a square free number. Page [13] obtained in 1935 an asymptotic formula for $T(n)$, the number of representations of n as the sum of a prime and a square-free integer. His error estimate was improved by Walfisz [21] in 1936 to produce the result:

$$T(n)=\prod_{p\nmid n}\left\{1-\frac{1}{p(p-1)}\right\}Lin+0(n/\log^H n)\quad\text{as}\quad n\to\infty$$

where the constant implied by the 0-estimate is dependent on H.

This result was extended to the r-free integers in 1949 by Mirsky [12]. In 1970, K. W. Feng [6] found an asymptotic formula for the number of representations of n as the sum of a prime and a (k, r) integer. His error term was improved by Subbarao and Suryanarayana [16], who used their improvement to prove a new representation result: Every sufficiently large integer can be written as the sum of a prime and a number of the form $a\cdot b^k$, where $k\geq 3$ is fixed $b>1$, a is square-free, and $(a, b)=1$.

In this paper, we improve this representation result. We find an asymptotic formula for $T(k, r;n)$, the number of representations of n as the sum of a prime and a (k, r)-free integer, together with three error estimates, one with no assumptions, an improvement under the Page hypothesis (which we describe in (§2)), and a second improvement under the extended Riemann hypothesis. We use the asymptotic formula to show that every sufficiently large integer n is the sum of a prime and an integer of the form $a\cdot b^k$, where $k\geq 3$ is fixed, $(a, b)=1$, a and b are both square-free and $b>1$. We give an estimate for the number of such representations. This is our improvement over Subbarao and Suryanarayana's result. The corresponding result for $k=2$ remains an open problem.

First we require several lemmas.

§2. **Some Lemmas and Notation.** Notation: To simplify our notation, the following abbreviations of conditions under summation signs are used. Here p is any prime number and all other letters represent positive integers.

D1:$P\nmid n$	M:$p\equiv n(\text{mod } a^kb^r)$	S4:$n\leq x$
D2:$p\nmid n$	P1:$(m, n)=1$	S5:$np\leq x$
D3:$p\nmid a$	P2:$(n, p)=1$	S5A:$n\leq x/p$
E1:$a^kb^rc=m$	P3:$(a, b)=1$	S6:$b\leq y$
E2:$a^kb^rc=n$	P4:$(a^kb^r, n)=1$	S7:$b\leq z$
E3:$a^kb^rc=p^\alpha$	P5:$(a, n)=1$	S8:$a^kb^rc\leq x$
E4:$a^kb^rc+p=n$	P6:$(b, n)=1$	S8A:$a^kc\leq x/b^r$
E5:$p+m=n$	P7:$(a^kb^r, n)>1$	S8B:$b\leq(x/a^kc)^{1/r}$
E6:$d^rl=n$	P8:$(a^kb^r, n)=p$	S9:$a^kc\leq\rho^{-r}$
E7:$d^rl=p^\alpha$	P9:$(d^{k-r}, l)\in Q_{k-r}$	S10:$a^kb^r\leq x$
G1:$b>z$	Q:$a\in Q_2$	S10A:$a^k\leq x/b^r$
G2:$b>\rho z$	S1:$p<n$	S11:$a\leq\rho^{-r/k}$
G3:$a^kb^r>x$	S2:$a^kb^rc<n$	
LG:$z<b\leq y$	S3:$b<\rho z$	

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For example $\sum_{S4,P1} \mu(n)$ means the summation is taken over all n such that $n \le x$ and $(m, n) = 1$.

We write $\mu(n)$ for the well known Mobius μ function and $\zeta(s)$ for the Riemann Zeta function. The letter p always denotes a prime.

LEMMA 2.1. *If we define* $M(x)$ *as follows*:

$$M(x) = \sum_{n \le x} \mu(n), \tag{2.2}$$

then we have

$$M(x) = 0(x\delta(x)), \qquad (x \to \infty) \tag{2.3}$$

where

$$\begin{aligned} \delta(x) &= \exp\{-A \log^{3/5} x \cdot (\log\log x)^{-1/5}\}, \qquad x \ge 3 \\ \delta(x) &= \delta(3), \qquad 0 < x < 3 \end{aligned} \tag{2.4}$$

where $A > 0$ *is an absolute constant. This result is due to A. Walfisz* [21].

We recall $\phi_s(n)$, the generalized Jordan totient function, is defined to be: $\phi_s(n) = n^s \prod_{p|n} (1 - p^{-s})$ where s need not be an integer.

We use this function in the following lemma:

LEMMA 2.5: *Define* $M(x, m)$ *as follows*:

$$M(x, m) = \sum_{S4,P1} \mu(n) \tag{2.6}$$

then we have, for any fixed ε, $0 < \varepsilon < 1$,

$$M(x, m) = 0(x \cdot \delta(x) \cdot m^{1-\varepsilon}/\phi_{1-\varepsilon}(m)) \quad as \quad x \to \infty \tag{2.7}$$

where the constant implied by the 0*-estimate depends only on* ε.

Proof. We claim that when x is sufficiently large, $\delta(x/s) \le s^{\varepsilon}\delta(x)$, or equivalently:

$$(x/s)^{\varepsilon}\delta(x/s) \le x^{\varepsilon}\delta(x).$$

To prove this, it is sufficient to prove $x^{\varepsilon}\delta(x) \to \infty$ as $x \to \infty$ and that $d/dx(x^{\varepsilon}\delta(x)) > 0$ for x sufficiently large. That $x^{\varepsilon}\delta(x) \to \infty$ as $x \to \infty$ follows from the definition of $\delta(x)$.

$$\frac{d}{dx}(x^{\varepsilon}\delta(x)) = x^{\varepsilon-1}\delta(x)(\varepsilon - A \log^{-(2/5)} x(\tfrac{3}{5}(\log\log x)^{-(1/5)} - \tfrac{1}{5}(\log\log x)^{-(6/5)}))$$

and since ε is fixed, $\varepsilon > 0$, $d/dx(x^{\varepsilon}\delta(x)) > 0$ for x sufficiently large. Thus for x sufficiently large, $\delta(x/s) \le s^{\varepsilon}\delta(x)$.

