Can. J. Math.Vol. 44 (3), 1992 pp. 483–504

SLICINGS, SELECTIONS AND THEIR APPLICATIONS

N. GHOUSSOUB, B. MAUREY AND W. SCHACHERMAYER

0. Introduction. In the past few years, much progress have been made on several open problems in infinite dimensional Banach space theory. Here are some of the most recent results:

1) The existence of *boundedly complete basic sequences* in a large class of Banach spaces including the ones with the so-called *Radon-Nikodym property* ([G-M2], [G-M4]).
2) The embedding of separable reflexive Banach spaces into reflexive spaces with basis ([Z]).
3) The existence of *long sequences of projections* and hence of *locally uniformly convex norms* in the duals of *Asplund spaces.* ([F-G])

We have chosen these problems because—as we are going to show in this paper—their solutions turned out to be closely related. Indeed, the solution of 3) is based on a recent selection theorem due to Jayne and Rogers [J-R], while for problem 2), Zippin devised an ad-hoc method for selecting points in certain weak*-compact subsets of dual Banach spaces and asked (in the first version of his paper) whether a general selection principle can be established in a non-linear setting. We later realized that the *slicing* methods (and the disguised *selections*) used in [G-M2] and [G-M4] to deal with problem 1) can be used to answer Zippin's query which, in turn, contains the selection result of Jayne and Rogers.

We refer to that selection—in Theorem (A) below—as *the Dessert selection* because as we shall see later, the point that is selected in a set K, will be—roughly—the one that corresponds to "the last bite" of K, in some well ordered procedure of "eating up" the whole space. We then consider what happens if we choose to select the point that corresponds to "the first bite". In that case we obtain what we called *The Hors-d'œuvre selection* which is in some sense "dual" to the Dessert selection.

The Hors-d'œuvre selection turned out to be the appropriate extension of the classical selection theorems of Kuratowski and Ryll-Nardzewski [K-R]. In the bitopological setting, it can be chosen to select points of continuity relative to the set in question. In the convex compact setting (resp. in the Radon-Nikodym case, resp. in the Analytic Radon-Nikodym case), it can be chosen to select extreme points (resp. strongly exposed points) (resp. plurisubharmonic barriers) from the closed sets.

Received by the editors May 3, 1990.
AMS subject classification: 46A55, 46B20, 49A27.

https://doi.org/10.4153/CJM-1992-031-6 Published online by Cambridge University Press

In the first section of this paper, we introduce a general setting which allows us to cover many—already used—slicing procedures and from which we can prove the selection theorems discussed above. In section II, we deduce various classical selection results while in Section III, we give a streamlined proof of the theorem of Zippin mentioned above.

I. The selection theorems. Let X be a topological space and let Δ be a metric on X. We shall say that X is *Δ-fragmentable* if for every non-empty closed subset F of X and every $\varepsilon > 0$, there exists a closed subset G of X such that $F \setminus G$ is non-empty and has a Δ-diameter less than ε. We denote the class of non-empty compact subsets of X by $\mathcal{K}(X)$. The definition of a *slice-upper (or lower) semi-continuous* multivalued map will be given below.

We shall start by stating the main results of this paper.

THEOREM (A) (THE DESSERT SELECTION). *Assume X is a topological space that is fragmentable by a metric Δ; then there exists a selection s: $\mathcal{K}(X) \longrightarrow X$ such that:*

a) $s(K) \in K$ for every $K \in \mathcal{K}(X)$.

b) If $K_1 \subset K_2$ and $s(K_2) \in K_1$ then $s(K_1) = s(K_2)$.

c) If (K_i) is a decreasing net in $\mathcal{K}(X)$ and if $K = \cap_i K_i$, then $\lim_i \Delta\big(s(K_i), s(K)\big) = 0$.

d) If Γ is a slice-upper semi-continuous multivalued mapping from a metric space (Z, d) into $\mathcal{K}(X)$, then $z \longrightarrow s\big(\Gamma(z)\big)$ is a Baire-1 function from Z into (X, Δ).

Furthermore, if (X, Δ) is a convex subset of a topological vector space, then $z \longrightarrow s\big(\Gamma(z)\big)$ is a pointwise limit of a sequence of continuous functions.

Note that if (X, d) itself is a metric space and if we equip $\mathcal{K}(X)$ with the induced Hausdorff metric, the above theorem implies then that the selection map s: $\mathcal{K}(X) \longrightarrow (X, \Delta)$ is a Baire-1 map.

For the next result, we shall denote by $\mathcal{F}(X)$ the class of all Δ-complete subsets of X.

THEOREM (B) (THE HORS-D'OEUVRE SELECTION). *Assume (X, τ) is a completely regular topological space that is Δ-fragmentable by a metric that induces a finer topology on X; then there exists a selection s^*: $\mathcal{F}(X) \longrightarrow X$ such that:*

a) For every $F \in \mathcal{F}(X)$, $s^*(F)$ is a point of $(\tau - \Delta)$-continuity relative to F.

b) If $F_1 \subset F_2$ and $s^*(F_2) \in F_1$ then $s^*(F_1) = s^*(F_2)$.

c) If (F_i) is an increasing net in $\mathcal{F}(X)$ then $\big(s^*(F_i)\big)$ is Δ-Cauchy. Moreover, if $F = \Delta$-closure of $\cup_i F_i$ is Δ-complete, then $\lim_i \Delta\big(s^*(F_i), s^*(F)\big) = 0$.

d) If Γ is a slice-lower semi continuous multivalued mapping from a metric space (Z, d) into $\mathcal{F}(X)$, then $z \longrightarrow s^*\big(\Gamma(z)\big)$ is a Baire-1 function from Z into (X, Δ).

Furthermore, if X is a convex subset of a topological vector space, then $z \longrightarrow s^*\big(\Gamma(z)\big)$ is a pointwise limit of a sequence of continuous functions.

THEOREM (C) (THE INJECTIVE SLICING). *Assume X is Δ-fragmentable, then there exist a totally ordered space B and a map $\varphi: X \longrightarrow B$ such that:*

https://doi.org/10.4153/CJM-1992-031-6 Published online by Cambridge University Press

a) *φ is one to one and upper semi-continuous (i.e. $X_s = \{\varphi \geq s\}$ is closed for every $s \in B$).*

b) *For $K \in \mathcal{K}(X)$, φ achieves its maximum at a unique point in K, equal to the dessert selection $s(K)$.*

c) *When t decreases to s in B, the Δ-diam of $X_s \setminus X_t$ tends to zero. Hence the inverse mapping of φ is right-continuous.*

Assume in addition that (X, τ) is completely regular and that (X, Δ) is complete while defining a topology finer than τ. Then,

d) *Every net (x_i) in X such that $\big(\varphi(x_i)\big)$ is decreasing is necessarily Δ-convergent to an $x \in X$ that verifies $\varphi(x) = \lim_i \varphi(x_i)$.*

e) *For $F \in \mathcal{F}(X)$, φ achieves its minimum at a unique point in F, equal to the Hors d'œuvre selection $s^*(F)$.*

For the next result, we recall that a Banach space Y is said to have the *Radon-Nikodym Property* (R.N.P) (resp. *the Analytic Radon-Nikodym Property* (A.R.N.P)) if every uniformly bounded Y-valued martingale (resp. Analytic martingale) converges almost surely (See [B], [E] and [G-M4] for details). A point x in a closed bounded subset F of Y is said to be *strongly exposed* (resp. *a strong barrier*) in F if there exists a continuous linear functional (resp. a Lipschitz plurisubharmonic function) f such that every maximizing sequence for f in F converges necessarily to x (In particular f attains its maximum on F at the point x). Such a function f is called a *strongly exposing functional* (resp. *a plurisubharmonic barrier*). It is well known that closed bounded subsets of spaces with the R.N.P (resp. A.R.N.P) have many strongly exposed (resp. strong Barrier) points. In the following theorem we show that the selection of such points can be done in a measurable fashion.

If Y is a Banach space, we shall equip Y^* with its natural norm. We denote by PSH(Y) the convex cone of plurisubharmonic and Lipschitz functions on Y equipped with the following norm:

$$\|\varphi\| = \max\big\{|\varphi(0)|, \sup\{\,|\varphi(x) - \varphi(y)| / \|x - y\| \;;\, x \neq y\}\,\big\}.$$

Let $B_\rho(Y^*) = \{y^* \in Y^* \;;\, \|y^*\| \leq \rho\}$ and $\mathrm{PSH}_\rho(Y) = \{\varphi \in \mathrm{PSH}(Y) \;;\; \|\varphi\| \leq \rho\}$.

In Section I.7 we shall prove a result that implies the following.

THEOREM (D) (EXTREMAL SELECTIONS). *Let X be the unit ball of a real (resp. complex) Banach space Y with the Radon-Nikodym (resp. the Analytic Radon-Nikodym) Property. Let $\mathcal{F}(X)$ denote the class of closed subsets of X. Then there exist a selection $s^*\colon \mathcal{F}(X) \longrightarrow X$ and a Baire-1 map $r\colon X \longrightarrow B_1(Y^*)$ (resp. $r\colon X \longrightarrow \mathrm{PSH}_1(Y)$) such that:*

a) *For every $F \in \mathcal{F}(X)$, $s^*(F)$ is strongly exposed in F by $r\big(s^*(F)\big)$.*

b) *If $F_1 \subset F_2$ and $s^*(F_2) \in F_1$ then $s^*(F_1) = s^*(F_2)$.*

c) *If (F_i) is an increasing net in $\mathcal{F}(X)$ and if $F =$ norm-closure of $\cup_i F_i$, then $s^*(F_i)$ converges to $s^*(F)$.*

https://doi.org/10.4153/CJM-1992-031-6 Published online by Cambridge University Press

d) *If* Γ *is a slice-lower semi-continuous multivalued mapping from a metric space* (Z, d) *into* $\mathcal{F}(X)$*, then* $z \longrightarrow s^*\big(\Gamma(z)\big)$ *is a pointwise limit of a sequence of continuous functions.*

Suppose now f is a bounded below lower semi-continuous function on a complete metric space (X, Δ). A well known theorem of Ekeland [Ek] asserts that for any $\varepsilon > 0$ and any closed subset F of X on which f is not identically $+\infty$, there exists a point $x_0 \in F$ that minimizes the function $f + \varepsilon\Delta(x_0, .)$ on F. Such a point will be called an ε*-Ekeland point for* f *on* F. In the following theorem, we show that the selection of Ekeland points from the closed subsets of X can be done in a measurable fashion. In the sequel, we shall denote by $\mathcal{F}_f(X)$ the class of non empty closed subsets of X on which f is not identically $+\infty$.

