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Stable finiteness of twisted group rings and noisy linear cellular automata

Xuan Kien Phung

Abstract. For linear nonuniform cellular automata (NUCA) which are local perturbations of linear CA over a group universe G and a finite-dimensional vector space alphabet V over an arbitrary field k, we investigate their Dedekind finiteness property, also known as the direct finiteness property, i.e., left or right invertibility implies invertibility. We say that the group G is L^1-surjunctive, resp. finitely L^1-surjunctive, if all such linear NUCA are automatically surjective whenever they are stably injective, resp. when in addition k is finite. In parallel, we introduce the ring $D^1(k[G])$ which is the Cartesian product $k[G] \times (k[G])[G]$ as an additive group but the multiplication is twisted in the second component. The ring $D^1(k[G])$ contains naturally the group ring $k[G]$ and we obtain a dynamical characterization of its stable finiteness for every field k in terms of the finite L^1-surjunctivity of the group G, which holds, for example, when G is residually finite or initially subamenable. Our results extend known results in the case of CA.

1 Introduction

In this paper, we investigate and establish the relation between some extensions of two well-known conjectures in symbolic dynamics and ring theory, namely, Gottschalk's surjunctivity conjecture and Kaplansky's stable finiteness conjecture. More specifically, given a group G, a field k, and a finite set A, Kaplansky conjectured [22] that the group ring $k[G]$ is stably finite, i.e., every one-sided invertible element of the ring of square matrices of size $n \times n$ with coefficients in $k[G]$ must be a two-sided unit, while Gottschalk's surjunctivity conjecture [19] states that every injective G-equivariant uniformly continuous self-map $A^G \circlearrowleft$ must be surjective. It is known that every one-sided unit of $\mathbb{C}[G]$ must be a two-sided unit [22]. Moreover, both conjectures are known for the wide class of sofic groups introduced by Gromov (see [1, 4, 16, 20, 23, 35, 39]) but they are still open in general. As an application of our main results, we obtain an extension of the known equivalence (cf. [35, Theorem B], [36, Theorem B]) between Kaplansky's stable finiteness and a weak form of Gottschalk's surjunctivity conjecture. More precisely, we establish the equivalence between the surjunctivity property of locally disturbed linear cellular automata (CA) and the stable finiteness of some twisted group rings (Theorem B).

To state the main results, let us recall some notions of symbolic dynamics. Given a discrete set A and a group G, a *configuration* $c \in A^G$ is a map $c : G \to A$. Two

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configurations $x, y \in A^G$ are *asymptotic* if $x|_{G\setminus E} = y|_{G\setminus E}$ for some finite subset $E \subset G$. The *Bernoulli shift* action $G \times A^G \to A^G$ is defined by $(g, x) \mapsto gx$, where $(gx)(h) = x(g^{-1}h)$ for $g, h \in G$, $x \in A^G$. We equip the *full shift* A^G with the *prodiscrete topology*. For $x \in A^G$, we define $\Sigma(x) = \overline{\{gx : g \in G\}} \subset A^G$ as the smallest closed subshift containing x. Following an idea of von Neumann and Ulam [27], a CA over the group G (the *universe*) and the set A (the *alphabet*) is a self-map $A^G \circlearrowleft$ which is G-equivariant and uniformly continuous (cf. [6, 21]). One refers to group elements $g \in G$ as to the cells of the universe. When different cells can evolve according to different local transition maps, we obtain *nonuniform CA* (NUCA). More precisely, we have (cf. [14, 15], [33, Definition 1.1]) the following:

Definition 1.1 Let G be a group, and let A be a set. Let $M \subset G$ be a subset, and let $S = A^{A^M}$ be the set of all maps $A^M \to A$. Given $s \in S^G$, the NUCA $\sigma_s : A^G \to A^G$ is defined for all $x \in A^G$ and $g \in G$ by the formula

$$\sigma_s(x)(g) = s(g)((g^{-1}x)|_M).$$

The set M is called a *memory* and $s \in S^G$ the *configuration of local defining maps* of σ_s. Every CA is thus a NUCA with finite memory and constant configuration of local defining maps. Following [33], we say that σ_s is *invertible* if it is bijective and the inverse map σ_s^{-1} is a NUCA with *finite* memory. Moreover, σ_s is *left-invertible*, resp. *right-invertible*, if $\tau \circ \sigma_s = \mathrm{Id}$, resp. $\sigma_s \circ \tau = \mathrm{Id}$, for some NUCA $\tau : A^G \to A^G$ with *finite* memory. The NUCA σ_s is *pre-injective* if $\sigma_s(x) = \sigma_s(y)$ implies $x = y$ whenever $x, y \in A^G$ are asymptotic, and σ_s is *post-surjective* if for all $x, y \in A^G$ with y asymptotic to $\sigma_s(x)$, then $y = \sigma_s(z)$ for some $z \in A^G$ asymptotic to x. We say that σ_s is *stably injective* if σ_p is injective for every $p \in \Sigma(s)$. Similarly, σ_s is *stably post-surjective* if σ_p is post-surjective for every $p \in \Sigma(s)$.

If A is a vector space, A^G is naturally a vector space with component-wise operations and we call a NUCA $\tau : A^G \to A^G$ *linear* if it is also a linear map of vector spaces. Clearly, τ is a linear NUCA if and only if its local transition maps are all linear. Such linear NUCA with finite memory are interesting dynamical objects since they satisfy the shadowing property [29, 37].

Definition 1.2 Given a group G and a vector space V, we denote by $\mathrm{LNUCA}_c(G, V)$ the space of all linear NUCA $\tau : V^G \to V^G$ with finite memory which admit asymptotically constant configurations of local defining maps, i.e., $\tau \in \mathrm{LNUCA}_c(G, V)$ if there exist finite subsets $M, E \subset G$ and $s \in \mathcal{L}(V^M, V)^G$ such that $\tau = \sigma_s$ and $s(g) = s(h)$ for all $g, h \in G\setminus E$.

Let G be a group, and let k be a field, it is not hard to deduce from [33, Theorem 6.2] that $\mathrm{LNUCA}_c(G, k^n)$ is a k-algebra whose multiplication is given by the composition of maps and whose addition is component-wise.