Since replacing m by $\gamma(m)$, the largest square free divisor of m, does not alter either side of (2.7), we may, without loss of generality, assume m to be square free. To prove the lemma, we shall induct on the number of prime divisors of m. If $m=1$, (2.7) follows directly from Lemma 2.1. Assume Lemma 2.5 holds for m having $\leq j$ prime divisors, and $m^*=p\cdot m$ has $(j+1)$ prime divisors. We have:

$$M(x,m^*)=M(x,p\cdot m)=\sum_{S4,P1}\mu(n)-\sum_{S4,D1,P1}\mu(n)$$

$$=M(x,m)-\sum_{S5,P1}\mu(p\cdot n)=M(x,m)+\sum_{S5A,P1,P2}\mu(n)$$

$$=M(x,m)+M(x/p,m^*). \tag{2.8}$$

Iterating (2.8), we find

$$M(x,m^*)=\sum_{k=0}^{c}M(x/p^k,m) \tag{2.9}$$

where $c=[\log x/\log p]$. By the inductive hypothesis and (2.7), we have:

$$M(x/p^k,m)=0((x/p^k)\delta(x/p^k)m^{1-\varepsilon}/\phi_{1-\varepsilon}(m))$$

But we may assume $\delta(x/p^k)<p^{k\varepsilon}\cdot\delta(x)$. Thus

$$M(x/p^k,m)=0((x/p^{k-k\varepsilon})\delta(x)\cdot m^{1-\varepsilon}/\phi_{1-\varepsilon}(m)). \tag{2.10}$$

Combining (2.9) and (2.10) we find:

$$M(x,m^*)=0\left(\sum_{k=0}^{c}(x/p^{k-k\varepsilon})\delta(x)m^{1-\varepsilon}/\phi_{1-\varepsilon}(m)\right)$$
$$=0\left(x\cdot\delta(x)\cdot(m^{1-\varepsilon}/\phi_{1-\varepsilon}(m))\cdot\sum_{k=0}^{\infty}p^{-k(1-\varepsilon)}\right)$$
$$=0(x\cdot\delta(x)\cdot(m^{1-\varepsilon}/\phi_{1-}(m))\cdot(p^{1-\varepsilon}/p^{1-\varepsilon}-1))$$
$$=0(x\cdot\delta(x)\cdot(m^*)^{1-\varepsilon}/\phi_{1-\varepsilon}(m^*)).$$

Thus (2.7) holds for m^* and Lemma 2.5 follows by induction.

REMARK. The above proof can be refined to improve the estimate given in the lemma. See, for example, [18], Lemma 3.5. For our purpose, however, the stated result is adequate.

LEMMA 2.11. *For ε fixed, $0<\varepsilon<1$, $k\geq 2$ and $z>0$, we have:*

$$\sum_{G1,P3}\mu(b)/b^k=0(z^{-(k-1)}\delta(z)a^{1-\varepsilon}/\phi_{1-\varepsilon}(a)) \quad as \quad z\to\infty \tag{2.12}$$

where the constant implied by the 0-estimate depends only on ε.

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Proof.

$$\sum_{G1,P3} \mu(b)/b^k = k \sum_{G1,P3} \mu(b) \int_b^\infty y^{-(k+1)}\,dy = k\int_z^\infty y^{-(k+1)}\Big(\sum_{LG,P3} \mu(b)\Big)\,dy$$

$$(2.13) \qquad = k\int_z^\infty y^{-(k+1)}\Big(\sum_{SG,P3} \mu(b)\Big)\,dy - k\int_z^\infty y^{-(k+1)}\Big(\sum_{S7,P3} \mu(b)\Big)\,dy$$

$$= k\int_z^\infty M(y,a)y^{-(k+1)}\,dy - kM(z,a)\int_z^\infty y^{-(k+1)}\,dy.$$

Applying (2.7) to (2.13), we have:

$$\sum_{G1,P3} \mu(b)/b^k = 0\Big(k\int_z^\infty \delta(y)\cdot(a^{1-\varepsilon}/\phi_{1-\varepsilon}(a))y^{-k}\,dy\Big) + 0(\delta(z)(a^{1-\varepsilon}/\phi_{1-\varepsilon}(a))z^{-(k-1)}).$$

Since δ is monotonically decreasing, we may replace $\delta(y)$ in the first 0-estimate above to obtain:

$$\sum_{G1,P3} \mu(b)/b^k = 0\Big(\delta(z)(a^{1-\varepsilon}/\phi_{1-\varepsilon}(a))\int_z^\infty ky^{-k}\,dy\Big) + 0(\delta(z)(a^{1-\varepsilon}/\phi_{1-\varepsilon}(a))z^{-(k-1)})$$
$$= 0(\delta(z)\cdot(a^{1-\varepsilon}/\phi_{1-\varepsilon}(a))z^{-(k-1)}).$$

LEMMA 2.14 (Cf., Titchmarsh [19], Theorem 14–26(A), p. 316). *If the Riemann hypothesis is true, then for* $x \geq 1$,

$$(2.15) \qquad M(x) = 0(x^{1/2}\omega(x)) \quad as \quad x\to\infty$$

where

$$(2.16) \qquad \begin{aligned}\omega(x) &= \exp\{A\log x(\log\log x)^{-1}\}, \qquad x\geq 16\\ &= \exp\{A\log 16(\log\log 16)^{-1}\}, \qquad x<16.\end{aligned}$$

and A is an absolute positive constant.