THEOREM (E) (THE OPTIMAL SELECTION). *Let* (X, Δ) *be a complete metric space and let* ε *be a strictly positive real number. Then, for every lower semi-continuous function* $f: X \longrightarrow [0, +\infty]$, *there exists a selection* $s_f^*: \mathcal{F}_f(X) \longrightarrow X$ *such that:*

a) *For every* $F \in \mathcal{F}_f(X)$, $s_f^*(F)$ *is an* ε*-Ekeland point for* f *relative to* F.

b) *If* $F_1 \subset F_2$ *and* $s_f^*(F_2) \in F_1$ *then* $s_f^*(F_1) = s_f^*(F_2)$.

c) *If* (f_i) *is a decreasing family of l.s.c functions from* X *into* $[0, +\infty]$ *and if* f *is the l.s.c envelope of* $\inf_i f_i$*, then for any* $F \in \mathcal{F}_{f_0}(X)$*, the net* $s_{f_i}^*(F)$ *converges to* $s_f^*(F)$.

d) *If* f *is continuous and if* (F_i) *is an increasing net in* $\mathcal{F}_f(X)$*, then* $s_f^*(F_i)$ *converges to* $s_f^*(F)$*, where* $F = \Delta$*-closure of* $\cup_i F_i$.

e) *If* Γ *is a slice-lower semi-continuous multivalued mapping from a metric space* (Z, d) *into* $\mathcal{F}_f(X)$*, then* $z \longrightarrow s_f^*\big(\Gamma(z)\big)$ *is a Baire-*1 *function from* Z *into* X.

Furthermore, if X *is a convex subset of a topological vector space, then* $z \longrightarrow s_f^*\big(\Gamma(z)\big)$ *is a pointwise limit of a sequence of continuous functions.*

In the process of proving the above theorems we shall introduce and discuss various concepts which may have an independent interest.

I.1. Slicings. Let X be a topological space. A *slicing of* X is a mapping $f: X \longrightarrow A$ where A is a totally ordered set and where f is upper semi-continuous (u.s.c) in the sense that for every $\alpha \in A$, the set $X_\alpha = \{x \in X\,; f(x) \geq \alpha\}$ is a closed subset of X.

REMARKS I.1.1. With the above definition we have:

a) Either $\bigcap_\alpha X_\alpha = \emptyset$, or $f(X)$ has a greatest element.

b) For every non-empty compact subset K of X, f achieves its maximum on K.

Indeed a) is obvious while to prove b), it is enough to consider $I = f(K)$ and to notice that $\{K_i = K \cap X_i\,;\ i \in I\}$ is a decreasing family of non-empty compact subsets of X, therefore the intersection L of this family is non-empty. It is then clear that $f(x) = \max f(K)$ for every $x \in L$.

If α and β in A are such that no γ in A satisfies $\alpha < \gamma < \beta$, we say that β is the successor of α in A. When no confusion can occur we denote this β by $\alpha + 1$.

If every element in A has a successor, we see that the set $\{f > \alpha\} = \{f \geq \alpha + 1\}$ is closed. We shall say that $f: X \longrightarrow A$ is *a discrete slicing* if for every $\alpha \in A$, the set $\{f > \alpha\}$ is a closed subset of X. This set $\{f > \alpha\}$ will be also denoted by $X_{\alpha+}$.

We say that *f is a well-ordered slicing* if A is well ordered (w.o slicing in short). For a discrete slicing it might be easier to think about the difference sets $D_\alpha = X_\alpha \setminus X_{\alpha+} = \{f = \alpha\}$. It is clear these difference sets cover X and determine completely the slicing f.

REMARK I.1.2. In every case (discrete, w.o) we may assume the difference sets non-empty by replacing A by $f(X)$.

Products of slicings. If $f: X \longrightarrow A$ is a discrete slicing and $g: X \longrightarrow B$ a slicing, we may consider the slicing $f \times g: X \longrightarrow A \times B$ where $A \times B$ is equipped with the lexicographic order. Indeed, if $(\alpha, \beta) \in A \times B$, the set $\{x \in X ; f \times g(x) \geq (\alpha, \beta)\}$ agrees then with the closed set $\{f > \alpha\} \cup (\{f \geq \alpha\} \cap \{g \geq \beta\})$.

If g is also discrete, then $f \times g$ is discrete and the difference sets are given by $D_{(\alpha,\beta)} = D_\alpha \cap D_\beta = \{f = \alpha\} \cap \{g = \beta\}$. If both are w.o slicings then $f \times g$ is a w.o slicing.

REMARK I.1.3. Let $f: X \longrightarrow A$ be a discrete slicing of X and $g: X \longrightarrow B$ a mapping from X into a totally ordered set B. If the restriction of g to every difference set $\{f = \alpha\}$ defines a slicing (resp. discrete slicing) of $\{f = \alpha\}$ then $f \times g$ is a slicing (resp. discrete slicing) of X.

I.2. Slice-generating classes of sets. Let $\mathcal{C}$ be a class of subsets of X. We say that *$\mathcal{C}$ is slice-generating* if for every non-empty closed subset F of X, there exists a closed subset G of X such that $F \setminus G$ is non-empty and belongs to $\mathcal{C}$.

PROPOSITION I.2.1. *Assume that $\mathcal{C}$ is hereditary. Then $\mathcal{C}$ is a slice-generating class if and only if there is a well ordered slicing f of X with difference sets $(D_\alpha)_{\alpha \in A}$ belonging to $\mathcal{C}$.*

PROOF. Assume first that $f: X \longrightarrow A$ is a well ordered slicing with difference sets $D_\alpha = \{f = \alpha\}$ belonging to $\mathcal{C}$. If F is a non-empty closed subset of X, let α be the smallest element of $f(F)$ and $G = X_{\alpha+1} = \{f \geq \alpha + 1\}$. We have that $F \setminus G$ is non-empty and contained in D_α, thus $F \setminus G$ belongs to $\mathcal{C}$ since the latter class is hereditary.

The other direction will be proved by a standard transfinite argument. The well-ordered set A will be some ordinal. We define by transfinite induction a decreasing family $(X_\alpha)_\alpha$ of closed subsets of X, for α ordinal, in the following way:

a) $X_0 = X$
b) If $\alpha = \beta + 1$ and X_β is non-empty, let $X_\alpha = G$ where G is a closed subset of X such that $X_\beta \setminus G \in \mathcal{C}$ is non empty.
c) If α is a limit ordinal, let $X_\alpha = \cap_{\beta < \alpha} X_\beta$.

There exists some ordinal γ such that $X_\gamma = \emptyset$. We let $A = [0, \gamma)$ and define f as follows:

https://doi.org/10.4153/CJM-1992-031-6 Published online by Cambridge University Press

For $x \in X$, let $\alpha(x)$ be the smallest α such that $x \notin X_\alpha$. By condition c), α must be a successor, say $\alpha = \beta + 1$; we set $f(x) = \beta$. The difference sets for f are the sets $X_\beta \setminus X_{\beta+1}$, hence they belong to $\mathcal{C}$.

I.2.2. Examples of slice-generating classes. 1) The typical example of a slice-generating class of sets appears when we have a topological space X that is fragmentable by a metric Δ. In this case, the class $\mathcal{C}_\varepsilon$ of all subsets of X of Δ-diameter less than ε is slice-generating for every $\varepsilon > 0$. According to Proposition I.2.1, there exists for every $\varepsilon > 0$ a w.o. slicing f_ε of X with difference sets of Δ-diameter less than ε.

The most trivial case is when the metric Δ generates the topology τ of X (*i.e.* X is a metric space) while the most general case corresponds to when the identity Id: $(X, \tau) \longrightarrow (X, \Delta)$ is a Baire-1 map. Here are some examples:

a) If X is the unit ball of a Banach space equipped with the weak topology, then it will be fragmentable by the norm if the space has the so-called *Point of Continuity Property* (P.C.P). This happens for instance when the space is reflexive [E-W], [G-M1].

b) If X is the unit ball of a dual Banach space equipped with the weak*-topology, then it will be norm-fragmentable if it has the *Radon-Nikodym Property* (R.N.P). This happens for instance when X is norm separable [B].

$\mathcal{F}$–Slice-generating classes of sets. In many cases the sets X_α of our slicing may belong to a special subclass of closed sets. For example

i) If (X, Δ) is a metrizable convex compact subset of a locally convex topological vector space, then for each $\varepsilon > 0$, the class is slice-generating. See for example [G-M3]. In that case, it is possible to produce a "convex compact slicing" (The X_α's are convex and compact).

ii) More generally, if X is a closed convex bounded Radon-Nikodym subset of a Banach space E equipped with the weak topology then the class

$$\mathcal{S}_\varepsilon = \left\{ S \neq \emptyset \,;\, \operatorname{diam}(S) < \varepsilon,\ S = F \cap \{\, \ell > 0 \} \right\}$$

where F is closed convex and $\ell \in E^* + \mathbb{R}$ is also slice-generating [B]. In that case the X_α's are closed and convex.

iii) If X is a closed bounded subset of a quasi-Banach space with the Analytic Radon-Nikodym property (A.R.N.P) equipped with the quasi-norm, then the class

$$\mathcal{S}_\varepsilon = \left\{ S \neq \emptyset \,;\, \operatorname{diam}(S) < \varepsilon,\ S = F \cap \{\, \ell > 0 \} \right\}$$

where F is closed and ℓ plurisubharmonic is also slice-generating [G-M5]. In that case the X_α's are closed and pseudoconvex (*i.e.* an intersection of a family of sets of the form $\{\varphi \geq 0\}$ with φ belonging to PSH(X).

To exploit this additional information about a slicing, one can easily adapt the results of this section to the following framework:

Let $\mathcal{F}$ be a class of closed subsets of X, containing X and stable by intersection, and let $\mathcal{C}$ be a class of subsets of X; we say that *$\mathcal{C}$ is $\mathcal{F}$-slice-generating* if for every non empty $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ such that $F \setminus G$ is non-empty and belongs to $\mathcal{C}$. We may also say that a slicing f of X is an $\mathcal{F}$-slicing when all sets $\{f \geq \alpha\}$ belong to $\mathcal{F}$.