In parallel, we can define a generalization of the group ring $k[G]$, namely, $D^1(k[G])$, which is given as the product $D^1(k[G]) = k[G] \times (k[G])[G]$ with component-wise addition but where the multiplication is given by

$$(\alpha_1, \beta_1) * (\alpha_2, \beta_2) = (\alpha_1\alpha_2, \alpha_1\beta_2 + \beta_1\alpha_2 + \beta_1\beta_2).$$

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Here, the product $\alpha_1\alpha_2$ is computed with the multiplication rule in the group ring $k[G]$ so that $k[G]$ is naturally a subring of $D^1(k[G])$ via the map $\alpha \mapsto (\alpha, 0)$. However, $\alpha_1\beta_2, \beta_1\alpha_2, \beta_1\beta_2$ are *twisted products* (see Definition 5.1) which are different from the products computed with the multiplication rule of the group ring $(k[G])[G]$ with coefficients in $k[G]$.

By [5], there exists a canonical ring isomorphism between $M_n(k[G])$ and the ring $\mathrm{LCA}(G, k^n)$ of all linear CA $(k^n)^G \to (k^n)^G$. Extending the above isomorphism, we can also interpret $\mathrm{LNUCA}_c(G, k^n)$ algebraically in terms of the ring $M_n(D^1(k[G]))$ as follows (see Theorem 6.2 and Proposition 7.1):

Theorem A *For every field k and every infinite group G, there exists a canonical isomorphism* $\mathrm{LNUCA}_c(G, k^n) \simeq M_n(D^1(k[G]))$ *for every* $n \geq 1$.

In [5] and [36], respectively, the authors study the *L-surjunctivity* and the *finite L-surjunctivity* of a group, namely, a group G is L-surjunctive, resp. finitely L-surjunctive, if for every finite-dimensional vector space V, resp. finite vector space V, every injective $\tau \in \mathrm{LCA}(G, V)$ is also surjective. It was shown that all sofic groups are L-surjunctive [5, 20]. Notably, we know from [5] that a group G is L-surjunctive if and only if $k[G]$ is stably finite for every field k. Moreover, results in [36] show that L-surjunctivity and finite L-surjunctivity are equivalent notions. In this vein, we introduce the following various notions of surjunctivity in the case of linear NUCA.

Definition 1.3 Let G be a group. We say that G is *L^1-surjunctive*, resp. *finitely L^1-surjunctive*, if for every finite-dimensional vector space V, resp. for every finite vector space V, every stably injective $\tau \in \mathrm{LNUCA}_c(G, V)$ is also surjective.

In the line of some recent results which establish the multifold interaction between symbolic dynamics, group theory, and ring theory such as [2, 10, 31, 35, 36], etc. our main result is the following:

Theorem B *For every infinite group G, the following are equivalent:*

(i) *G is L^1-surjunctive;*
(ii) *G is finitely L^1-surjunctive;*
(iii) *for every field k, the ring $D^1(k[G])$ is stably finite;*
(iv) *for every finite field k, the ring $D^1(k[G])$ is stably finite;*
(v) *G is dual L^1-surjunctive;*
(vi) *G is finitely dual L^1-surjunctive.*

Here, a group G is *dual L^1-surjunctive*, resp. *finitely dual L^1-surjunctive*, if for every finite-dimensional vector space V, resp. for every finite vector space V, every stably post-surjective $\tau \in \mathrm{LNUCA}_c(G, V)$ is pre-injective.

The dual surjunctivity is studied in [3] where it was shown that every post-surjective CA over a sofic universe and a finite alphabet is also pre-injective. See also [32] for some extensions. As an application of Theorem B, we obtain the following result which extends [33, Theorem B] and [37, Theorem D] to cover the case of initially subamenable group universes (see Section 2) and arbitrary finite-dimensional vector space alphabets.

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Theorem C *All initially subamenable groups and all residually finite groups are L^1-surjunctive and dual L^1-surjunctive.*

We deduce immediately from Theorems B and C the following result on the stable finiteness of twisted group rings.

Corollary 1.1 *Let G be a residually finite group or an initially subamenable group. Then for every field k, the ring $D^1(k[G])$ is stably finite.*

The paper is organized as follows: We recall in Section 2, the definition of initially subamenable groups and residually finite groups. Section 3 collects the construction of various induced local maps of NUCA. Then we establish the equivalence of the left-invertibility and the stable injectivity of elements of the class $\mathrm{LNUCA}_c(G, V)$, where V is any finite-dimensional vector space (Theorems 4.2 and 4.3). The construction of the twisted group ring $D^1(k[G])$ is given in Section 4. We then present the proof of Theorem A as a consequence of Theorem 6.2 and Proposition 7.1, respectively, in Sections 5 and 6. The dynamical characterization of the direct finiteness of the ring $M_n(D^1(k[G]))$ in terms of the direct finiteness of $\mathrm{LNUCA}_c(G, k^n)$. The proof of the main result Theorem B is contained in Section 8. Finally, in Section 9, we prove Theorem C as an application of Theorem B.

2 Initially subamenable groups and residually finite groups

2.1 Amenable groups

Amenable groups were defined by von Neumann [26]. A group G is *amenable* if the Følner's condition [18] is satisfied: for every $\varepsilon > 0$ and $T \subset G$ finite, there exists $F \subset G$ finite such that $|TF| \leq (1+\varepsilon)|F|$. Finitely generated groups of subexponential growth and solvable groups are amenable. However, all groups containing a subgroup isomorphic to a free group of rank 2 are non-amenable (see, e.g., [38] for some more details). The celebrated Moore and Myhill Garden of Eden theorem [24, 25] was generalized to characterize amenable groups (cf. [2, 9, 12, 28, 32]) and asserts that a CA with finite alphabet over an amenable group universe is surjective if and only if it is pre-injective.

More generally, we say that a group G is *initially subamenable* if for every $E \subset G$ finite, there exist an amenable group H and an injective map $\varphi : E \to H$ such that $\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in E$ with $gh \in E$. Initially subamenable groups are sofic but the converse does not hold [13]. Note also that finitely presented initially subamenable groups are residually amenable but there exist initially subamenable groups which are not residually amenable [17].

2.2 Residually finite groups

We say that a group G is *residually finite* if for every finite subset $F \subset G$, there exists a finite group H and a surjective group homomorphism $\varphi : G \to H$ such that the restriction $\varphi|_F : F \to H$ is injective. All finitely generated abelian groups and, more generally, all finitely generated linear groups are residually finite. Note that free groups are non-amenable but residually finite.