LEMMA 2.17. *If the Riemann hypothesis is true, then for* $x \geq 1$,

$$(2.18) \qquad M(x,m) = 0(x^{1/2}\omega(x)m^{1/2}/\phi_{1/2}(m)) \quad as \quad x\to\infty$$

where the constant implied by the 0*-estimate is independent of m and x.*

Proof. We employ an inductive argument similar to the one for Lemma 2.5. Again, we may assume m is square free. If $m=1$, Lemma 2.17 follows directly from Lemma 2.14. Let us assume Lemma 2.17 holds for m having $\leq j$ prime divisors and let $m^* = m\cdot p$ have $(j+1)$ prime divisors. Then, as in Lemma 2.5,

$$M(x,m^*) = \sum_{k=0}^{c} M(x/p^k, m)$$

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where $c=[\log x/\log p]$. By the inductive hypothesis, we find:

$$M(x, m^*)=0\left(\sum_{k=0}^{c} ((x/p)^{1/2}\omega(x/p)\cdot m^{1/2}/\phi_{1/2}(m))\right)$$
$$=0\left(x^{1/2}\omega(x)(m^{1/2}/\phi_{1/2}(m))\sum_{k=0}^{\infty} p^{-(1/2)}\right)$$

since $\omega(x)$ is monotonically increasing. Thus

$$M(x, m^*)=0(x^{1/2}\omega(x)(m^{1/2}/\phi_{1/2}(m))\cdot p^{1/2}/p^{1/2}-1$$
$$=0(x^{1/2}\omega(x)(m^*)^{1/2}/\phi_{1/2}(m^*))$$

and the lemma follows by induction.

LEMMA 2.19. *Given $z\geq 1$, $k\geq 2$, and the Riemann hypothesis then:*

$$(2.20)\quad \sum_{G1,P3} \mu(b)/b^k=0(\omega(z)\cdot z^{-(1/2)}\cdot z^{-(k-1)}\cdot a^{1/2}/\phi_{1/2}(a)) \quad as \quad z\to\infty$$

where the constant implied by the 0*-estimate is independent of z and a.*

Proof. Since a straight forward differentiation shows $\omega(z)\cdot z^{-(1/2)}$ is monotonically decreasing for z sufficiently large, we follow the proof of Lemma 2.11, replacing $\delta(z)$ by $\omega(z)\cdot z^{-(1/2)}$, using Lemma 2.17 in place of Lemma 2.5 and easily derive (2.20).

LEMMA 2.21. *For $1<r\leq k\leq\infty$, we have:*

$$(2.22)\qquad \chi_{(k;r)}(n)=\sum_{E2,P3,Q}\mu(b)$$

and

$$(2.23)\qquad \chi_{(k;r)}(n)=\sum_{E6,P9}\mu(d)$$

where Q_l is the set of l-free integers.

Proof. Since $\mu(n)$ is multiplicative, both $\sum_{E2,P3,Q}\mu(b)$ and $\sum_{E6,P9}\mu(d)$ are multiplicative.

Thus we may assume $n=P^\alpha$, a prime power. But it may readily be verified that

$$\sum_{E3,P3,Q}\mu(b)=\sum_{E7,P9}\mu(d)=\begin{cases}0 & r\leq\alpha<k\\ 1 & \text{otherwise.}\end{cases}$$

Lemma 2.21 follows.

REMARK. The above proof holds even when $k=\infty$, in that case notice that $a=1$ in (2.22).

We use the standard notation $\pi(x; u, v)=$ the number of primes $\leq x$ which are $\equiv v \pmod u$, where $(u, v)=1$.

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LEMMA 2.24. *If* $(u, v)=1$, *then for any* H:

$$\pi(x; u, v)=\frac{Li(x)}{\phi(u)}+0\left(\frac{x}{\log^H x}\right) \quad as \quad x\to\infty \tag{2.25}$$

where the constant implied by the 0-*estimate depends only on* H.

This is due to Van der Corput [20], Footnote 4, pp. 279–280.

For the convenience of the reader, we shall now state the Page hypothesis, which appears in Lemma 2.27 below:

HYPOTHESIS 2.26 (Page Hypothesis). The greatest real zero (which we denote by σ) possessed by any Dirichlet L-function with modulus q satisfies $\sigma<1-A\ (\log q)^{-1}$ where A is an absolute positive constant.

REMARK. Page has shown there is at most one real primitive character which does not satisfy this hypothesis, and he believes the hypothesis is probably true.

LEMMA 2.27. *If* $(u, v)=1$ *and the Page Hypothesis* 2.26 *is true, then*:

$$\pi(x; u, v)=\frac{Li(x)}{\phi(u)}+0\{x\exp(-\beta\sqrt{\log x})\} \quad as \quad x\to\infty \tag{2.28}$$

where β *is an absolute positive constant and the* 0-*estimate is independent of* u *and* v.

The proof is due to Page [13], p. 135.

LEMMA 2.29. *If* $(u, v)=1$ *and the extended Riemann hypothesis is true, then for* $u<x$:

$$\pi(x; u, v)=\frac{Li(x)}{\phi(u)}+0\left(\frac{x}{(x^{1/2}/\log x)}\right) \quad as \quad x\to\infty \tag{2.30}$$

where the 0-*estimate is independent of* u *and* v.

The proof is due to Titchmarch [19], Theorem 6, p. 427.

LEMMA 2.31. *There exists an integer* n_0 *such that for all* $n>n_0$, *there is a prime* $p<2\log n$ *such that* $p\mid n$.

Proof. By the prime number theorem, $\sum_{p\le x}\log p\sim x$ as $x\to\infty$. Thus $\exists n_0$ such that for all $n>n_0$,

$$\sum_{p<2\log n}\log p>\log n. \tag{2.32}$$

Let us assume that $n>n_0$ such that for all $p<2\log n$, $p\mid n$. Then we have $\prod_{p<2\log n}p\mid n$ which implies $\prod_{p<2\log n}p<n$ which implies $\sum_{p<2\log n}\log p<\log n$. This contradicts (2.32), and thus Lemma 2.31 is proved.

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§3. The main results.