$\mathcal{F}$-derivations. The notion of $\mathcal{F}$-slice generating class of subsets of X is not precise enough to deal with the extremal selections of Theorem (D). We introduce for this purpose the notion of *$\mathcal{F}$-derivation* on X; we call $\mathcal{F}$-derivation on X, a multimapping D that associates to every non-empty subset $F \in \mathcal{F}$ a *non-empty* class $D(F)$ of subsets E of F in such a way that every $E \in D(F)$ verifies $E \in \mathcal{F}$ and $E \neq F$. It is easy to see a connection between this notion and $\mathcal{F}$-slice generating classes: if D is an $\mathcal{F}$-derivation, then $\mathcal{C} = \{F \setminus E\,;\, F \in \mathcal{F}, E \in D(F)\}$ is an $\mathcal{F}$-slice-generating class. Conversely, if $\mathcal{C}$ is an $\mathcal{F}$- slice generating class, we get a derivation D_1 by setting

$$D_1(F) = \{G \subset F\,;\, G \in \mathcal{F} \quad \text{and} \quad \emptyset \neq F \setminus G \in \mathcal{C}\}\ ;$$

unfortunately, if we apply these two steps successively, we don't get in general that $D_1 = D$.

The proof of Proposition I.2.1 extends with essentially no change to this setting, and yields the following:

PROPOSITION I.2.3. *Let D be an $\mathcal{F}$-derivation on X. Then, there exists a w.o $\mathcal{F}$-slicing f of X such that $X_{\alpha+1} \in D(X_\alpha)$ for every α.*

In particular, if $\mathcal{C}$ is an $\mathcal{F}$-slice generating class of subsets of X, then there exists a w.o $\mathcal{F}$-slicing f of X such that every difference set belongs to $\mathcal{C}$. Conversely, if such a slicing f exists, then the class $\mathcal{D}$ consisting of all subsets of elements of $\mathcal{C}$ is $\mathcal{F}$-slice generating in X.

It is clear that in example i) (resp. ii), resp. iii)) mentioned above, $\mathcal{F}$ can be taken to be the class of convex compact (resp. convex closed, resp. pseudoconvex closed) sets and that natural $\mathcal{F}$-derivations can be associated to these examples.

Slice-constant functions. We say that a function h from X into an arbitrary set Y is *slice-constant* if there exists a discrete slicing $f\colon X \longrightarrow A$ of X with difference sets $(D_\alpha)_{\alpha\in A}$ on which h is constant. According to Proposition I.2.1, that happens for instance if the class $\mathcal{C}$ of subsets of X on which h is constant is slice- generating.

We say that a function between two topological spaces X and Y is *Baire*-1 when the inverse image of every open subset of Y is an F_σ subset of X.

LEMMA I.2.4. *Assume that X is a metric space, then any slice-constant function h from X into a topological space Y is Baire-1. Moreover, if Y is a convex subset of a topological vector space, then h is the pointwise limit of a sequence of continuous functions from X to Y.*

PROOF. For the first assertion, let $f\colon X \longrightarrow A$ be a discrete slicing of X such that h is constant on the difference sets $D_\alpha = \{f = \alpha\}$. Clearly it is enough to prove that every union $Z = \cup_{\alpha\in I} D_\alpha$ (where I is contained in A) is an F_σ in X. For every $n \geq 0$

consider $D_\alpha^n = \{x \in D_\alpha \; ; \; d(x, X_{\alpha+}) \geq 2^{-n}\}$. Then $Z^n = \cup_{\alpha \in I} D_\alpha^n$ is closed; indeed if (x_k) is contained in Z^n and x_k tends to x, we get that $f(x_k) \leq f(x)$ for $k \geq k_0$, since $x \notin X_{f(x)+}$. Now if $2^{-n} \leq d(x_k, X_{f(x_k)+}) \leq d(x_k, X_{f(x)+})$ we get that $d(x, X_{f(x)+}) \geq 2^{-n}$ and hence $x \in Z^n$. Since $Z = \cup_n Z^n$, it follows that Z is an F_σ in X.

For the second assertion, consider for every $n \geq 0$ the set

$$U_\alpha^n = \{x \in X \; ; \; d(x, X_\alpha) < 2^{-n})\} \setminus X_{\alpha+1}.$$

This open set contains D_α and therefore (U_α^n) is an open covering of X. Since X is paracompact, we may find a partition of unity (φ_α^n) such that φ_α^n vanishes outside U_α^n. If y_α denotes the constant value of h on D_α, consider $h_n(x) = \Sigma_{\alpha \in A} \varphi_\alpha^n(x) y_\alpha$. It is easy to check that $h_n(x) = h(x)$ when $n \geq n(x)$.

I.3. Two partial selections associated to a slicing.

The last bite. Let f be a fixed discrete slicing of X. Recall that $\mathcal{K}(X)$ denotes the class of non-empty compact subsets of X.

For every $K \in \mathcal{K}(X)$ we know that f achieves its maximum on K (Remark I.1.1). We define

$$\text{ind}(K) = \max f(K) \text{ and } L(K) = K \cap X_\alpha, \text{ where } \alpha = \text{ind}(K).$$

Note that we also have that $L(K) = K \cap (X_\alpha \setminus X_{\alpha+}) = K \cap D_\alpha$.

PROPOSITION I.3.1. *Let f be a discrete slicing of X. Then*

a) $L(K)$ is a non-empty compact subset of K.

b) $K_1 \subset K_2$ implies that $\text{ind}(K_1) \leq \text{ind}(K_2)$.

c) If K_1 is contained in K_2 and $x \in K_1 \cap L(K_2)$, then $\text{ind}(K_1) = \text{ind}(K_2)$ *and* $x \in L(K_1) \subset L(K_2)$.

d) If f is a w.o slicing and if (K_i) is a decreasing net in $\mathcal{K}(X)$ with intersection $K = \cap_{i \in I} K_i$, then $\text{ind}(K_i)$ *becomes eventually constant. It follows that $L(K_i)$ is eventually decreasing and that there exists i_0 such that $L(\cap_i K_i) = \cap_{i > i_0} L(K_i)$.*

PROOF. The first two assertions are obvious. To prove c), it is enough to notice that $\text{ind}(K_2) = f(x) \leq \max f(K_1)$ which implies that $\text{ind}(K_1) = \text{ind}(K_2)$ and $x \in L(K_1)$.

For d), assume that $(K_i)_{i \in I}$ is a decreasing net and that $f : X \longrightarrow A$ is a w.o slicing of X. Let $i_1 \in I$ and let $i_0 \in I$ be such that $\alpha_0 = \text{ind}(K_{i_0})$ is the smallest element of $B = \{\text{ind}(K_i) \; ; \; i \geq i_1\}$. If $i \geq i_0$ then $\text{ind}(K_i) \leq \text{ind}(K_{i_0})$ and hence $\text{ind}(K_i) = \alpha_0$. It follows that for $i \geq i_0$, $L(K_i) = K_i \cap X_{\alpha_0}$, and the rest is straightforward.

The first bite. Let f be a fixed w.o slicing of X. For every set F, define

$$\text{ind}^*(F) = \inf f(F) \text{ and } L^*(F) = F \setminus X_{\alpha+1}, \text{ where } \alpha = \text{ind}^*(F).$$

Note that we also have $L^*(F) = F \cap D_\alpha$.

The proof of the following proposition is similar to the preceding one and is left to the interested reader.

PROPOSITION I.3.2. *Let f be a w.o slicing of X. Then*

a) *For any non-empty subset F of X, $L^*(F)$ is a non-empty subset of F.*

b) *$F_1 \subset F_2$ implies that* $\operatorname{ind}^*(F_1) \geq \operatorname{ind}^*(F_2)$.

c) *If F_1 is contained in F_2 and $x \in F_1 \cap L^*(F_2)$, then* $\operatorname{ind}^*(F_1) = \operatorname{ind}^*(F_2)$ *and* $x \in L^*(F_1) \subset L^*(F_2)$.

d) *If (F_i) is an increasing net of subsets of X, then $\big(\operatorname{ind}^*(F_i)\big)$ becomes eventually constant and $\big(L^*(F_i)\big)$ becomes eventually increasing.*

Slice-u.s.c and l.s.c multimappings. Now let Γ be a multimapping from some topological space Z into X. Recall that Γ is said to be *upper* (resp. *lower*) *semi-continuous* if for every closed subset F of X, the set $\{z \in Z\ ;\ \Gamma(z) \cap F \neq \emptyset\}$ (resp. $\{z \in Z\ ;\ \Gamma(z) \subset F\}$) is closed.

We shall say that Γ is *slice-upper semi-continuous* (resp. *slice-lower semi-continuous*) if there exists a discrete slicing $g\colon Z \longrightarrow B$ such that the restriction of Γ to every difference set of g is u.s.c (resp. l.s.c). This is the case for example when the class of subsets Y of Z such that the restriction of Γ to Y is upper (resp. lower) semi-continuous is slice-generating in Z.

LEMMA I.3.3. *Let Z be topological and let Γ be a slice-upper semi-continuous multimapping from Z into $\mathcal{K}(X)$. Then the function $z \longrightarrow \operatorname{ind}\big(\Gamma(z)\big)$ is slice-constant, and the multimapping $z \longrightarrow L\big(\Gamma(z)\big)$ is slice-upper semi-continuous.*

PROOF. Let $g\colon Z \longrightarrow B$ be a discrete slicing such that Γ is u.s.c on every difference set E_β. Then $\varphi(z) = \operatorname{ind}\big(\Gamma(z)\big)$ defines a discrete slicing of E_β, for every $\beta \in B$. This is because $\{\varphi \geq \alpha\} \cap E_\beta = \{z \in E_\beta\ ;\ \Gamma(z) \cap X_\alpha \neq \emptyset\}$ is a closed subset of E_β and similarly for $\{\varphi > \alpha\}$. Using Remark I.1.3 we know that $g \times \varphi$ defines a discrete slicing of Z. We see then that φ is slice-constant on Z. On a difference set $D_{\beta,\alpha} = \{g = \beta\} \cap \{\varphi = \alpha\}$ we have $z \in E_\beta$ and $L\big(\Gamma(z)\big) = \Gamma(z) \cap X_\alpha$, thus $z \longrightarrow L\big(\Gamma(z)\big)$ is u.s.c on $D_{(\beta,\alpha)}$.

LEMMA I.3.4. *Assume that X is equipped with a w.o slicing. Let Z be topological and let Γ be a slice-lower semi-continuous multimapping from Z into the subsets of X. Then the function $z \longrightarrow \operatorname{ind}^*\big(\Gamma(z)\big)$ is slice-constant, and the multimapping $z \longrightarrow L^*\big(\Gamma(z)\big)$ is slice-lower semi-continuous.*

PROOF. As above let $g\colon Z \longrightarrow B$ be a discrete slicing such that Γ is l.s.c on every difference set E_β. Then $\varphi(z) = \operatorname{ind}^*\big(\Gamma(z)\big)$ defines a discrete slicing of E_β, for every $\beta \in B$. This is because $\{\varphi \geq \alpha\} \cap E_\beta = \{z \in E_\beta\ ;\ \Gamma(z) \subset X_\alpha\}$ is a closed subset of E_β and similarly for $\{\varphi > \alpha\}$. Using Remark I.1.3 we know that $g \times \varphi$ defines a discrete slicing of Z. We see then that φ is slice-constant on Z. On a difference set $D_{\beta,\alpha} = \{g = \beta\} \cap \{\varphi = \alpha\}$, we have $z \in E_\beta$ and $L^*\big(\Gamma(z)\big) = \Gamma(z) \setminus X_{\alpha+1}$, thus $z \longrightarrow L^*\big(\Gamma(z)\big)$ is l.s.c on $D_{(\beta,\alpha)}$.