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3 Induced local maps of NUCA

To fix the notation, for all sets $E \subset F$ and $\Lambda \subset A^F$, we denote $\Lambda_E = \{x|_E : x \in \Lambda\} \subset A^E$. Let G be a group, and let A be a set. For every subset $E \subset G$, $g \in G$, and $x \in A^E$, we define $gx \in A^{gE}$ by setting $gx(gh) = x(h)$ for all $h \in E$. In particular, $gA^E = \{gx : x \in A^E\} = A^{gE}$.

Let M be a subset of a group G. Let A be a set, and let $S = A^{A^M}$. For every finite subset $E \subset G$ and $w \in S^E$, we define a map $f^+_{E,w} : A^{EM} \to A^E$ by setting

$$f^+_{E,w}(x)(g) = w(g)((g^{-1}x)|_M), \tag{3.1}$$

for all $x \in A^{EM}$ and $g \in E$ (see, e.g., [9, Lemma 3.2], [28, Proposition 3.5], [30, Section 2.2] for the case of CA).

In the above formula, note that $g^{-1}x \in A^{g^{-1}EM}$ and $M \subset g^{-1}EM$ since $1_G \in g^{-1}E$ for $g \in E$. Therefore, the map $f^+_{E,w} : A^{EM} \to A^E$ is well defined.

Consequently, for every $s \in S^G$, we have a well-defined induced local map $f^+_{E,s|_E} : A^{EM} \to A^E$ for every finite subset $E \subset G$ which satisfies

$$\sigma_s(x)(g) = f^+_{E,s|_E}(x|_{EM})(g), \tag{3.2}$$

for all $x \in A^G$ and $g \in E$. Equivalently, we have, for all $x \in A^G$, that

$$\sigma_s(x)|_E = f^+_{E,s|_E}(x|_{EM}). \tag{3.3}$$

4 Left-invertibility of stably injective linear NUCA

For the proof of the main result of the section Theorem 4.2, we shall need the following useful technical lemma.

Lemma 4.1 *Let G be a finitely generated infinite group, and let V be a finite-dimensional vector space. Let $\tau \in \mathrm{LNUCA}_c(G, V)$ be a stably injective linear NUCA, and let $\Gamma := \tau(V^G)$. Then there exists a finite subset $N \subset G$ such that the following condition holds:*

(C) *for any $d \in \Gamma$ and $g \in G$, the element $\tau^{-1}(d)(g) \in V$ depends only on the restriction $d|_{gN}$.*

Proof Since τ is a linear NUCA with finite memory, there exists a finite subset $M \subset G$ and $s \in S^G$, where $S = \mathcal{L}(V^M, V)$ such that $\tau = \sigma_s$. By hypothesis, s is asymptotic to a constant configuration $c \in S^G$. Up to enlarging M, we can also suppose that $s|_{G\setminus M} = c|_{G\setminus M}$ and that $1_G \in M$. Since the group G is finitely generated, thus countable, it admits an increasing sequence of finite subsets $M = E_0 \subset \cdots \subset E_n \ldots$ such that $G = \cup_{n\in\mathbb{N}} E_n$.

Suppose on the contrary that there does not exist a finite subset $N \subset G$ which satisfies condition (C). Then, by linearity, there exist, for each $n \in \mathbb{N}$, a configuration $d_n \in \Gamma$ and an element $g_n \in G$ such that for $c_n = \tau^{-1}(d_n)$ (which is well-defined since τ is injective), we have

$$d_n|_{g_n E_n} = 0^{g_n E_n} \quad \text{and} \quad c_n(g_n) \neq 0.$$

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Consequently, by letting $x_n = g_n^{-1}c_n$ and $y_n = g_n^{-1}d_n$, we infer from [33, Lemma 5.1] that $\sigma_{g_n^{-1}s}(x_n) = y_n$ and

$$y_n|_{E_n} = 0^{E_n} \quad \text{and} \quad x_n(1_G) \neq 0.$$

Since s is asymptotic to a constant configuration $c \in S^G$ by hypothesis, the set $T = \{s(g) : g \in G\}$ is actually a finite subset of $S = \mathcal{L}(V^M, V)$. It follows that $\Sigma(s) \subset T^G$ is a compact subspace. Therefore, up to restricting to a subsequence, we can suppose without loss of generality that the sequence $(g_n^{-1}s)_{n\in\mathbb{N}}$ converges to a configuration $t \in T^G \subset S^G$ with respect to the prodiscrete topology.

By [33, Lemma 8.1], we know that $\Sigma(s) = \{gs : g \in G\} \cup \{c\}$. Note that if s is constant then the lemma results from [5]. Hence, we can suppose in the sequel that s is not a constant configuration. In particular, T and $\Sigma(s)$ are not singletons. We distinguish two cases according to whether $t = c$ or not.

Case 1: $t = gs$ for some $g \in G$. Then, since G is infinite and s is asymptotic but not equal to c, we can, up to restricting to a subsequence again, assume without loss of generality that $g_n^{-1} = g$ for all $n \in \mathbb{N}$. Up to replacing s by gs, we can also suppose that $g = 1_G$ so that $\sigma_s(x_n) = y_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, consider the following linear subspace of V^{E_nM}:

$$I_n := \operatorname{Ker}(f^+_{E_n, s|_{E_n}}) \subset V^{E_nM}.$$

Observe that $x_n|_{E_nM} \in I_n \backslash \{0^{E_nM}\}$. Note also that, for all $n \le m \le k$, the projection $p_{nm} : A^{E_m} \to A^{E_n}$ induces a linear map $\pi_{nm} : I_m \to I_n$ and $\pi_{nk}(I_k) \subset \pi_{nm}(I_m)$. Hence, for each $n \in \mathbb{N}$, we obtain a decreasing sequence of linear subspaces $(\pi_{nm}(I_m))_{m\ge n}$ of I_n. Hence, $(\pi_{nm}(I_m))_{m\ge n}$ is stationary and there exists a linear subspace $J_n \subset I_n \subset A^{E_nM}$ such that $\pi_{nm}(I_m) = J_n$ for all m large enough.

Observe that $\pi_{nm}(J_m) \subset J_n$ for all $m \ge n$. We claim that the restriction linear map $q_{nm} : J_m \to J_n$ is surjective for all $m \ge n$. Indeed, let $y \in J_n$ and let $k \ge m$ be sufficiently large such that $q_{nk}(I_k) = J_n$ and $q_{mk}(I_k) = J_m$. Thus, $q_{nk}(x) = y$ for some $x \in I_k$. As $q_{nk} = q_{nm} \circ q_{mk}$, we have $q_{nm}(y') = y$, where $y' = q_{mk}(x) \in J_m$. The claim is proved.