THEOREM 3.1. *For $2 \le r < k \le \infty$, we have:*

(3.2) $$Q_{(k;r)}(x) = x \cdot c_{(k;r)} + 0\left(x \cdot \delta_r(x) \frac{k}{k-r}\right) \quad as \quad x \to \infty$$

where

(3.3) $$c_{(k;r)} = \prod_p (1 - p^{-r} + p^{-k})$$

and

(3.4) $$\delta_r(x) = \exp(-c_r(\log^{3/5} x)(\log\log x)^{-(1/5)})$$

where c_r depends only on r and the 0-estimate is uniform in k and r.

Proof. From Lemma 2.21 it follows that:

(3.5) $$Q_{(k;r)}(x) = \sum_{n \le x} \chi_{(k;r)}(n) = \sum_{\text{S8,P3,Q}} \mu(b).$$

Let $z = x^{1/r}$ and $\rho = \rho(z)$ be a function of z, $\rho < 1$, such that $(\rho \cdot z) \to \infty$ as $z \to \infty$ (ρ will be chosen later). Then if $a^k b^r c \le x$, we cannot have both $b > \rho z$ and $a^k c > \rho^{-r}$. Thus we have:

(3.6) $$Q_{(k;r)}(x) = \sum_{\text{S8,P3,Q,S3}} \mu(b) + \sum_{\text{S8,P3,Q,S9}} \mu(b) - \sum_{\text{S3,S9,P3,Q}} \mu(b)$$
$$= S_1 + S_2 + S_3 \text{ (say)}.$$

We consider each of these sums separately

$$S_1 = \sum_{b<\rho z} \mu(b) \sum_{\text{S8A,P3,Q}} 1$$
$$= \sum_{b<\rho z} \mu(b) \sum_{\text{S10A,P3,Q}} [x/a^k b^r]$$
$$= \sum_{b<\rho z} \mu(b) \sum_{\text{S10A,P3,Q}} ((x/a^k b^r) + 0(1))$$
$$= \sum_{b<\rho z} \mu(b) \left\{0(x^{1/k}/b^{r/k}) + \sum_{\text{P3,Q}} (x/a^k b^r) + 0\left(\sum_{a=[x^{1/k}/b^{r/k}]}^{\infty} (x/a^k b^r)\right)\right\}$$
$$= x \sum_{b<\rho z} \mu(b)/b^r \sum_{\text{P3,Q}} 1/a^k + 0\left(x^{1/k} \sum_{b=1}^{\rho z} b^{-r/k}\right) + 0\left(\sum_{b<\rho z} \sum_{a=[x^{1/k}/b^{r/k}]}^{\infty} (x/a^k b^r)\right).$$

A simple calculation shows, since $x^{1/r} = z$, we have:

$$0\left(x^{1/k} \sum_{b=1}^{\rho z} 1/b^{r/k}\right) = 0\left(z \cdot \rho^{1-r/k} \cdot \frac{k}{k-r}\right)$$

and

$$0\left(\sum_{b<\rho z} \sum_{a=[x^{1/k}/b^{r/k}]}^{\infty} (x/a^k b^r)\right) = 0\left(z \cdot \rho^{1-r/k} \cdot \frac{k}{k-r}\right)$$

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Thus

$$S_1 = x \cdot \sum_{b=1}^{\infty} (\mu(b)/b^r) \sum_{P3,Q} 1/a^k + 0\left(x \sum_{b \geq \rho z} (\mu(b)/b^r) \sum_{P3,Q} 1/a^k\right) + 0\left(z\rho^{1-r/k} \cdot \frac{k}{k-r}\right) \tag{3.7}$$

We now examine the first 0-term more closely:

$$x \sum_{b \geq \rho z} (\mu(b)/b^r) \sum_{P3,Q} 1/a^k = x \sum_{a \in Q_2} a^{-k} \sum_{G2,P3} \mu(b)/b^r. \tag{3.8}$$

Applying Lemma 2.11 to (3.8) with $\varepsilon = 1/2$, we have

$$\begin{aligned} x \sum_{b \geq \rho z} (\mu(b)/b^r) \sum_{P3,Q} a^{-k} &= x \sum_{a \in Q_2} 0(\delta(\rho z) \cdot a^{(1/2)-k}/\rho^{r-1} z^{r-1} \phi_{1/2}(a)) \\ &= 0(x \cdot \delta(\rho z) \cdot \rho^{1-r} \cdot x^{-1} \cdot z \sum_{a=1}^{\infty} (a^{(1/2)-k}/\phi_{1/2}(a)) \\ &= 0(z\delta(\rho z)\rho^{1-r}) \end{aligned} \tag{3.9}$$

since $\sum_{a=1}^{\infty} (a^{(1/2)-k}/\phi_{1/2}(a))$ converges uniformly for $k \geq 3$. Thus by (3.7) and (3.9), we have:

$$\begin{aligned} S_1 &= x \sum_{b=1}^{\infty} (\mu(b)/b^r) \sum_{P3,Q} 1/a^k + 0(z\delta(\rho z)\rho^{1-r}) + 0\left(z\rho^{1-r/k} \cdot \frac{k}{k-r}\right) \\ &= x \cdot c_{(k;r)} + 0(z\delta(\rho z)\rho^{1-r}) + 0\left(z\rho^{1-r/k} \cdot \frac{k}{k-r}\right), \end{aligned} \tag{3.10}$$

by a simple argument and (3.3). We next consider S_2:

$$S_2 = \sum_{P3,Q,S9,S8} \mu(b) = \sum_{S9,Q} \sum_{S8B,P3} \mu(b) \tag{3.11}$$

Applying Lemma 2.5 to the inner sum of (3.11) we obtain (again with $\varepsilon = \frac{1}{2}$):

$$\begin{aligned} S_2 &= \sum_{S9,Q} 0\{(x/a^k c)^{1/r} \delta((x/a^k c)^{1/r}) a^{1/2}/\phi_{1/2}(a)\} \\ &= 0\left\{x^{1/r} \sum_{S9,Q} (a^k c)^{-(1/r)} \delta((x/a^k c)^{1/r}) a^{1/2}/\phi_{1/2}(a)\right\}. \end{aligned} \tag{3.12}$$