I.4. Two selections associated to a fragmentation. Suppose that X is a topological space and let Δ be a metric on X. We shall call *fragmentation of* X any sequence of w.o slicings $(f_n)_n$ of X such that for every integer n the difference sets of the slicing f_n have Δ-diameter less than 2^{-n}. If each f_n is also an $\mathcal{F}$-slicing, we shall call it then *an* $\mathcal{F}$*-fragmentation of* X.

The dessert selection associated to a fragmentation $(f_n)_n$. Let L_n be the partial selection operator associated with f_n. Define inductively a sequence of operators on $\mathcal{K}(X)$ by

$$S_0(K) = K \text{ and } S_{n+1}(K) = L_{n+1}\big(S_n(K)\big).$$

Then for every n, $S_n(K)$ is compact, non-empty and contained in some difference set for f_n, hence has Δ-diameter less than 2^{-n}. Furthermore, this sequence $\big(S_n(K)\big)$ is decreasing. It follows that $\cap_n S_n(K)$ contains exactly one point, which we will denote by $s(K)$. The function $s\colon \mathcal{K}(X) \longrightarrow X$ will be called *the dessert selection* associated to the fragmentation $(f_n)_n$.

The hors d'œuvre selection associated to a fragmentation $(f_n)_n$. Recall that $\mathcal{F}(X)$ denotes the class of all Δ-complete subsets of X. Let now L_n^* be the first-partial selection operator associated with f_n. Define a sequence of operators on $\mathcal{F}(X)$ by

$$S_0(F) = F \text{ and } S_{n+1}^*(F) = L_{n+1}^*\big(S_n^*(F)\big).$$

Then for every n, $S_n^*(F)$ is non-empty and contained in some difference set for f_n, hence it has a Δ-diameter less than 2^{-n}. Furthermore, this sequence $\big(S_n^*(F)\big)$ is decreasing. If $\bar{S}_n^*(F)$ denotes the Δ-closure of $\big(S_n^*(F)\big)$, it follows that $\cap_n \bar{S}_n^*(F)$ contains exactly one point that belongs necessarily to F and which we will denote by $s^*(F)$. The function $s^*\colon \mathcal{F}(X) \longrightarrow X$ will be called *the hors-d'œuvre selection associated to the fragmentation* $(f_n)_n$.

PROOF OF THEOREM (A). Assume that X is fragmentable by the metric Δ, then—as mentioned above—the class $\mathcal{C}_\varepsilon$ of all subsets of X of Δ-diameter less than ε is slice-generating for every $\varepsilon > 0$. According to Proposition I.2.1, there exists for every integer n a w.o. slicing f_n of X with difference sets of Δ-diameter less than 2^{-n}. Let $s\colon \mathcal{K}(X) \longrightarrow X$ be *the dessert selection* associated to the fragmentation $(f_n)_n$. We shall now verify that s satisfies the properties announced in Theorem (A).

a) is clear.

To prove b) assume that $s(K_2) \in K_1$ and $K_1 \subset K_2$. Then, according to Proposition I.3.1 d), we have $s(K_2) \in L_1(K_1) \subset L_1(K_2)$, hence $s(K_2) \in L_2\big(S_1(K_2)\big)$ and $S_1(K_1) \subset S_1(K_2)$. By an inductive application of that observation, it follows that $s(K_2) \in S_n(K_1) \subset S_n(K_2)$ for every $n \geq 1$, therefore $s(K_2) = s(K_1)$.

c) If $(K_i)_{i\in I}$ is a decreasing net in $\mathcal{K}(X)$ with intersection K, we prove inductively using Proposition I.3.1 c) that for every $n \geq 1$ there exists i_n such that $i > i_n$ yields that $\mathrm{ind}_n\big(S_{n-1}(K_i)\big) = \mathrm{ind}_n(S_{n-1}(K)$, that $\big(S_n(K_i)\big)$ is decreasing for $i > i_n$ and that

$\bigcap_{i>i_n} S_n(K_i) = S_n(K)$; it follows that for $i > i_n$, $s(K_i)$ and $s(K)$ belong to the same difference set for f_n, and therefore $\Delta\big(s(K_i), s(K)\big) \leq 2^{-n}$.

d) We see by induction, using Lemma I.3.3 that $g_n: z \longrightarrow \mathrm{ind}_n(S_{n-1}(\Gamma(z))$ is slice-constant and that $z \longrightarrow S_n\big(\Gamma(z)\big)$ is slice-u.s.c for every $n \geq 1$. Let x^n_α be an arbitrary point in $\{f_n = \alpha\}$ provided this latter set is non-empty. Then $h_n(z) = x^n_{g_n(z)}$ is slice-constant, thus a Baire-1 function, in view of Lemma I.2.5. Moreover, these functions converge Δ-uniformly to $s\big(\Gamma(z)\big)$ since $f_n\big(h_n(z)\big) = g_n(z) = f_n\Big(s\big(\Gamma(z)\big)\Big)$ implies that $\Delta\Big(h_n(z), s\big(\Gamma(z)\big)\Big) < 2^{-n}$.

In the following, we show that fragmentability is essentially a necessary condition for the existence of dessert selections.

PROPOSITION I.4.1. *Let (X, d) be a complete metric space and suppose that for some metric $\Delta \geq d$ there exists a selection s: $\mathcal{K}(X) \longrightarrow X$ such that for any sequence $(K_n)_n$ in $\mathcal{K}(X)$ that decreases to a singleton $\{x\}$ we have that* $\lim_n \Delta\big(s(K_n), x\big) = 0$. *Then the identity map $(X, d) \longrightarrow (X, \Delta)$ is a Baire-1 function and therefore (X, d) is Δ-fragmentable.*

PROOF. It is enough to show that for every compact $K \subset (X, d)$ that is homeomorphic to $\{-1, +1\}^{\mathbb{N}}$, the restriction id: $(K, d) \longrightarrow (K, \Delta)$ is a Baire-1 map, since then we get from standard Baire theory that the inverse mapping will have a residual set of points of continuity, which will clearly imply that (X, d) is Δ-fragmentable.

For each n and for any sequence $(\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$ denote by $F_{\varepsilon_1,\ldots,\varepsilon_n}$ the set $\big\{t \in \{-1, 1\}^{\mathbb{N}}\,;\, t_i = \varepsilon_i, i = 1, \ldots, n\big\}$.

It is now enough to notice that the functions i_n: $(K, d) \longrightarrow (K, \Delta)$ defined by

$$i_n = \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in\{-1,+1\}^n} s(F_{\varepsilon_1,\ldots,\varepsilon_n}) \cdot \chi_{F_{\varepsilon_1,\ldots,\varepsilon_n}}$$

are continuous and that the hypothesis implies their pointwise convergence to the identity.

I.5. Strong fragmentability. In order to prove the other theorems we are going to use a slightly stronger hypothesis than the Δ-fragmentability of X. First we note that for any fragmentation $(f_n)_n$, we can associate a *decreasing fragmentation* $(g_n)_n$ by just taking $g_n = f_1 \times f_2 \times \cdots \times f_n$. It is clear that the dessert (resp. the hors-d'œuvre) selections associated to the fragmentations $(f_n)_n$ and $(g_n)_n$ are identical. Hence (unless we are dealing with $\mathcal{F}$-slicings) we can always assume that a fragmentation consists of a decreasing sequence of slicings. However, for the remaining results, we shall need a slightly stronger notion of fragmentablity that will help us choose fragmentations with some control on the way two consecutive slicings are related.

This hypothesis—that we shall call (SF) (for Δ-*strongly fragmentable*)—will read as follows:

(SF): Δ defines a finer topology on X in such a way that for every pair F, H of closed subsets of X such that $F \setminus H$ is non-empty and for every $\varepsilon > 0$ there exists a closed subset G of X such that $F \setminus G \neq \emptyset$, has Δ-diameter less than ε while its Δ-closure is disjoint from H.

Here are some examples where we have strong fragmentability:

EXAMPLE I.5.1. If X is a completely regular space that is fragmented by a metric which induces a finer topology on X, then it is clear that hypothesis (SF) is satisfied.

If for example X is the unit ball of a Banach space with P.C.P, equipped with the weak topology, then X verifies (SF) with respect to the norm.

More generally, if $\mathcal{F}$ is a class of closed subsets of X, containing X and stable by intersection (as in I.2.4), we may define the following property:

(SF-$\mathcal{F}$): Δ defines a finer topology on X in such a way that for every pair F, H of sets in $\mathcal{F}$ such that the set $F \setminus H$ is non-empty and for every $\varepsilon > 0$ there exists a set $G \in \mathcal{F}$ such that $F \setminus G \neq \emptyset$, has Δ-diameter less than ε while its Δ-closure is disjoint from H.

EXAMPLE I.5.2. If X is the unit ball of a Banach space with R.N.P (resp. A.R.N.P) and if $\mathcal{F}$ is the class of closed convex (resp. closed pseudoconvex) subsets of X, then X verifies (SF-$\mathcal{F}$) (see Section I.7 for the details).

Let us discuss some consequences of (SF).

PROPOSITION I.5.3. *Let X be a topological space and let $\mathcal{F}$ be a class of closed subsets of X stable by intersection and containing the whole space X. Assume X verifies (SF-$\mathcal{F}$) with respect to a metric Δ. Then, there exists a fragmentation $(f_n)_n$ of $\mathcal{F}$-slicings of X with difference sets in classes $(\mathcal{C}_n)_n$ that verify the following property:*

For each n, the class $\mathcal{C}_{n+1}$ consists of subsets C of X on which f_n is constant equal to $\alpha(C) = \alpha$ say, and such that the Δ-closure of C is disjoint from $\{f_n \geq \alpha + 1\}$.