We choose $k \in \mathbb{N}$ large enough such that $\pi_{0k}(I_k) = J_0$. Let $z_0 = \pi_{0k}(x_k) \in J_0$ then $z_0(1_G) \neq 0$. We define by induction a sequence $(z_n)_{n\in\mathbb{N}}$, where $z_n \in J_n$ for all $n \in \mathbb{N}$ as follows. Given $z_n \in J_n$ for some $n \in \mathbb{N}$, there exists by the surjectivity of the map $q_{n,n+1}$ an element

$$z_{n+1} \in q_{n,n+1}^{-1}(z_n) \subset J_{n+1} \subset A^{E_{n+1}M}.$$

We thus obtain a configuration $c \in V^G$ defined by $z|_{E_nM} = z_n$ for all $n \in \mathbb{N}$. Since $G = \cup_{n\in\mathbb{N}} E_nM$, the configuration z is well-defined.

By construction, we have for all $n \in \mathbb{N}$ that

$$\tau(z)|_{E_n} = f^+E_n, s|_{E_n}(z|_{E_nM}) = f^+E_n, s|_{E_n}(z_n) = 0^{E_n}.$$

Therefore, $\tau(z) = 0^G$ but $z(1_G) \neq 0$ which then contradicts the injectivity of the linear NUCA τ.

Case 2: $t = c$. Then, since $\lim_{n\to\infty} g_n^{-1}s = t$ and $s \neq c$, we deduce immediately that $g_n \to \infty$ when $n \to \infty$, i.e., for every finite subset $E \subset G$, there exists $N \in \mathbb{N}$ such that $g_n \notin E$ for all $n \ge N$. Consequently, by restricting to a suitable subsequence, we can

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suppose without loss of generality that $g_nE_nM \cap M = \varnothing$ for all $n \in \mathbb{N}$. As $s|_{G\setminus M} = c|_{G\setminus M}$, it follows that $(g_n^{-1}s)|_{E_nM} = c|_{E_nM}$ for all $n \in \mathbb{N}$. Since c is constant, we deduce that $\sigma_c(x_n)|_{E_n} = 0^{E_n}$ and $x_n(1_G) \neq 0$. We infer from the stable injectivity of σ_s that σ_c is injective. Therefore, a similar argument as in *Case 1* applied for σ_c and the sequence $(x_n)_{n\in\mathbb{N}}$ leads to a contradiction.

Consequently, there must exist a finite subset $N \subset G$ which satisfies condition (C) and the proof is thus complete. ∎

Our next results Theorems 4.2 and 4.3 extend the results [37, Theorem 10.1] and [33, Theorem 7.1] for NUCA over finite alphabet to the class LNUCA_c over an arbitrary finite-dimensional vector space.

Theorem 4.2 *Let G be a group, and let V be a finite-dimensional vector space. Let $\tau \in \text{LNUCA}_c(G, V)$ be a stably injective linear NUCA. Then τ is left-invertible, i.e., there exists $\sigma \in \text{LNUCA}_c(G, V)$ such that $\sigma \circ \tau = \text{Id}$.*

Proof As the linear NUCA τ has finite memory, we can find a finite subset $M \subset G$ and $s \in S^G$ where $S = \mathcal{L}(V^M, V)$ such that $\tau = \sigma_s$. By hypothesis, the configuration s is asymptotic to a constant configuration $c \in S^G$. Hence, we can, up to enlarging M, suppose that $s|_{G\setminus M} = c|_{G\setminus M}$ and that $1_G \in M$.

Assume first that G is a finitely generated infinite group. Then we infer from Lemma 4.1 that there exists a finite subset $N \subset G$ such that for any $d \in \tau(V^G)$ and $g \in G$, the element $\tau^{-1}(d)(g) \in V$ depends only on the restriction $d|_{gN}$. Up to enlarging M and N, we can clearly suppose that $M = N$. Consequently, for each $g \in G$, we have a well-defined map $\varphi_g : \tau(V^G)_{gM} \to V$ given by $d|_{gM} \mapsto \tau^{-1}(d)(g)$ for every $d \in V^G$.

Since τ is linear and $\tau(V^G)_{gM}$ is a linear subspace of V^{gM}, it follows that φ_g is also a linear map and we can extend φ_g to a linear map $\tilde{\varphi}_g : V^{gM} \to V$ which coincides with φ_g on $\tau(V^G)_{gM}$. Let $\phi_g : V^M \to V^{gM}$ be the canonical automorphism induced by the bijection $M \simeq gM$, $h \mapsto gh$. Let us define an configuration $t \in S^G$ where $S = \mathcal{L}(V^M, V)$ by setting $t(g) = \tilde{\varphi}_g \circ \phi_g : V^M \to M$ for every $g \in G$. It is immediate from the construction that for every $c \in V^G$, $g \in G$, and $d = \tau(c) \in \tau(V^G)$, we have

$$\sigma_t(\sigma_s(c))(g) = \sigma_t(d)(g) = t(g)((g^{-1}d)|_M) = \tau^{-1}(d)(g) = c(g).$$

Therefore, $\sigma_t \circ \sigma_s = \text{Id}$ and we conclude that $\tau = \sigma_s$ is left-invertible. In fact, since $s|_{G\setminus M} = c|_{G\setminus M}$, the linear spaces $W = \phi_g^{-1}(\tau(V^G)_{gM}) = \phi^{-1}(f^+gM, s|_{gM}(V^{gM^2}))$ coincide as linear subspaces of V^M for all $g \in G\setminus MM^{-1}$. Let us fix a direct sum decomposition $V^M = W \oplus U$ of V^M. Thus, if we define $\tilde{\varphi}_g$ by setting $\tilde{\varphi}_g(v) = 0$ for all $v \in \phi_g(U)$ and $\tilde{\varphi}_g(v) = \varphi_g(v)$ if $v \in \phi_g(W)$ and extend by linearity on the whole space V^{gM}, then it is clear that t is also asymptotically constant, which completes the proof of the theorem in the case when G is a finitely generated infinite group.