But we have, since $\delta(x)$ is monotonically decreasing, and $a^k c \leq \rho^{-r}$, that

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$(1/a^kc)^{1/r} \geq \rho$ and $\delta((x/a^kc)^{1/r}) \leq \delta(\rho x^{1/r}) = \delta(\rho z)$. Thus, from (3.12) we have:

$$(3.13)\quad \begin{aligned} S_2 &= 0\left\{x^{1/r}\delta(\rho z) \sum_{a \leq \rho^{-r/k}} a^{-k/r}(a^{1/2}/\phi_{1/2}(a)) \sum_{c \leq \rho^{-r}a^{-k}} c^{-1/r}\right\} \\ &= 0\left\{z\delta(\rho z) \sum_{a \leq \rho^{-r/k}} a^{-k/r}(a^{1/2}/\phi_{1/2}(a))(\rho^{-r}a^{-k})^{1-(1/r)}\right\} \end{aligned}$$

$$(3.14)\quad \begin{aligned} S_2 &= 0\left\{z\delta(\rho z)\rho^{1-r} \sum_{a \leq \rho^{-r/k}} a^{-k}a^{1/2}/\phi_{1/2}(a)\right\} \\ &= 0(z\delta(\rho z)\rho^{1-r}) \end{aligned}$$

since $\sum_{a=1}^{\infty} a^{-k}a^{1/2}/\phi_{1/2}(a)$ converges uniformly for $k \geq 3$. Finally, we have:

$$(3.15)\quad S_3 = \sum_{S3,S9,P3,Q} \mu(b) = \sum_{S9,Q} \sum_{S3,P3} \mu(b).$$

Applying Lemma 2.5 to (3.15), we obtain for $\varepsilon = \frac{1}{2}$:

$$(3.16)\quad \begin{aligned} S_3 &= \sum_{S9,Q} 0\{\rho z\delta(\rho z)a^{1/2}/\phi_{1/2}(a)\} \\ &= 0\left\{\sum_{S10,Q} \rho z \cdot \delta(\rho z) \cdot (a^{1/2}/\phi_{1/2}(a)) \cdot \rho^{-r}/a^k\right\} \\ &= 0\left\{z \cdot \delta(\rho z) \cdot \rho^{1-r} \sum_{S11,Q} a^{-k}a^{1/2}/\phi_{1/2}(a)\right\} = 0(z \cdot \delta(\rho z)\rho^{1-r}) \end{aligned}$$

since $\sum_{a=1}^{\infty} a^{-k}a^{1/2}/\phi_{1/2}(a)$ converges uniformly for $k \geq 3$.

Thus, substituting (3.10), (3.14) and (3.16) into (3.6), we have:

$$(3.17)\quad Q_{(k;r)}(x) = xc_{(k;r)} + 0(z \cdot \rho(\rho z) \cdot \rho^{1-r}) + 0\left(z \cdot \rho^{1-(r/k)} \cdot \frac{k}{k-r}\right)$$

Setting $\rho = \rho(x) = (\delta(x^{1/2r}))^{1/r}$, from (3.17) we obtain (3.2). The calculations follow the procedure of Subbarao and Suryanarayana [16]. Q.E.D.

THEOREM 3.18. *If* $2 \leq r < k \leq \infty$ *and the Riemann hypothesis is true, then*:

$$(3.19)\quad Q_{(k;r)}(x) = x \cdot c_{(k;r)} + 0\left(x^{1/r}\omega(x)x^{-(k-r)/r(2kr+k-2r)} \cdot \frac{k}{k-r}\right) \quad \text{as} \quad x \to \infty$$

where $\omega(x)$ *is given by* (2.16), *and the* 0-*estimate is uniform in* k *and* r.

Proof. Following the proof of Theorem 3.1, replacing $\delta(x)$ by $(\omega(x)x^{-(1/2)})$, (which is, as we remarked before, monotonically decreasing for x sufficiently large), and using Lemmas 2.17 and 2.19 in place of Lemmas 2.5 and 2.11

respectively, we obtain:

$$
\begin{aligned}
Q_{(k;r)}(x) &= x\cdot c_{(k;r)} + 0(z\cdot\omega(pz)\cdot(\rho z)^{-(1/2)}\cdot\rho^{1-r}) + 0\left(z\cdot\rho^{1-(r/k)}\cdot\frac{k}{k-r}\right) \\
&= x\cdot c_{(k;r)} + 0(z^{1/2}\omega(x)\rho^{(1/2)-r}) + 0\left(z\cdot\rho^{1-(r/k)}\cdot\frac{k}{k-r}\right)
\end{aligned} \tag{3.20}
$$

Setting $\rho = z^{-1/(1+2r-(2r/k))}$, we obtain (3.19) from (3.20).

REMARK. The proof of Theorems 3.1 and 3.18 when $k=\infty$ requires only a minor modification of the proofs given above.

THEOREM 3.21. *If* $2\le r\le k<\infty$, *then for any* H,

$$
\begin{aligned}
T(k,r;n) &= Li(n)\prod_{p\nmid n}\left\{1-\frac{1}{p-1}\left(\frac{1}{p^{r-1}}-\frac{1}{p^{k-1}}\right)\right\} \\
&\quad + 0\left(\frac{n\cdot(k/k-r)}{\log^{H(r-1)/r}(n)}\right) \quad as \quad n\to\infty
\end{aligned} \tag{3.22}
$$

where the constant implied by the 0*-estimate depends only on* H.