PROOF. Assume that f_n is a given w.o $\mathcal{F}$-slicing of X. It follows from (SF-$\mathcal{F}$) that the class $\mathcal{C}_{n+1}$ of all subsets C of X with $\Delta - \operatorname{diam}(C) < 2^{-n-1}$, on which f_n is constant equal to $\alpha(C) = \alpha$ say, and such that the Δ-closure of C is disjoint from $\{f_n \geq \alpha + 1\}$, is $\mathcal{F}$-slice-generating . Indeed, it is enough to apply (SF-$\mathcal{F}$) to F, and $H = \{f_n > \alpha\}$ where $\alpha = \inf f_n(F)$ to obtain $C = F \setminus G$ contained in $\{f_n = \alpha\}$, with Δ-diameter less than 2^{-n-1} and whose Δ-closure is disjoint from $\{f_n > \alpha\}$. It follows from Proposition I.2.1 that we can choose for f_{n+1} a slicing such that all difference sets belong to $\mathcal{C}_{n+1}$.

A fragmentation verifying the conclusion of the above proposition will be called a *strong $\mathcal{F}$-fragmentation*. For the sequel, we shall repeatedly use the following property enjoyed by a strong fragmentation.

LEMMA I.5.4. *Assume X verifies (SF-$\mathcal{F}$) with respect to a metric Δ and let (f_n) be a strong fragmentation of X, then for any sequence (x_k) in X such that $\lim_k \Delta(x_k, x) = 0$ and $f_{n+1}(x_k) = \beta$ for every k, we have that $f_n(x_k)$ is also constant and equal to $f_n(x)$.*

PROOF. Indeed, by the above construction the set $\{f_{n+1} = \beta\}$ is contained in some difference set $\{f_n = \alpha\}$, thus $\big(f_n(x_k)\big)_k$ is constant. We obtain $f_n(x) \geq \alpha$ since $\{f_n \geq \alpha\}$ is Δ-closed. On the other hand, since the Δ-closure of $\{f_{n+1} = \beta\}$ is disjoint from $\{f_n > \alpha\}$, we get that $f_n(x) \leq \alpha$.

https://doi.org/10.4153/CJM-1992-031-6 Published online by Cambridge University Press

PROOF OF THEOREM (B). We let $(f_n)_n$ be a strong fragmentation of X and we consider *the hors-d'œuvre selection* s^*: $\mathcal{F}(X) \longrightarrow X$ associated to it. If L_n^* is the first-partial selection operator associated with f_n and if (S_n^*) the sequence of operators on $\mathcal{F}(X)$ defined in Section I.4 by

$$S_0(F) = F \text{ and } S_{n+1}^*(F) = L_{n+1}^*\big(S_n^*(F)\big)$$

then for every $F \in \mathcal{F}(X)$, the sequence $S_n^*(F)$ is "strongly decreasing ", that is $\bar{S}_{n+1}^*(F) \subset S_n^*(F)$ for every n. Furthermore, we have that $\cap_n S_n^*(F) = s^*(F)$. We shall now show that s^* satisfies the properties a), b), c) and d) announced in Theorem (B).

a) It is enough to notice that for every F, $s^*(F)$ has a fundamental set of τ-neighborhoods (relative to F) of arbitrarily small Δ-diameter.

b) assume that $s^*(F_2) \in F_1$ and $F_1 \subset F_2$. Since (f_n) is a strong fragmentation, we have that $s^*(F_2) \in L_1^*(F_2)$. According to Proposition I.3.2 d), we have that $s^*(F_2) \in L_1^*(F_1) \subset L_1^*(F_2)$, hence $s^*(F_2) \in L_2^*\big(S_1^*(F_2)\big)$ and $S_1^*(F_1) \subset S_1^*(F_2)$. By an inductive application of that observation, it follows that $s^*(F_2) \in S_n^*(F_1) \subset S_n^*(F_2)$ for every $n \geq 1$, therefore $s^*(F_2) = s^*(F_1)$.

c) If $(F_i)_{i\in I}$ is an increasing net in $\mathcal{F}(X)$, we can prove inductively using Proposition I.3.2 c) that for every $n \geq 1$ there exists i_n such that $j \geq i > i_n$ yields that $\text{ind}_n^*\big(S_{n-1}^*(F_i)\big) = \text{ind}_n^*\big(S_{n-1}^*(F_j)\big)$ and that $\big(S_n^*(F_i)\big)$ is increasing for $i > i_n$; it follows that for $j \geq i > i_n$, $s^*(F_i)$ and $s^*(F_j)$ belong to the same difference set for f_n, and therefore $\Delta\big(s^*(F_i), s^*(F_j)\big) \leq 2^{-n}$.

If now $F = \overline{\cup_i F_i}$ is Δ-complete, then for every n, there is i_n such that for every $i \geq i_n$, we have $s^*(F) \in S_n^*(F) \subset \overline{S_n^*(\cup_i F_i)} \subset \overline{\cup_{i\geq i_n} S_n^*(F_i)}$. This clearly implies that $s^*(F)$ is the limit of $s^*(F_i)$.

d) We see by induction, using Lemma I.3.4 that $g_n: z \longrightarrow \text{ind}_n^*(S_{n-1}^*(\Gamma(z))$ is slice-constant and that $z \longrightarrow S_n^*\big(\Gamma(z)\big)$ is slice-l.s.c for every $n \geq 1$. Let x_α^n be an arbitrary point in $\{f_n = \alpha\}$ provided, of course, this latter set is non-empty. Then $h_n(z) = x_{g_n(z)}^n$ is slice-constant, thus a Baire-1 function, in view of Lemma I.2.4. Moreover, these functions converge Δ-uniformly to $s^*\big(\Gamma(z)\big)$ since $f_n\big(h_n(z)\big) = g_n(z) = f_n\big(s^*\big(\Gamma(z)\big)\big)$ implies that $\Delta\big(h_n(z), s^*\big(\Gamma(z)\big)\big) < 2^{-n}$.

REMARK I.5.5. The above proof shows that if F is Δ-complete, then s^* selects the same point for F and for its τ-closure in X.

I.6. Injective slicing associated to a fragmentation. Suppose again that X is a topological space and that Δ is a metric on it. Let $(f_n)_n$ be a fragmentation on X. For every integer n, let A_n be the w.o set associated to the slicing f_n . Consider the product $B_n = A_1 \times \cdots \times A_n$ with the lexicographic order. It is not hard to realize that the operator S_n defined in I.4 is the partial selection operator associated with the slicing $f_1 \times \cdots \times f_n$. Denote by 0 the smallest element of each A_n, by B the infinite product $A_1 \times \cdots \times A_n \times \cdots$ equipped with the lexicographic order and embed B_n in B by adding 0 after the n-th place. Now $\varphi_n = f_1 \times \cdots \times f_n$ will be considered as a slicing of X with ordered set B; we also have a slicing $\varphi = f_1 \times \cdots \times f_n \times \cdots$ that we shall call *the injective slicing associated to the fragmentation* $(f_n)_n$.

PROOF OF THEOREM (C). Since X is Δ-fragmentable, we can proceed as in Theorem (A) to get a fragmentation $(f_n)_n$. Let now φ be the *injective slicing* associated to $(f_n)_n$. We shall prove that it verifies the assertions claimed in Theorem (C).

a) If $\varphi(x) = \varphi(y)$ then $f_n(x) = f_n(y)$ and hence $\Delta(x, y) < 2^{-n}$ for every n.

b) Since φ is a slicing of X, φ achieves its maximum on every compact $K \in \mathcal{K}(X)$. This maximal point is necessarily unique since φ is one to one.

c) Let $s \in B$, $s = (s_1, s_2, \ldots, s_n, \ldots)$ and $t_n = (s_1, s_2, \ldots, s_{n-1}, s_n + 1, 0, 0, \ldots)$ where $s_n + 1$ denotes the successor of s_n in A_n. If $s < t \leq t_n$ and $x, y \in X_s \setminus X_t$ then $s \leq \varphi(x), \varphi(y) < t \leq t_n$ implies that $f_n(x) = f_n(y)$ and $\Delta(x, y) < 2^{-n}$. It follows that if $\big(\varphi(x_i)\big)$ is a decreasing net in B, then (x_i) is Δ-Cauchy in X.

d) Suppose now that X is completely regular and that Δ induces a finer topology on X. This clearly implies that hypothesis (SF) is satisfied and hence by Proposition I.5.3, we can find a strong fragmentation $(f_n)_n$. Let φ be the *injective slicing* associated to $(f_n)_n$. If now $\big(\varphi(x_i)\big)$ is a decreasing net in B, we let s be its greatest lower bound in B, $s = (s_1, s_2, \ldots, s_n, \ldots)$. We know already that (x_i) is a Δ-Cauchy net in X. If (X, Δ) is complete, this net converges to some $x \in X$. For $i > i_n$, we have that $f_n(x_i)$ is constant, equal to s_n. It follows from Lemma I.5.4 that $f_n(x) = s_n$. Since this is true for every $n \geq 1$, we conclude that $\varphi(x) = s$.

Finally e) follows immediately from d).

REMARK I.6.1. If X is the unit ball of a separable dual Banach space, the sets B_n are then countable and one can choose B order isomorphic to $[0, 1]$. One may then perform the construction in such a way that $\varphi: X \longrightarrow [0, 1]$ is u.s.c, onto, quasi-concave, and with a right continuous inverse.

I.7 Extremal and optimal selections. This section is devoted to the proof of Theorems (D) and (E). To avoid repetition we shall work in the following general framework:

Let (X, Δ) be a complete metric space and let $\mathcal{A}$ be a set of Δ-Lipschitz real-valued functions on X equipped with a metric δ that is at least as strong as the metric of uniform convergence on X. We shall assume that $(\mathcal{A}, \delta)$ is complete.

For $F \subset X$, $\varphi \in \mathcal{A}$ and $t > 0$, we shall write

$$S(F, \varphi, t) = \{x \in F\,;\, \varphi(x) > \sup \varphi(F) - t\}.$$

Let us say that (X, Δ) is *$\mathcal{A}$-uniformly dentable* if for every non-empty set $F \subset X$, the set of $\varphi \in \mathcal{A}$ such that $\lim_{t\downarrow 0} \Delta\text{ - diam} S(F, \varphi, t) = 0$ is dense in $\mathcal{A}$.

A point x in a closed subset F of X is said to be *$\mathcal{A}$-strongly exposed in F* if there exists $\varphi \in \mathcal{A}$ such that every maximizing sequence for φ in F converges necessarily to x. (In particular φ attains its maximum on F at the point x). Such a function φ is called *strongly exposing*.