The case when G is a finite group is trivial since every injective endomorphism of a finite-dimensional vector space is an automorphism. Let us consider the general case where G is an infinite group. Let H be the subgroup of G generated by M. Let $G/H = \{gH : g \in G\}$ be the set of all right cosets of H in G. By identifying $x \in A^G$ with $(x|_u)_{u\in G/H}$, we obtain a factorization $A^G = \prod_{u\in G/H} A^u$. Moreover, $\sigma_s =$

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$\prod_{u\in G/H}\sigma_s^u$, where $\sigma_s^u : A^u \to A^u$ is given by $\sigma_s^u(y) = \sigma_s(x)|_u$ for all $y \in A^u$ and any $x \in A^G$ extending y. Similarly, we have $\sigma_c = \prod_{u\in G/H}\sigma_c^u$.

For every coset $u \in G/H$, let us choose $g_u \in G$ such that $g_H = 1_G$. Then, if $u \neq H$, we have $s|_u = c|_u$ and $\sigma_s^u = \sigma_c^u$ is conjugate to the restriction CA $\sigma_{c|_H} = \sigma_c^H : A^H \to A^H$ by the uniform homeomorphism $\phi_u : A^u \to A^H$ given by $\phi_u(y)(h) = y(g_u h)$ for all $y \in A^u$ and $h \in H$ (cf. the discussion following [11, Lemma 2.8]). Hence, σ_s and σ_c are left-invertible (resp. injective) if and only if so are $\sigma_{s|_H}$ and $\sigma_{c|_H}$ (see also [7, Theorem 1.2]). Consequently, the general case follows from the case when G is finite or when G is a finitely generated infinite group. The proof is thus complete. ∎

Conversely, we show that left-invertibility implies stable injectivity for linear NUCA with finite memory whose configuration of local defining maps is asymptotically constant.

Theorem 4.3 *Let G be a group, and let V be a finite-dimensional vector space. Suppose that $\tau \in \mathrm{LNUCA}_c(G, V)$ is a left-invertible linear NUCA. Then τ stably injective.*

Proof As in the proof of Theorem 4.2, we can suppose without loss of generality that G is a finitely generated infinite group. Since τ is a linear NUCA with finite memory and left-invertible, we can find a finite subset $M \subset G$ and $s, t \in S^G$, where $S = \mathcal{L}(V^M, V)$ such that $\tau = \sigma_s$ and $\sigma_t \circ \sigma_s = \mathrm{Id}$. In particular, we deduce immediately that σ_s is injective.

As s is asymptotically constant, we infer from [33, Lemma 8.1] that $\Sigma(s) = \{gs : g \in G\} \cup \{c\}$ for some constant configuration $c \in S^G$. Note that by [33, Lemma 5.1], the injectivity of σ_{gs} for all $g \in G$ follows from the injectivity of σ_s. We must show that σ_c is injective. For this, we can suppose, up to enlarging M, that $s|_{G\setminus M} = c|_{G\setminus M}$. Since G is infinite, there exists $g \in G$ such that $gM \cap M = \varnothing$. It follows that $s|_{gM} = c|_{gM}$. On the other hand, we infer from the identity $\sigma_t \circ \sigma_s = \mathrm{Id}$ that

$$t(g) \circ f^{+}_{gM, s|_{gM}} = \pi_{gM^2, g},$$

where $\pi_{F,E} : V^F \to V^E$ denotes the canonical projection induced by any inclusion of sets $E \subset F$. Consequently, $t(g) \circ f^{+}_{gM, c|_{gM}} = \pi_{gM^2, g}$. Since c is constant, we deduce that $\sigma_d \circ \sigma_c = \mathrm{Id}$, where $d \in S^G$ is the constant configuration defined by $d(h) = t(g)$ for all $h \in G$. In particular, σ_c is injective and we conclude that σ_s is stably injective. The proof is thus complete. ∎

5 The twisted group ring $D^1(k[G])$

Given a group G and a ring R (with unit), recall that the group ring $R[G]$ is the R-algebra which admits G as a basis and whose multiplication is defined by the group product on basis elements and the distributive law.

Definition 5.1 Let k be a ring, and let G be a group. We define $D^1(k[G])$ as the Cartesian product

$$D^1(k[G]) = k[G] \times (k[G])[G].$$

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Elements of $D^1(k[G])$ are couples (α, β), where $\alpha \in k[G]$ is called the *regular part* and $\beta \in (k[G])[G]$ is called the *singular part* of (α, β). The addition operation of $D^1(k[G])$ is component-wise:

$$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2).$$

The multiplication operation $* : D^1(k[G]) \times D^1(k[G]) \to D^1(k[G])$ is defined as follows:

$$(\alpha_1, \beta_1) * (\alpha_2, \beta_2) = (\alpha_1\alpha_2, \alpha_1\beta_2 + \beta_1\alpha_2 + \beta_1\beta_2). \tag{5.1}$$

Here, $\alpha_1\alpha_2$ is computed with the multiplication rule in the group ring $k[G]$. However, for $\alpha \in k[G]$ and $\beta, \gamma \in (k[G])[G]$, we define, by abuse of notation, the *twisted products* $\alpha\beta$, $\beta\alpha$, and $\beta\gamma$ as elements of $(k[G])[G]$ as follows, which should be distinguished from the multiplication rule of the group ring $(k[G])[G]$ with coefficients in $k[G]$. Let $g, h \in G$, we set:

$$\begin{aligned}(\alpha\beta)(g)(h) &= \sum_{t\in G} \alpha(t)\beta(gt)(t^{-1}h),\\ (\beta\alpha)(g)(h) &= \sum_{t\in G} \beta(g)(t)\alpha(t^{-1}h),\\ (\beta\gamma)(g)(h) &= \sum_{t\in G} \beta(g)(t)\gamma(gt)(t^{-1}h).\end{aligned} \tag{5.2}$$

It is not hard to check that the above product rule (5.2) is associative and distributive with respect to addition. For example, with the above α, β, γ, we have $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ since for all $g, h \in G$:

$$\begin{aligned}(\alpha(\beta\gamma))(g)(h) &= \sum_{t\in G} \alpha(t) \sum_{q\in G} \beta(gt)(q)\gamma(gtq)(q^{-1}t^{-1}h)\\ &= \sum_{t\in G} \alpha(t) \sum_{r\in G} \beta(gt)(t^{-1}r)\gamma(gr)(r^{-1}h) \qquad (r = tq)\\ &= \sum_{r\in G}\sum_{t\in G} \alpha(t)\beta(gt)(t^{-1}r)\gamma(gr)(r^{-1}h)\\ &= ((\alpha\beta)\gamma)(g)(h).\end{aligned}$$

The following lemma tells us that $D^1(k[G])$ is indeed a ring with unit.