Proof. To avoid repetitions, we shall assume all 0-estimates to be as $n\to\infty$ unless stated otherwise. We have:

$$
T(k,r;n) = \sum_{\substack{p\\ E5}} \chi_{(k;n)}(m). \tag{3.23}
$$

Applying Lemma 2.21 to (3.23), we find:

$$
\begin{aligned}
T(k,r;n) &= \sum_{E5}\sum_{E1,P3,Q} u(b) \\
&= \sum_{E4,P3,Q}\mu(b) = \sum_{E4,P3,Q,S10}\mu(b) + \sum_{E4,P3,,Q,G3}\mu(b)
\end{aligned} \tag{3.24}
$$

where $x=x(n)$ is a function of n, $x(n)<n$, monotonically increasing to infinity as $x\to\infty$, its precise valuation to be determined later. Considering the first sum from (3.24), we find:

$$
\begin{aligned}
\sum_{E4,P3,Q,S10}\mu(b) &= \sum_{S10,P3,Q,P4}\mu(b)\sum_{S1,M}1 + \sum_{S10,P3,Q,P7}\mu(b)\sum_{S1,M}1 \\
&= \Sigma_1 + \Sigma_2 \text{ (say).}
\end{aligned} \tag{3.25}
$$

Let us consider Σ_2. If $n=(a^kb^r)c+p$ and $(n,a^k,b^r)>1$, we must have $(n,a^kb^r)=p$. Thus

$$
\Sigma_2 = \sum_{E4,S10,P8,P3,Q}\mu(b).
$$

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For a given a and b, if there is a representation $n = a^k b^r c + p$ with $(a^k b^r, n) > 1$, this uniquely fixes p (namely $p = (a^k b^r, n)$) and this in turn fixes c. Thus

$$\begin{aligned}\Sigma_2 &= 0\Big(\sum_{a^k b^r \le x} 1\Big) \\ &= 0\Big(\frac{k}{k-r} \cdot x^{1/r}/x\Big) \qquad (3.26) \\ &= 0\Big(\frac{k}{k-r}\Big(\frac{n}{x}\Big) \cdot x^{1/r}\Big).\end{aligned}$$

We now turn to Σ_1:

$$\Sigma_1 = \sum_{S10,P4,P3,Q} \sum_{S1,M} 1. \tag{3.27}$$

Applying (2.25) to (3.27), we find:

$$\begin{aligned}\Sigma_1 &= \sum_{S10,P4,P3,Q} \mu(b)\Big(\frac{Li(n)}{\phi(a^k b^r)} + 0\Big(\frac{n}{\log^H(n)}\Big)\Big) \\ &= Li(n) \sum_{P4,P3,Q} \Big(\frac{1}{a^{k-1}\phi(a)} \frac{\mu(b)}{b^{r-1}\phi(b)}\Big) \\ &\quad - Li(n) \sum_{P4,P3,Q,G3} \Big(\frac{1}{a^{k-1}\phi(a)} \frac{\mu(b)}{b^{r-1}\phi(b)}\Big) \\ &\quad + 0\Big(\frac{n}{\log^H(n)} \sum_{a^k b^r \le x} 1\Big) = \Sigma_3 + \Sigma_4 + \Sigma_5 \text{ (say).}\end{aligned}$$

We consider Σ_3:

$$\Sigma_3 = Li(n) \sum_{P5,Q} \frac{1}{a^{k-1}\phi(a)} \sum_{P6,P3} \frac{\mu(b)}{b^{r-1}\phi(b)}.$$

The inner sum, above, can be expanded as an Euler product as follows:

$$\begin{aligned}\sum_{P6,P3} \frac{\mu(b)}{b^{r-1}\phi(b)} &= \prod_{D2,D3} \Big(1 - \frac{1}{p^r - p^{r-1}}\Big) \\ &= \prod_{p \nmid n} \Big(1 - \frac{1}{p^r - p^{r-1}}\Big) \prod_{p \mid a} \Big(1 - \frac{1}{p^r - p^{r-1}}\Big)^{-1} \\ &= \prod_{p \nmid n} \Big(1 - \frac{1}{p^r - p^{r-1}}\Big) \prod_{p \mid a} \Big(\frac{p^r - p^{r-1}}{p^r - p^{r-1} - 1}\Big),\end{aligned}$$

since $a \nmid n$. Thus we have:

$$\begin{aligned}\Sigma_3 &= Li(n) \sum_{P5,Q} \frac{1}{a^{k-1}\phi(a)} \prod_{p \nmid n} \Big(1 - \frac{1}{p^r - p^{r-1}}\Big) \prod_{p \mid a} \Big(\frac{p^r - p^{r-1}}{p^r - p^{r-1} - 1}\Big) \\ &= Li(n) \prod_{p \nmid n} \Big(1 - \frac{1}{p^r - p^{r-1}}\Big) \sum_{P5,Q} \frac{1}{a^{k-1}\phi(a)} \prod_{p \mid a} \frac{p^r - p^{r-1}}{p^r - p^{r-1} - 1}\end{aligned}$$

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Since in the above sum $a \in Q_2$, we may also expand this sum as an Euler product to obtain:

$$\Sigma_3 = Li(n) \prod_{p \nmid n} \left(1 - \frac{1}{p^r - p^{r-1}}\right) \prod_{p \nmid n} 1 + \frac{p^r - p^{r-1}}{p^{k-1}(p-1)(p^r - p^{r-1} - 1)}$$
$$(3.28) \qquad = Li(n) \prod_{p \nmid n} \left\{1 - \frac{1}{p-1}\left(\frac{1}{p^{r-1}} - \frac{1}{p^{k-1}}\right)\right\}.$$

We now estimate Σ_4

$$(3.29) \qquad \Sigma_4 = 0\left(Li(n) \sum_{a^k b^r > x} a^{-k} \cdot \frac{a}{\phi(a)} \cdot b^{-r} \cdot \frac{b}{\phi(b)}\right).$$