If we denote by $\mathcal{F}$ the class of subsets of X that are intersections of families of sets of the form $\{\varphi \le \theta\}$ with φ belonging to $\mathcal{A}$ and $\theta \in \mathbb{R}$, it is easy to see that if (X, Δ) is *$\mathcal{A}$-uniformly dentable* then for every $\varepsilon > 0$, the class $\mathcal{C}_\varepsilon$ of sets of the form $S(F, \varphi, t)$ and whose Δ-diameter is less than ε is an $\mathcal{F}$-slice generating class.

THEOREM I.7.1. *Let (X, Δ) be a complete metric space that is $\mathcal{A}$-uniformly dentable. Let $\mathcal{F}(X)$ denote the class of non empty closed subsets of X. Then there exists a selection $s^*\colon \mathcal{F}(X) \longrightarrow X$ and a Baire-1 map $r\colon X \longrightarrow \mathcal{A}$ such that:*

a) For every $F \in \mathcal{F}(X)$, $s^*(F)$ is strongly exposed in F by $r\big(s^*(F)\big)$.

b) If $F_1 \subset F_2$ and $s^*(F_2) \in F_1$ then $s^*(F_1) = s^*(F_2)$.

c) If (F_i) is an increasing net in $\mathcal{F}(X)$ and if $F = \Delta$-closure of $\cup_i F_i$, then $s^*(F_i)$ converges to $s^*(F)$.

d) If Γ is a slice-lower semi-continuous multivalued mapping from a metric space (Z, d) into $\mathcal{F}(X)$, then $z \longrightarrow s^*\big(\Gamma(z)\big)$ is a Baire-1 function. Furthermore, if (X, Δ) is a convex subset of a topological vector space, then this map is the pointwise limit of a sequence of continuous functions.

PROOF. If f is a discrete slicing of X, we shall say that a function φ on X is f-measurable if φ is constant on the difference sets of f.

We shall construct inductively sequences $(D^{(n)}, f_n, \varphi^{(n)}, \varepsilon^{(n)})$ where $D^{(n)}$ is an $\mathcal{F}$-derivation, f_n a w.o slicing of X adapted to $D^{(n)}$ (in the sense of I.2), $\varphi^{(n)}$ is an f_n-measurable function from X to $\mathcal{A}$ and $\varepsilon^{(n)}$ an f_n-measurable positive function on X.

Start with any $\varphi_0 \in \mathcal{A}$. Assuming $(D^{(n)}, f_n, \varphi^{(n)}, \varepsilon^{(n)})$ already constructed, we shall explain how to pass to the $(n+1)$-th step. For $F \in \mathcal{F}$, $D^{(n+1)}F$ will consist of all subsets of F of the form $F \cap \{\psi \le \theta\}$ verifying the following properties:

i) $\psi \in \mathcal{A}$, $S = F \cap \{\psi > \theta\}$ is non-empty and its Δ-diameter is less than 2^{-n-1}.

ii) If $\alpha = \mathrm{ind}_n^*(F)$, then $\sup\{\psi(y)\,; f_n(y) \ge \alpha + 1\} < \theta$. Thus $\bar{S}$ is contained in the difference set $\{f_n = \alpha\}$.

iii) $\delta\big(\psi, \varphi^{(n)}(x)\big) < \varepsilon^{(n)}(x)$ for every $x \in S$.

We shall show first that the class $D^{(n+1)}F$ is non-empty whenever $F \in \mathcal{F}$ is non-empty, that is $D^{(n+1)}$ is an $\mathcal{F}$-derivation.

For that let $\alpha = \mathrm{ind}_n^*(F)$ and let $\varphi \in \mathcal{A}$ be the constant value of $\varphi^{(n)}$ and $\varepsilon > 0$ the constant value of $\varepsilon^{(n)}$ on the difference set $\{f_n = \alpha\}$. For every n and any ordinal α, we denote by $X_\alpha^{(n)}$ the set $\{f_n \ge \alpha\}$.

By construction, $X_{\alpha+1}^{(n)} = X_\alpha^{(n)} \cap \{\varphi \le \theta\}$ for some θ. Since $F \cap \{\varphi > \theta\}$ is non-empty, we deduce that $\sup \varphi(F) = \tau > \theta$. Since X is $\mathcal{A}$-uniformly dentable, there exists $\psi \in \mathcal{A}$ such that

$$\lim_{t \downarrow 0} \operatorname{diam} S(F, \psi, t) = 0 \quad\text{and}\quad \delta(\psi, \varphi) < \min\Big\{\varepsilon, \frac{\tau - \theta}{4}\Big\}.$$

This implies that

$$\sup \psi(X_{\alpha+1}^{(n)}) \le \theta + \frac{\tau - \theta}{4} = \frac{\tau + 3\theta}{4} \quad\text{and}\quad \sigma = \sup \psi(F) \ge \tau - \frac{\tau - \theta}{4} > \frac{\tau + 3\theta}{4}.$$

https://doi.org/10.4153/CJM-1992-031-6 Published online by Cambridge University Press

It follows that for $t > 0$ small enough, the set $S = F \cap \{\psi > \sigma - t\}$ is non-empty, its diameter is less than 2^{-n-1}, $\sup \psi(X^{(n)}_{\alpha+1}) < \sigma - t$, and $\delta\left(\psi, \varphi^{(n)}(x)\right) < \varepsilon^{(n)}(x)$ for any $x \in S$. Hence the set $F \cap \{\psi \leq \sigma - t\}$ belongs to $D^{(n+1)}F$ and $D^{(n+1)}$ is an $\mathcal{F}$-derivation.

According to I.2.3, there exists a w.o $\mathcal{F}$-slicing f_{n+1} of X such that for every β, $X^{n+1}_{\beta+1} \in D^{(n+1)}(X^{n+1}_\beta)$. This means that for every β,

$$\{f_{n+1} \geq \beta + 1\} = \{f_{n+1} \geq \beta\} \cap \{\psi_\beta \leq \theta_\beta\}$$

where properties i), ii) and iii) are satisfied.

Define $\varphi^{(n+1)}$ on X by $\varphi^{(n+1)}(x) = \psi_{f_{n+1}(x)}$. This function is clearly f_{n+1}-measurable and satisfies $\delta\left(\varphi^{(n+1)}(x), \varphi^{(n)}(x)\right) < \varepsilon^{(n)}(x)$ for every $x \in X$. Next we set

$$\varepsilon^{(n+1)}(x) = \frac{1}{4}\min\left\{\varepsilon^{(n)}(x), \theta_\beta - \sup\{\psi_\beta(y)\,; f_n(y) \geq \alpha + 1\}\right\}$$

where $\alpha = f_n(x)$, $\beta = f_{n+1}(x)$ and $\psi_\beta = \varphi^{(n+1)}(x)$. We have that $0 < \varepsilon^{(n+1)}(x) < \frac{1}{4}\varepsilon^{(n)}(x)$ for every $x \in X$.

This finishes the construction of the fragmentation (f_n). Let now s^* be its corresponding hors d'œuvre selection. Note that the sequence $\varphi^{(n)}$ converges uniformly to a Baire-1 map r from X to $\mathcal{A}$.

Assertions b), c) and d) follow from the properties of an hors d'œuvre selection (Theorem (C)). It remains to check that for every $F \in \mathcal{F}$, the function $r\big(s^*(F)\big)$ strongly exposes F at $s^*(F)$.

Indeed, let $x = s^*(F)$. Since $\varepsilon^{(k+1)}(x) < \frac{1}{4}\varepsilon^{(k)}(x)$ for every k, we get that

$$\delta\left(r(x), \varphi^{(n+1)}(x)\right) < 2\varepsilon^{(n+1)}(x).$$

Let $\beta = f_{n+1}(x)$. By construction we have

$$2\varepsilon^{(n+1)}(x) \leq \left(\theta_\beta - \sup\{\varphi^{(n+1)}(x)(y)\,; f_n(y) \geq f_n(x) + 1\}\right)/2$$

and $\varphi^{(n+1)}(x)(x) \geq \theta_\beta$. It follows that

$$r(x)(x) > \sup\{r(x)(y)\,; f_n(y) \geq f_n(x) + 1\}.$$

This shows that if $\alpha = f_n(x)$, then the set $S\big(F, r(x), t\big)$ is disjoint from $X^n_{\alpha+1}$ for $t > 0$ small enough, hence it is contained in $\{f_n = \alpha\}$ and its diameter is smaller than 2^{-n}. This finishes the proof of Theorem I.7.1.

REMARK I.7.2. Note that we started the construction with any function $\varphi_0 \in \mathcal{A}$ and we can make sure (by a more careful control on the functions $\varepsilon^{(n)}$) that the range of the map r is contained in any prescribed open ball in $\mathcal{A}$ that is centered at φ_0.

PROOF OF THEOREM (D). If Y is a Banach space with the R.N.P (resp. the A.R.N.P), we let $X = \text{Ball}(Y)$ and $\mathcal{A}$ will denote Y^* (resp PSH(Y)). The class $\mathcal{F}$ will consist of the closed convex (or pseudoconvex) subsets of X. Δ will be the norm of Y. It is well known that in these cases, X is Y^*-uniformly dentable [B] (resp. PSH(Y)-uniformly dentable [G-M5]). The rest follows from Theorem I.7.1.

PROOF OF THEOREM (E). Let (X,Δ) be a complete metric space. We can and shall assume without loss of generality that $\Delta \leq 1$. Consider the space $\tilde{X} = X \times [0,+\infty)$ equipped with the metric $\tilde{\Delta}\big((x,t),(y,s)\big) = \Delta(x,y) + |t-s|$. It is clearly a complete metric space. Let $\mathcal{A}$ be the class of functions on $\tilde{X}$ of the form $\tilde{\varphi}(y,t) = \varphi(y) + t$ with $\varphi \in \mathrm{Lip}(X)$ and $\delta(\tilde{\varphi},\tilde{0}) \leq 1$, where

$$\delta(\tilde{\varphi},\tilde{\psi}) = \max\{\,\|\tilde{\varphi}-\tilde{\psi}\|_\infty, \|\tilde{\varphi}-\tilde{\psi}\|_{Lip}\}.$$

The space $(\mathcal{A},\delta)$ is complete. To show that $\tilde{X}$ is $\mathcal{A}$-uniformly dentable, let $\tilde{\varphi} \in \mathcal{A}$ and let F be a closed subset F of $\tilde{X}$. For $\tau > 0$, consider any $(x_0,t_0) \in F$ such that

$$(1-\tau)\varphi(x_0) + t_0 < \inf\{\,(1-\tau)\varphi(x) + t\,;\,(x,t) \in F\} + \tau^2.$$

Let $\varphi_\tau = (1-\tau)\varphi + \tau\Delta(.\,,x_0)$. It is easy to see that the slice

$$S = \{\,(y,t) \in F\,;\,\tilde{\varphi}_\tau(y,t) < \tilde{\varphi}_\tau(y_0,t_0) + \tau^2\}$$

is non empty and has a diameter less than 2τ. Since $\delta(\tilde{\varphi}_\tau,\tilde{\varphi}) \leq 2\tau$, it follows that $(\tilde{X},\tilde{\Delta})$ is $\mathcal{A}$-uniformly dentable. Theorem I.7.1 applies and we get a selection s^* on $\mathcal{F}(\tilde{X})$.