Lemma 5.1 *For every group G and every ring k, the set $D^1(k[G])$ equipped with the addition and multiplication operations as defined in Definition 5.1 is a ring with unit $(1_G, 0)$ and neutral element $(0, 0)$.*

Proof Since the addition is component-wise and $k[G]$ and $(k[G])[G]$ are abelian groups, $D^1(k[G])$ is also an abelian group. It is clear that $(\alpha, \beta) * (1_G, 0) = (1_G, 0) * (\alpha, \beta) = (\alpha, \beta)$ for all $(\alpha, \beta) \in D^1(k[G])$. Moreover, the associativity of the multiplication is satisfied since for all $(\alpha_i, \beta_i) \in D^1(k[G])$ $(i = 1, 2, 3)$, we find that

$$\begin{aligned}&((\alpha_1, \beta_1) * (\alpha_2, \beta_2)) * (\alpha_3, \beta_3) = (\alpha_1\alpha_2, \alpha_1\beta_2 + \beta_1\alpha_2 + \beta_1\beta_2) * (\alpha_3, \beta_3)\\ &= (\alpha_1\alpha_2\alpha_3, \alpha_1\alpha_2\beta_3 + \alpha_1\beta_2\alpha_3 + \beta_1\alpha_2\alpha_3 + \beta_1\beta_2\alpha_3 + \alpha_1\beta_2\beta_3 + \beta_1\alpha_2\beta_3 + \beta_1\beta_2\beta_3)\\ &= (\alpha_1\alpha_2\alpha_3, \alpha_1\alpha_2\beta_3 + \alpha_1\beta_2\alpha_3 + \alpha_1\beta_2\beta_3 + \beta_1\alpha_2\alpha_3 + \beta_1\alpha_2\beta_3 + \beta_1\beta_2\alpha_3 + \beta_1\beta_2\beta_3)\\ &= (\alpha_1, \beta_1) * (\alpha_2\alpha_3, \alpha_2\beta_3 + \beta_2\alpha_3 + \beta_2\beta_3)\\ &= (\alpha_1, \beta_1) * ((\alpha_2, \beta_2) * (\alpha_3, \beta_3)).\end{aligned}$$

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Finally, we see without difficulty that the distributivity of $D^1(k[G])$ follows from the distributivity of $k[G]$ and $(k[G])[G]$. Hence, we conclude that $D^1(k[G])$ is a ring with unit $(1_G, 0)$ and neutral element $(0, 0)$. ∎

The next lemma says that the generalized group ring $D^1(k[G])$ contains naturally the group ring $k[G]$ as the subring of regular elements, i.e., elements whose singular parts are zero.

Lemma 5.2 *Let k be a ring, and let G be a group. Then we have a canonical embedding of rings* $\varphi : k[G] \hookrightarrow D^1(k[G])$ *given by the formula* $\varphi(\alpha) = (\alpha, 0)$ *for all* $\alpha \in k[G]$.

Proof The map φ is trivially injective. Moreover, it is a direct consequence of the definition of the addition of multiplication operations of $D^1(k[G])$ that $\varphi(\alpha_1) + \varphi(\alpha_2) = \varphi(\alpha_1 + \alpha_2)$ and $\varphi(\alpha_1\alpha_2) = \varphi(\alpha_1\alpha_2)$ for all $\alpha_1, \alpha_2 \in k[G]$. ∎

Observe that the group ring $k[G]$ is naturally a Hopf k-algebra where the antipodal map is induced by $g \mapsto g^{-1}$ and the comultiplication map is induced by $g \mapsto g \otimes g$ for all $g \in G$. In general, our construction of the twisted group ring $D^1(k[G])$ can be suitably extended in various ways where we replace $k[G]$ by an arbitrary Hopf k-algebra A as follows. We denote the comultiplication map of A by $\Delta : A \to A \otimes A$. Let us first define $D^1(A) = A \oplus (A \otimes A)$ as a k-module. Suppose that we have defined bilinear product rules $\alpha * \beta, \beta * \alpha, \beta * \gamma \in A \otimes A$, where $\alpha \in A$ and $\beta, \gamma \in A \otimes A$, which satisfy the associativity $(u * v) * w = u * (v * w)$ for all $u, v, w \in A \cup (A \otimes A)$ where we set $x * y := xy \in A$ if $x, y \in A$. Then a straightforward application of the proof of Lemma 5.1 shows that the space $D^1(A)$ will become an unital k-algebra via the following multiplication rule defined in a similar fashion as in (5.1):

$$(\alpha_1, \beta_1) * (\alpha_2, \beta_2) = (\alpha_1\alpha_2, \alpha_1 * \beta_2 + \beta_1 * \alpha_2 + \beta_1 * \beta_2).$$

As an immediate example, let $\varphi : A \otimes A \to A \otimes A$ be a k-algebra homomorphism. For $\alpha \in A$ and $\beta, \gamma \in A \otimes A$, consider the following bilinear product rules:

$$\alpha * \beta := \varphi(\Delta(\alpha))\beta \in A \otimes A,$$
$$\beta * \alpha := \beta\varphi(\Delta(\alpha)) \in A \otimes A,$$
$$\beta * \gamma := \beta\gamma \in A \otimes A.$$

Then clearly $(u * v) * w = u * (v * w)$ for all $u, v, w \in A \cup (A \otimes A)$ and $D^1(A)$ is thus an unital k-algebra. The rules described by the formula (5.2) provide another nontrivial example of dynamical origin (see Section 6) when A is the group ring $k[G]$ for some group G. We suspect that such constructions may lead to further interesting investigations concerning Hopf algebras.

6 Non-uniform linear NUCA $(k^n)^G \circlearrowleft$ and $D^1(M_n(k)[G])$

Let k be a field, and let G be a group. Let us fix an integer $n \geq 1$ and denote $V = k^n$. Recall that $\mathrm{LNUCA}_c(G, k^n)$ is the k-algebra of all linear NUCA with finite memory $\tau : (k^n)^G \to (k^n)^G$ which admit asymptotically constant configurations of local defining maps. The multiplication of $\mathrm{LNUCA}_c(G, k^n)$ is given by the composition of maps and whose addition is component-wise.