But $Li(n) = 0(n/\log n)$. Also, $ab < n$ and $\lim(\phi(n)(\log\log n)/n) = e^{-\gamma}$ (cf., Hardy and Wright [8], p. 267). Thus, with $(a, b) = 1$, we have $a/\phi(a) \cdot b/\phi(b) = ab/\phi(ab) = 0(\log n)$. Applying these results to (3.29), we have

$$(3.30) \qquad \Sigma_4 = 0\left(n \cdot \sum_{a^k b^r > x} \frac{1}{a^k b^r}\right) = 0(n \cdot x^{1/r}/x).$$

Estimating Σ_5, we have:

$$(3.31) \qquad \Sigma_5 = 0\left(\frac{n}{\log^H(n)} \sum_{a^k b^r \le x} 1\right) = 0\left(\frac{k}{k-r} \cdot n \cdot x^{1/r}/\log^H(n)\right).$$

Finally, we estimate the second sum in (3.24)

$$(3.32) \qquad \sum_{E4,P3,Q,G3} \mu(b) = 0\left(\sum_{S2,G3} 1\right) = 0\left(\sum_{a^k b^r > x} \frac{n}{a^k b^r}\right)$$
$$= 0(n \cdot x^{1/r}/x).$$

Combining (3.26), (3.28), (3.30), (3.31) and (3.32), we obtain:

$$(3.33) \qquad T(k, r; n) = Li(n) \prod_{p \nmid n} \left\{1 - \frac{1}{(p-1)}\left(\frac{1}{p^{r-1}} - \frac{1}{p^{k-1}}\right)\right\}$$
$$+ 0\left(\frac{k}{k-r} \cdot \frac{nx^{1/r}}{x}\right) + 0\left(\frac{k}{k-r} \cdot \frac{nx^{1/r}}{\log^H(n)}\right).$$

Setting $x = \log^H n$ in (3.33), we obtain (3.22).

THEOREM 3.34. *If* $2 \le r < k < \infty$ *and the Page Hypothesis is true, then*:

$$(3.35) \qquad T(k, r; n) = Li(n) \prod_{p \nmid n} \left\{1 - \frac{1}{(p-1)}\left(\frac{1}{p^{r-1}} - \frac{1}{p^{k-1}}\right)\right\}$$
$$+ 0\left(\frac{k}{k-r} \cdot n \exp\left\{-A\left(\frac{r-1}{r}\right)\sqrt{\log n}\right\}\right\} \quad as \quad n \to \infty$$

where the constant implied by the 0*-estimate is independent of* k *and* r.

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Proof. Proceeding as above, using (2.28) in place of (2.25), we obtain:

$$(3.36)\quad \begin{aligned} T(k, r; n) = Li(n) \prod_{p \nmid n} \left\{1 - \frac{1}{(p-1)}\left(\frac{1}{p^{r-1}} - \frac{1}{p^{k-1}}\right)\right\} + 0\left(\frac{k}{k-r} \cdot \frac{nx^{1/r}}{x}\right) \\ + 0\left(\frac{k}{k-r} \cdot n \cdot x^{1/r} \cdot \exp(-A\sqrt{\log n})\right) \quad \text{as} \quad n \to \infty. \end{aligned}$$

Setting $x = \exp(A\sqrt{\log n})$ in (3.36), we obtain (3.35).

THEOREM 3.37. *If $2 \le r < k < \infty$ and the extended Riemann hypothesis is true, then:*

$$(3.38)\quad \begin{aligned} T(k, r; n) = Li(n) \prod_{p \nmid n} \left\{1 - \frac{1}{(p-1)}\left(\frac{1}{p^{r-1}} - \frac{1}{p^{k-1}}\right)\right\} \\ + 0\left(\frac{k}{k-r}(n^{(r+1)/2r}(\log n)^{(r-1)/r}\right) \quad as \quad r \to \infty. \end{aligned}$$

where the constant implied by the 0-estimate is independent of k and r.

Proof. As above, using (2.30) in place of (2.25), we obtain:

$$(3.39)\quad \begin{aligned} T(k, r; n) = Li(n) \prod_{p \nmid n} \left\{1 - \frac{1}{p-1}\left(\frac{1}{p^{r-1}} - \frac{1}{p^{k-1}}\right)\right\} \\ + 0\left(\frac{k}{k-r} \cdot \frac{nx^{1/r}}{x}\right) + 0\left(\frac{k}{k-r} \cdot \frac{nx^{1/r}}{n^{1/2}/\log n}\right). \end{aligned}$$

Setting $x = n^{1/2}/\log n$ in (3.39) we obtain (3.38).

REMARKS. Setting $k = \infty$ in the statements of Theorems 3.21, 3.24 or 3.37 (with $k/k-r$ replaced by 1, and $1/p^\infty = 1/p^\infty - 1 = 0$) reduces these theorems to known results about the r-free integers. A proof for $k = \infty$ requires a minor modification of the proofs given above.

From the above results, we may easily deduce that every sufficiently large integer n can be expressed as the sum of a prime and a (k, r)-free integer. However, this is not a new result since a square-free number is always (k, r)-free and (as was noted in (§1)) in 1931 Esterman proved that every sufficiently large number is the sum of a prime and a square-free integer. The following result is new:

THEOREM 3.40. *Every sufficiently large integer n can be written as $n = p + ab^k$ where both a and b are square free, $(a, b) = 1$, $b > 1$ and k is any fixed integer, $k \ge 3$.*

Proof. Let $T(n)$ be the number of representations of n as a sum as given above. Clearly:

$$T(n) = T(k, 2; n) - T(k+1, 2; n).$$

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Applying Theorem 3.21, we find:

$$T(n) = Li(n)\left(\prod_{p\nmid n}\left(1-\frac{1}{p-1}\left(\frac{1}{p}-\frac{1}{p^{k-1}}\right)\right)\right. \tag{3.41}$$
$$\left.-\prod_{p\nmid n}\left(1-\frac{1}{p-1}\left(\frac{1}{p}-\frac{1}{p^{k}}\right)\right)\right)+0(n/\log^H(n)) \quad \text{as} \quad n\to\infty$$