To any lower semi-continuous function $f\colon X \longrightarrow [0,+\infty]$ and any $F \in \mathcal{F}_f$, we associate the (closed) epigraph $F_f = \{\,(x,t) \in F \times [0,+\infty)\,;\,f(x) \leq t\}$. Define now $s_f^*(F) = \pi_1\big(s^*(F_f)\big)$ where $\pi_1\colon \tilde{X} \longrightarrow X$ is the projection on the first coordinate.

To show that $s_f^*(F)$ is an ε- *Ekeland point for f on F*, we let $\tilde{\varphi}$ be the functional that exposes $s^*(F_f) = (x_0,t_0)$ in F_f. We can choose $\delta(\tilde{\varphi},\tilde{0}) \leq \varepsilon$ by Remark I.7.2. Since $-\varphi(x_0) - t_0 \geq -\varphi(x) - f(x)$ for any $x \in X$ and since $f(x_0) \leq t_0$, we get from the ε-Lipschitz property of φ that $f(x_0) - \varepsilon\Delta(x_0,x) \leq f(x)$ for any $x \in F$.

Assertions b), c) and e) follow from the corresponding properties b) and c) and d) in the selection of Theorem I.7.1. If now (F_i) and F are as in the hypothesis of d), then $\overline{\cup_i F_{f_i}} = F_f$ provided f is continuous. In this case, property c) of Theorem I.7.1 applies again and we get assertion d).

REMARK I.7.3. The above proof gives that $\tilde{X}$ is $\mathcal{A}$-uniformly dentable for the class $\mathcal{A}$ of functions $\tilde{\varphi}$ where φ is of the form $\sum_n \alpha_n\Delta^2(.\,,x_n)$ with $\alpha_n \geq 0$, $\sum_n \alpha_n = 1$ and $x_n \in X$. One can then select points à la Borwein-Preiss, that is, points that minimize perturbations of the form $f + \sum_n \alpha_n\Delta^2(.\,,x_n)$ [B-P].

REMARK I.7.4. Easy examples show that assertion d) in Theorem (E) above, does not hold if we only assume that f is lower semi-continuous.

If a Banach space Y has the R.N.P (resp. A.R.N.P), then the same holds for $Y \oplus \mathbb{R}$. The method used above to transfer the problem to the epigraph of the function yields the following strengthening of Theorem (D).

https://doi.org/10.4153/CJM-1992-031-6 Published online by Cambridge University Press

THEOREM I.7.5. *Let X be the unit ball of a real (resp. complex) Banach space Y with the Radon-Nikodym (resp. the Analytic Radon-Nikodym) Property. For each $\varepsilon > 0$ and any l.s.c function $f: X \longrightarrow [0, +\infty]$, there exists a selection $s^*_{f,\varepsilon}: \mathcal{F}_f(X) \longrightarrow X$ and a Baire-1 map $r_\varepsilon: X \longrightarrow B_\varepsilon(Y^*)$ (resp. $r_\varepsilon: X \longrightarrow \mathrm{PSH}_\varepsilon(Y)$) such that:*

a) *For every $F \in \mathcal{F}_f(X)$, $s^*_{f,\varepsilon}(F)$ is a point in F that is strongly exposed (from below) by $f + r_\varepsilon\big(s^*(F)\big)$.*

b) *If $F_1 \subset F_2$ and $s^*_{f,\varepsilon}(F_2) \in F_1$ then $s^*_{f,\varepsilon}(F_1) = s^*_{f,\varepsilon}(F_2)$.*

c) *If (f_i) is a decreasing family of l.s.c functions from X into $[0, +\infty]$ and if f is the l.s.c envelope of $\inf_i f_i$, then for any $F \in \mathcal{F}_{f_0}(X)$, the net $s^*_{f_i,\varepsilon}(F)$ converges to $s^*_{f,\varepsilon}(F)$.*

d) *If f is continuous and if (F_i) is an increasing net in $\mathcal{F}_f(X)$, then $\big(s^*_{f,\varepsilon}(F_i)\big)$ converges to $s^*_{f,\varepsilon}(F)$, where F is the closure of $\cup_i F_i$.*

e) *If Γ is a slice-lower semi-continuous multivalued mapping from a metric space (Z, d) into $\mathcal{F}_f(X)$, then $z \longrightarrow s^*_{f,\varepsilon}\big(\Gamma(z)\big)$ is a pointwise limit of a sequence of continuous functions.*

II. Comparison with the known selection theorems. In this section, we compare the above results with various well known selection theorems.

II.1. Jayne-Rogers selections. Let X be a Banach space and let $Y \subset X^*$ be norming for X. Say that X is *hereditarily Y-huskable* if for every bounded subset F of X and any $\varepsilon > 0$, there exists a $\sigma(X, Y)$-open set V such that $V \cap F \neq \emptyset$ and $\mathrm{diam}(F \cap V) < \varepsilon$. We shall denote by $\mathcal{K}^Y(X)$ the class of all $\sigma(X, Y)$-compact subsets of X. We can deduce from the above the following refinement of the results of Jayne-Rogers[J-R].

PROPOSITION II.1.1. *Let X be a hereditarily Y-huskable Banach space for some norming subspace Y in X^*. Then there exists a selection s: $\mathcal{K}^Y(X) \longrightarrow X$ such that:*

a) *$s(K) \in K$ for every $K \in \mathcal{K}^Y$.*

b) *If $K_1 \subset K_2$ and $s(K_2) \in K_1$ then $s(K_1) = s(K_2)$.*

c) *If (K_i) is a decreasing net to K in $\mathcal{K}^Y(X)$, then $\lim_i \|s(K_i) - s(K)\| = 0$.*

d) *If Γ is a slice-upper semi-continuous multivalued mapping from a metric space (Z, d) into $\mathcal{K}^Y(X)$, then $z \longrightarrow s\big(\Gamma(z)\big)$ is a Baire-1 function from Z into $(X, \|.\|)$, which means in this case that it is a pointwise norm-limit of a sequence of continuous functions.*

As we mentioned above, the two typical examples are:

i) The case of a Banach space with the P.C.P. In this case, $Y = X^*$ and we can then select from all weakly compact subsets of X.

ii) The case of a dual Banach space $X = Y^*$ with the R.N.P and we can then select from all weak*-compact subsets of Y^*.

In the latter case, it is interesting to notice that *maximal monotone maps*—like subdifferential maps and the norm-attainment maps— are weak*-upper semi-continuous maps from $Y \longrightarrow \mathcal{K}^*(Y^*)$ and therefore the above theorem can be applied to them to obtain single-valued selections. For details we refer to Jayne-Rogers [J-R] and to the recent lecture notes of Phelps [P].

II.2- Kuratowski and Ryll-Nardzewski selections. The hors-d'œuvre selection seems to be the appropriate extension of the classical selection theorems. It has the advantage of not requiring the compactness of the sets we are selecting from. If we apply it for instance, to a complete metric space X which is clearly fragmentable by its own metric Δ, we obtain the following refinement of a classical result.

COROLLARY II.2.1. *Assume (X,Δ) is a complete metric space, then there exists a selection $s^*\colon \mathcal{F}(X) \longrightarrow X$ such that:*

a) $s^*(F) \in F$ for every set $F \in \mathcal{F}(X)$.

b) If $F_1 \subset F_2$ and $s^*(F_2) \in F_1$ then $s^*(F_1) = s^*(F_2)$.

c) If (F_i) is an increasing net of closed sets and if F is the closure of $\cup_i F_i$, then $\lim_i s^*(F_i) = s^*(F)$.

d) If Γ is a slice-lower semi-continuous multivalued mapping from a metric space (Z,d) into $\mathcal{F}(X)$, then $z \longrightarrow s^*\big(\Gamma(z)\big)$ is a Baire-1 function from Z into X.

REMARK II.2.2. The proof of the above corollary can be slightly altered to give the result of Kuratowski and Ryll-Nardzewski [K-R] which says that if $\Gamma\colon (Z,d) \longrightarrow \mathcal{F}(X)$ is *of lower class α (i.e.* $\{z\,;\,\Gamma(z)\cap U \neq \emptyset\}$ is of additive class α for every open set in X), then there is a selector for Γ of class α.

II.3. Debs selection. If X is a metrizable convex compact subset of a locally convex topological vector space, then it is well known—and easy to see—that it is linearly homeomorphic to a compact convex subset of Hilbert space. Theorem (D) then gives the following refinement of a result of Debs [D].

COROLLARY II.4.1. *Let X be a metrizable convex compact subset of a locally convex topological vector space, and let $\mathcal{F}(X)$ denote the class of closed subsets of X. Then there exists a selection $s^*\colon \mathcal{F}(X) \longrightarrow X$ such that:*

a) For every $F \in \mathcal{F}(X)$, $s^*(F)$ is an extreme point of F.

b) If $F_1 \subset F_2$ and $s^*(F_2) \in F_1$ then $s^*(F_1) = s^*(F_2)$.

c) If (F_i) is an increasing net in $\mathcal{F}(X)$ and if $F = \overline{\cup_i F_i}$, then $\lim_i s^*(F_i) = s^*(F)$.

d) If Γ is a slice-lower semi-continuous multivalued mapping from a metric space (Z,d) into $\mathcal{F}(X)$, then $z \longrightarrow s^*\big(\Gamma(z)\big)$ is a pointwise limit of a sequence of continuous functions.

III. Zippin's theorem revisited. We shall now establish the following refinement of a recent Theorem of Zippin [Z]. I will denote the Cantor set.

THEOREM III.1. *If a bounded linear operator T from a separable Banach space X into $C(I)$ has an adjoint with separable range, then it factors through a Banach space Z with a shrinking basis. That is, there exist $T_1\colon X \longrightarrow Z$ and $T_2\colon Z \longrightarrow C(I)$ such that*

https://doi.org/10.4153/CJM-1992-031-6 Published online by Cambridge University Press

$T = T_2 T_1$. *In particular, every Banach space with a separable dual embeds in a Banach space with a shrinking basis.*

PROOF. Let $T: X \longrightarrow C(I)$ be a bounded linear operator such that $T^*\big(\mathcal{M}(I)\big)$ is norm separable. It is convenient to assume that T^* is one to one on the set of Dirac measures $\{\delta_t \,;\, t \in I\}$ (this can be done by adding an appropriate function to X). We shall assume that $T(X)$ contains a function that separates points in I; it will also be convenient at some point to assume that $T(X)$ contains the constant function 1.