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With every element $\omega = (\alpha, \beta) \in D^1(M_n(k)[G])$, we can associate a map $\tau^\omega : V^G \to V^G$ defined as follows:

$$\tau^\omega(x)(g) = \sum_{h\in G} \alpha(h)x(gh) + \sum_{h\in G} \beta(g)(h)x(gh) \quad \text{for all } x \in V^G, g \in G. \tag{6.1}$$

For every element $\gamma \in M_n(k)[G]$, we denote the *support* of γ as the finite subset $\operatorname{supp}(\gamma) = \{g \in G : \gamma(g) \neq 0\}$ of G. Given $\omega = (\alpha, \beta) \in D^1(M_n(k)[G])$, we define its *support* $\operatorname{supp}(\omega) \subset G$ by

$$\operatorname{supp}(\omega) = \cup_{g\in G} \operatorname{supp} \beta(g) \cup \operatorname{supp} \alpha. \tag{6.2}$$

Lemma 6.1 *The map $\tau^\omega : V^G \to V^G$ is a linear NUCA with finite memory. Moreover, τ^ω admits a configuration of local defining maps which is asymptotic to a constant configuration, i.e., $\tau^\omega \in \mathrm{LNUCA}_c(G, k^n)$.*

Proof Since $\alpha(g), \beta(g)(h) \in M_n(k)$ for all $g, h \in G$, it follows from (6.1) that τ^ω is a linear map. Let $M = \operatorname{supp}(\omega) \subset G$ (see (6.2)).

We define a configuration of local defining maps $s \in S^G$, where $S = \mathcal{L}(V^G, V)$ as follows. For every $g \in G$, let $s(g) \in S$ be the linear map determined for all $w \in V^M$ by

$$s(g)(w) = \sum_{h\in M} \alpha(h)w + \sum_{h\in M} \beta(g)(h)w.$$

Let $x \in V^G$ and $g \in G$. Then we infer from the definition (6.1) and the choice of M that

$$\begin{aligned}\sigma_s(x)(g) &= s(g)((g^{-1}x)|_M)\\ &= \sum_{h\in M} \alpha(h)x(gh) + \sum_{h\in M} \beta(g)(h)x(gh)\\ &= \sum_{h\in G} \alpha(h)x(gh) + \sum_{h\in G} \beta(g)(h)x(gh)\\ &= \tau^\omega(x)(g).\end{aligned}$$

We deduce that $\tau^\omega = \sigma_s$ is indeed a linear NUCA with finite memory. On the other hand, if we denote $E = \operatorname{supp} \beta$ then E is a finite subset of G and we have $s(g) = \alpha(g)$ for all $g \in G\backslash E$ by construction. Consequently, s is asymptotic to the constant configuration α^G. Thus, $\tau^\omega \in \mathrm{LNUCA}_c(G, k^n)$ and the proof is complete. ■

It turns out that the converse of the above lemma also holds. In other words, every linear NUCA over V^G with finite memory and asymptotically constant configuration of local defining maps arises uniquely as a map τ^ω described above. More specifically, the following results says that the map $\omega \mapsto \tau^\omega$ is a ring isomorphism when G is infinite.

Theorem 6.2 *Let k be a field, and let G be an infinite group. Then for every integer $n \geq 1$, the map $\Psi : D^1(M_n(k)[G]) \to \mathrm{LNUCA}_c(G, k^n)$ given by $\omega \mapsto \tau^\omega$ is a k-linear ring isomorphism.*

Proof Let $V = k^n$. We claim that Ψ is injective. Indeed, let $\omega = (\alpha, \beta) \in D^1(M_n(k)[G])$ be an element such that $\tau^\omega = 0$ as a map from V^G to itself. Let $M = \operatorname{supp}(\omega)$ (see (6.2)) then M is a finite subset of G. Since G is infinite, we can

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choose some $g_0 \in G\backslash M$. In particular, $\beta(g_0) = 0$ by the choice of M. Then for every $x \in V^G$, we find that $\tau^\omega(x)(g_0) = 0$ and it follows from (6.1) that

$$\sum_{h \in M} \alpha(h)x(g_0 h) = \sum_{h \in G} \alpha(h)x(g_0 h) = \tau^\omega(x)(g_0) = 0.$$

Since x is arbitrary, we deduce that $\alpha(h) = 0$ for all $h \in M$ and thus $\alpha = 0$ since $\operatorname{supp}(\alpha) \subset \operatorname{supp}(\omega) = M$. Consequently, we infer again from (6.1) that for all $g \in G$:

$$\sum_{h \in M} \beta(g)(h)x(gh) = \sum_{h \in G} \beta(g)(h)x(gh) = 0.$$

Thus, $\beta(g)(h) = 0$ for all $g, h \in G$. In other words, $\beta = 0$ and we conclude that $\omega = 0$. Hence, Ψ is indeed injective as claimed.

To check that Ψ is surjective, let $\sigma_s \in \mathrm{LNUCA}_c(G, V)$ where $s \in S^G$ for some $S = \mathcal{L}(V^M, V)$, where $M \subset G$ is a finite subset, such that s is asymptotic to a constant configuration $c \in S^G$. Up to enlarging M, we can also suppose that $s|_{G\backslash M} = c|_{G\backslash M}$.

Since $c(1_G) \in \mathcal{L}(V^M, V)$, there exist $\gamma_h \in \mathcal{L}(V, V) = M_n(k)$ for every $h \in M$ such that for all $w \in V^M$, we have $c(1_G)(w) = \sum_{h \in M} \gamma_h w$. Let us denote $\alpha = \sum_{h \in M} \gamma_h h \in M_n(k)[G]$.