Let:

$$\alpha(p)=1-\frac{1}{p-1}\left(\frac{1}{p}-\frac{1}{p^{k-1}}\right)$$
$$\beta(p)=1-\frac{1}{p-1}\left(\frac{1}{p}-\frac{1}{p^{k}}\right)$$
$$\gamma(p)=\frac{\alpha(p)}{\beta(p)}-1.$$

We have:

$$\begin{aligned}\beta(p)\gamma(p)&=\alpha(p)-\beta(p)\\&=1-\frac{1}{p-1}\left(\frac{1}{p}-\frac{1}{p^{k-1}}\right)-\left(1-\frac{1}{p-1}\left(\frac{1}{p}-\frac{1}{p^{k}}\right)\right)\\&=\frac{1}{p-1}\left(\frac{1}{p^k}-\frac{1}{p^{k+1}}\right)\\&=1/p^{k+1}\end{aligned}$$

Thus $\gamma(p)>1/p^{k+1}$

$$\frac{\alpha(p)}{\beta(p)}=1+\gamma(p)>1+1/p^{k+1}$$

Let $\beta=\prod_p\beta(p)$. We have $0<\beta<1$. Thus

$$\begin{aligned}\prod_{p\nmid n}\alpha(p)-\prod_{p\nmid n}\beta(p)&=\left(\prod_{p\nmid n}\beta(p)\right)\left(\left(\prod_{p\nmid n}\frac{\alpha(p)}{\beta(p)}\right)-1\right)\\&>\prod_{p\nmid n}\beta(p)\left(\left(\prod_{p\nmid n}(1+1/p^{k+1})\right)-1\right)\\&=\frac{\prod_p\beta(p)}{\prod_{p\mid n}\beta(p)}\left(\left(\prod_{p\nmid n}(1+1/p^{k+1})\right)-1\right)\\&>\beta\left(\left(\prod_{p\mid n}(1+1/p^{k+1})\right)-1\right)\\&>\beta((1+1/p^{k+1})-1)\\&=\beta/p^{k+1}\end{aligned}$$

where the prime above is any prime not dividing n. By Lemma 2.31, for n

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sufficiently large, we may assume $p<2\log n$. Thus

$$\prod_{p\nmid n}\alpha(p)-\prod_{p}\beta(p)>\beta/2^{k+1}(\log n)^{k+1}$$

Thus from (3.41) we have

$$\frac{T(n)}{\prod_{p\nmid n}\beta(n)-\prod_{p\nmid n}\alpha(p)}=1+0\left(\frac{n}{\log^{H(r-1)/n}(n)}2k\cdot\frac{\log^k n}{\beta}\cdot\frac{\log n}{n}\right)\quad\text{as}\quad n\to\infty$$
$$=1+0(\log^{-(H\cdot(r-1)/r)-k-2)}(n))\quad\text{as}\quad n\to\infty.$$

Since H is arbitrary, choose $H>(k+2)(r/r-1)$ and Theorem 3.40 follows.

§4. **Remarks.** I. In [2, section (5.2) and Theorem 4], L. Carlitz considered, in effect the following generalization of r-free integers. Let $s(p)$ be an integral valued function of the prime p such that $s(p)\geq 1$ for all primes p and indeed $s(p)\geq 2$ for all but possibly a finite number of primes. Then Carlitz considered integers n which are not divisible by $p^{s(p)}$ for any prime p. He did not estimate the number of such integers n not exceeding n. His interest was in estimating sums of the type

$$\sum_{n=n_1+\cdots+n_t} n_1^{h_1}\cdots n_t^{h_t}$$

where $h_1,\ldots,h_t$ are non negative integers and $p^{s_i(p)}\nmid n_i$ $(i=1,\ldots,t)$, each $s_i(p)$ being of the same type as $s(p)$ described above.

Some time ago, Carlitz suggested to the second author the study of integers n with the following more general property. Let $\alpha=(\alpha_1,\alpha_2,\ldots)$ and $\beta=(\beta_1,\beta_2,\ldots)$ where $1\leq\alpha_i\leq\beta_i\leq\infty$ for all i, with $\alpha_i>1$ for all but a finite number of values of i. Call a number n $(\alpha,\beta)-$free if in its unique representation

$$n=2^{\gamma_1}3^{\gamma_2}\cdots q_l^{\gamma_l}\ldots,$$

(where q_l is the lth prime in the natural sequence of primes), we have $\gamma_i<\alpha_i$ or $\gamma_i\geq\beta_i$.

Obviously our (k,r)-free integers are a specialized class of (α,β)-free integers.

We can show that if $Q_{\alpha,\beta}(x)$ denotes the number of (α,β)-free integers not exceeding x, then

$$Q_{\alpha,\beta}(x)\sim x\prod_{m=1}^{\infty}\left(1-\frac{1}{q_m^{\alpha_m}}+\frac{1}{q_m^{\beta_m}}\right).$$

We omit the details of proof which is similar to that of Theorem 3.1.

II. We give in (3.41) an explicit formula for the number of representations of n as the sum $p+ab^k$. We are naturally led to ask if the seemingly weaker question: can every sufficiently large integer be represented as the sum of a prime plus ab^2 where p is a prime, both a and b are square free, $(a,b)=1$ and $b>1$. Indeed, the result of Subbarao and Suryanarayana quoted above holds for $k=2$ (i.e., every sufficiently large integer is the sum of a prime plus ab^2, where $(a, b)=1$, and a is square free). This follows trivially from the result for $k=4$, since $n=p+a\cdot b^4=p+a\cdot(b^2)^2$. Our result does not admit such an easy generalization to $k=2$, and we leave this question open with the conjecture.

CONJECTURE 4.1. Every sufficiently large integer n can be represented as the sum of a prime and a number $a\cdot b^2$, where a and b are both square-free, $(a, b)=1$ and $b>1$.

Added in Proof: We can now prove this conjecture. The proof will be published later.

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