Consider on I the following metric:

$$\Delta(s,t) = \sup\Big\{ |\varphi(s) - \varphi(t)| \,;\, \varphi \in T\big(B(X)\big)\Big\}$$

If we consider $T^*: \mathcal{M}(I) \longrightarrow X^*$, we see that $\Delta(s,t) = \|T^*(\delta_s) - T^*(\delta_t)\|$, while the weak*-topology of X^* induces the usual topology on I, via the map $t \in I \longrightarrow T^*(\delta_t)$. Hence I is separable for the metric Δ, and the Δ-balls are closed subsets of I. It follows easily from Baire's theorem that every closed subset of I contains a non-empty relatively open subset $F \setminus G$ with small Δ-diameter. In other words, I is Δ-fragmentable. We may thus apply Theorem (A) and obtain the dessert selection s: $\mathcal{K}(I) \longrightarrow I$.

An atom of I will be a subset of the form $A = \{t \in I \,;\, t_i = \varepsilon_i, i = 1, \ldots, n\}$ for some n and for some sequence $(\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$. We define the size of this atom A as $|A| = 2^{-n}$.

Let π_0 be the trivial partition of I into a single atom equal to I itself. Define a sequence (π_n) of finite partitions of I into atoms in the following way: assume $n > 1$ and choose one atom A in π_{n-1} with the largest possible size in π_{n-1}; define a new partition π_n of I by splitting this atom A into two atoms of size $\frac{1}{2}|A|$. It is easy to check by induction that π_n is a partition of I into $n+1$ clopen subsets, and that π_n is a partition into atoms of equal size when $n = 2^k - 1$, for every $k \geq 0$. We define now for $n \geq 0$ and $f \in C(I)$

$$P_n f = \sum_{A \in \pi_n} f\big(s(A)\big) 1_A.$$

It is clear that $P_n f$ is continuous since the A's are clopen sets, that P_n is a projection (since $s(A) \in A$) and that $\|P_n\| \leq 1$. For $t \in I$, let $A_n(t)$ be the atom of π_n that contains t. We have $P_n f(t) = f\Big(s\big(A_n(t)\big)\Big)$. If $m \leq n$, then $t \in A_n(t) \subset A_m(t)$. Consider $u = s\big(A_m(t)\big)$; we have that $s\big(A_m(t)\big) = u \in A_n(u) \subset A_m(u) = A_m(t)$, hence $s\big(A_m(t)\big) \in A_n(u) \subset A_m(t)$. It follows from Theorem A.b) that $s\big(A_m(t)\big) = u = s\big(A_n(u)\big)$. Now

$$P_m P_n f(t) = P_n f\Big(s\big(A_m(t)\big)\Big) = P_n f(u) = f\Big(s\big(A_n(u)\big)\Big) = f(u) = P_m f(t).$$

This shows that $P_m P_n = P_m$ when $m \leq n$. Next we observe that $\mathrm{rank}(P_{n+1} - P_n) = 1$ for every $n \geq 0$. Indeed, let A be the atom of π_n that is split into two atoms A_1, A_2 of π_{n+1}. Let $t = s(A) \in A$, $i = 1$ or 2 such that $t \notin A_i$ and $u = s(A_i)$. Then for every $f \in C(I)$

$$(P_{n+1} - P_n) f = \big(f(u) - f(t)\big) 1_{A_i}.$$

Let us call B_{n+1} this special atom of $\pi_{n+1}, n \geq 0$ (with $B_0 = I$). It is clear from what we have said that $(e_n) = (1_{B_n})$ is a monotone basis for $C(I)$. We shall now prove the following:

CLAIM. If $W_0 = \cup_{n\geq 0} P_n T(B_X)$, then for every $\mu \in M(I)$ we have:

$$\lim_{k\to\infty} \sup_{w\in W_0} \langle (\mathrm{Id} - P_k)w, \mu \rangle = 0.$$

Let $x \in B_X$ and $w = P_n Tx$. Then $(\mathrm{Id} - P_k)w$ equals 0, that is $(P_n - P_k)Tx = (\mathrm{Id} - P_k)Tx - (\mathrm{Id} - P_n)Tx$; therefore, up to a factor 2, it is enough to control $\sup_{x\in B_X} \langle (\mathrm{Id} - P_k)Tx, \mu \rangle$. This is done in the following way:

$$\int (\mathrm{Id} - P_k)Tx\, d\mu = \sum_{A\in\pi_k} \int_A \Big(Tx(t) - Tx\big(s(A)\big)\Big)\, d\mu(t)$$
$$= \int \Big(Tx(t) - Tx\big(A_k(t)\big)\Big)\, d\mu(t) \leq \int \Delta\big(t, A_k(t)\big)\, d\mu(t).$$

According to Theorem (A), $\Delta\big(t, A_k(t)\big) \longrightarrow 0$ as $k \longrightarrow \infty$ (since $\{t\} = \cap_k A_k(t)$), while $\Delta(.,.) \leq 2$. Our claim follows then from Lebesgue's dominated convergence theorem.

We apply now the interpolation scheme from [D-F-J-P] to the set W = closure of W_0 in the space $C(I)$. We recall briefly this construction: for every $k \geq 0$, let j_k be the gauge of the convex subset $2^k W + 2^{-k} B$ of $C(I)$, where B denotes here the closed unit ball of $C(I)$. We denote by Z the vector subspace of $C(I)$ consisting of those f for which $\|f\|_Z^2 = \sum_{k\geq 0} j_k^2(f)$ is finite. It is not difficult to check that Z, equipped with the norm $\|.\|_Z$ is a Banach space, containing W and hence $T(X)$. We denote by T_1 the operator T acting now from $X \longrightarrow Z$ and we let T_2 be the natural injection of Z into $C(I)$. It is clear that $T = T_2 T_1$. It remains to prove that Z has a shrinking basis. To do that observe first that $P_n(W) \subset W$ for every $n \geq 0$. It then follows that $j_k\big(P_n(f)\big) \leq j_k(f)$ for every f and every $k \geq 0$, thus

$$\|P_n f\|_Z \leq \|f\|_Z.$$

Furthermore, the basis (e_n) of $C(I)$ is contained in Z. Indeed, if x is a function in X such that Tx separates points in I (this is our starting assumption), we see that $(P_n - P_{n-1})Tx = \lambda_n e_n, \lambda_n \neq 0$, for every $n > 0$ and $1 = e_0$ also belongs to $T(X)$ by assumption. It follows as before that, after normalization in the Z-norm, (z_n) is a monotone basis for Z. It remains to show that (z_n) is a shrinking basis for Z. This is equivalent to saying that for every $z^* \in Z^*$,

$$\sup_{z\in B_Z} \langle (\mathrm{Id} - P_k)z, z^* \rangle \longrightarrow 0 \text{ when } k \longrightarrow \infty.$$

It is a well known property of the factorization scheme that T_2^* has norm-dense range in Z^*. It is thus enough to check the above property when $z^* = T_2^*\mu$; that is to say $\lim_{k\to\infty} \sup_{z\in B_Z} \langle (\mathrm{Id} - P_k)z, \mu \rangle = 0$ for every $\mu \in M(I)$. By the definition of Z, we may write for every $n \geq 0$ and every $z \in B_Z$, $z = z_1 + z_2$, where $z_1 \in 2^n W$ and $\|z_2\|_\infty \leq 2^{-n}$, hence the latter assertion reduces to the claim proved above. This finishes the proof of the theorem.

REFERENCES

[B] R. D. Bourgin, *Geometric aspects of convex sets with the Radon-Nikodym property*, Lecture notes in Mathematics **993**, Springer-Verlag, 1983.

[B-P] J. Borwein and D. Preiss, *A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions*, Trans. Amer. Math. Soc. **303**(1987), 517–527.

[D-F-J-P] W. J. Davis, T. Figiel, W. B. Johnson and A. Pelczynski. *Factoring weakly compact operators*, J. Funct. Analysis **17**(1974), 311–327.

[D] G. Debs, *Sélections maximales d'une multi-application*, Séminaire d'initiation à l'analyse. Choquet Rogalski-St. Raymond, 1978/1979.

[E] G. A. Edgar, *Complex martingale convergence*, Springer-Verlag, Lecture Notes **1116**(1985), 38–59 .

[E-W] G. A. Edgar and R. F. Wheeler, *Topological properties of Banach spaces*, Pac. J. Math. **115**(1984).

[Ek] I. Ekeland, *Non convex minimization problems*, Bull. A.M.S. **1**(1979), 443–474.

[F-G] M. Fabian and G. Godefroy, *The dual of every Asplund space admits a projectional resolution of the identity*, Studia Math. (2) **91**(1989).

[G-M1] N. Ghoussoub and B. Maurey, *G_δ -embeddings in Hilbert space*, J. Funct. Anal. (1) **61**(1985), 72–97.

[G-M2] ———, *G_δ -embeddings in Hilbert space II*, J. Funct. Anal. (2) **78**(1988), 271–305.

[G-M3] ———, *H_δ -embeddings in Hilbert space and Optimization on G_δ -sets*, Memoirs of the A.M.S. **349** (1986).

[G-M4] ———, *A non-linear method for constructing certain basic sequences in Banach spaces*, Illinois J. Math. (3) **34**(1990), 607–613.

[G-M5] ———, *Plurisubharmonic martingales and barriers in complex quasi-Banach spaces*, Ann. Inst. Fourier (4) **39**(1989), 1007–1060.

[J-R] J. Jayne and A. Rogers, *Borel selectors for upper semi-continuous set valued maps*, Acta. Math. **155** (1985), 41–79.

[K-M] K. Kuratowski and A. Mostowski, *Set Theory with an introduction to descriptive set theory*, Studies in logic and the foundation of mathematics. North-Holland **86**(1976).

[P] R. R. Phelps, *Convex functions, monotone operators and differentiability*, **1364**, Lecture notes in Math., Springer-Verlag, 1989.

[Z] M. Zippin, *Banach spaces with separable duals*, Israel J. Math., (1988).

Department of Mathematics
University of Bristish Columbia
Vancouver, B.C. V6T 1Y4

U.F.R. de Mathématiques
Université Paris VII
75251 Paris Cedex 05
France

Institut für Mathematik
J. Kepler Universität
A-4040 Linz
Austria

https://doi.org/10.4153/CJM-1992-031-6 Published online by Cambridge University Press