For each $g \in M$, we define $\delta_g = s(g) - c(g) \in \mathcal{L}(V^M, V)$. By linearity, there exists uniquely $\delta_g(h) \in M_n(k) = \mathcal{L}(V, V)$ for $h \in M$ such that for all $w \in V^M$, we have

$$\delta_g(w) = \sum_{h \in M} \delta_g(h)w(h).$$

Hence, we obtain an element $\mu_g = \sum_{h \in M} \delta_g(h) \in M_n(k)[G]$ for every $g \in M$. Let us denote $\beta = \sum_{g \in M} \mu_g g \in (M_n(k)[G])[G]$ and $\omega = (\alpha, \beta) \in D^1(M_n(k)[G])$. We claim that $\tau^\omega = \sigma_s$. Indeed, for every $x \in V^G$ and $g \in G$, we find that

$$\begin{aligned}
\tau^\omega(x)(g) &= \sum_{h \in G} \alpha(h)x(gh) + \sum_{h \in G} \beta(g)(h)x(gh) \\
&= \sum_{h \in M} \gamma_h x(gh) + \sum_{h \in M} \mu_g(h)x(gh) \\
&= c(1_G)((g^{-1}x)|_M) + \sum_{h \in M} \delta_g(h)x(gh) \\
&= c(g)((g^{-1}x)|_M) + \delta_g((g^{-1}x)|_M) \\
&= s(g)((g^{-1}x)|_M) \\
&= \sigma_s(x)(g).
\end{aligned}$$

We conclude that $\sigma_s = \tau^\omega = \Psi(\omega)$ from which the surjectivity of the map Ψ follows. It is immediate from (6.1) that Ψ is a k-linear homomorphism of groups and Ψ sends the unit of $D^1(M_n(k)[G])$ to the unit of $\mathrm{LNUCA}_c(G, V)$.

To finish the proof, we have to check that for all elements $\omega = (\alpha, \beta)$ and $\omega' = (\alpha', \beta')$ of $D^1(M_n(k)[G])$, we have $\Psi(\omega * \omega') = \Psi(\omega) \circ \Psi(\omega')$. Indeed, using the formula $\omega * \omega' = (\alpha\alpha', \alpha\beta' + \beta\alpha' + \beta\beta')$ and (5.2), we can compute for all $x \in V^G$, $g \in G$, and $y = \tau^{\omega'}(x)$ that

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$$
\begin{aligned}
\Psi(\omega * \omega')(x)(g) &= \sum_{h\in G} (\alpha\alpha' + (\alpha\beta' + \beta\alpha' + \beta\beta')(g))(h)x(gh) \\
&= \sum_{h,t\in G} \big(\alpha(t)(\alpha'(t^{-1}h)+\beta'(gt)(t^{-1}h)) + \beta(g)(t)(\alpha'(t^{-1}h)+\beta'(gt)(t^{-1}h)\big)\, x(gh) \\
&= \sum_{t\in G}(\alpha(t)+\beta(g)(t)) \sum_{r\in G} (\alpha'(r)+\beta'(gt)(r))\, x(gtr) \qquad (r = t^{-1}h) \\
&= \sum_{t\in G} (\alpha(t)+\beta(g)(t))\, y(gt) \\
&= \tau^{\omega}(y)(g) \\
&= (\tau^{\omega}\circ\tau^{\omega'})(x)(g) \\
&= (\Psi(\omega)\circ\Psi(\omega'))\,(x)(g).
\end{aligned}
$$

It follows that $\Psi(\omega * \omega') = \Psi(\omega)\circ\Psi(\omega')$. The proof is thus complete. ∎

Observe that Theorem 6.2 does not hold when G is a finite group since the ring morphism Ψ fails to be injective in this case.

7 Stable finiteness of generalized group rings and stably *L*-surjunctive groups

The main goal of the present section is to show that the stable finiteness property of the generalized group ring $D^1(k[G])$ is equivalent to the surjunctivity property of the classes $\mathrm{LUNCA}_c(G, k^n)$ for every $n \geq 1$ (Theorem 7.2).

We begin with the following isomorphism between the ring $M_n(D^1(k[G]))$ of square matrices of size $n \times n$ with coefficients in the generalized group ring $D^1(k[G])$ and the ring $D^1(M_n(k[G]))$.

Proposition 7.1 *Let k be a ring, and let G be a group. Then for every integer $n \geq 1$, there exists a canonical ring isomorphism*

$$
D^1(M_n(k)[G]) \simeq M_n(D^1(k[G])). \tag{7.1}
$$

Proof By [28, Lemma 9.4], there exists a canonical isomorphism of rings $M_n(k)[G] \simeq M_n(k[G])$ given by $\sum_{g\in G} A(g)g \mapsto (\sum_{g\in G} A(g)_{ij}g)_{1\leq i,j\leq n}$. Consider the map $F\colon D^1(M_n(k)[G]) \to M_n(D^1(k[G]))$ defined as follows. For $x = (\alpha,\beta) \in D^1(M_n(k)[G])$, we can write $\beta = \sum_{g\in G}\beta(g)g \in (M_n(k)[G])[G]$ where $\beta(g) = (\beta(g)_{ij})_{1\leq i,j\leq n} \in M_n(k)[G]$ for all $g \in G$. Then we define $F(x) \in M_n(D^1(k[G]))$ by setting for all $1 \leq i, j \leq n$:

$$
F(x)_{ij} = (\alpha_{ij}, \sum_{g\in G} \beta(g)_{ij} g) \in D^1(k[G]).
$$

It is clear that F is a bijective homomorphism of groups and that $F((I_n, 0)) = J_n$, where $I_n \in M_n(k)[G]$ and $J_n \in M_n(D^1(k[G]))$ are identity matrices of $M_n(k)[G]$ and $M_n(D^1(k[G]))$, respectively.

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Now, let $x_i = (\alpha_i, \beta_i) \in D^1(M_n(k)[G])$ for $i = 1, 2$. Then $x_1 * x_2 = (\alpha_1\alpha_2, \alpha_1\beta_2 + \beta_1\alpha_2 + \beta_1\beta_2)$ and thus

$$F(x_1 * x_2)_{ij} = ((\alpha_1\alpha_2)_{ij}, \sum_{g\in G}((\alpha_1\beta_2 + \beta_1\alpha_2 + \beta_1\beta_2)(g))_{ij} g). \tag{7.2}$$

On the other hand, we find that

$$(F(x_1)F(x_2))_{ij} = \sum_{r=1}^{n} F(x_1)_{ir} * F(x_2)_{rj} = \sum_{r=1}^{n}((\alpha_1)_{ir}, (\beta_1)_{ir}) * ((\alpha_2)_{rj}, (\beta_2)_{rj}).$$

Therefore, if we denote $(F(x_1)F(x_2))_{ij} = (u, v)$ then

$$u = \sum_{r=1}^{n}(\alpha_1)_{ir}(\alpha_2)_{rj} = (\alpha_1\alpha_2)_{ij} \tag{7.3}$$

by the definition of matrix multiplication. Moreover, we deduce from the definition of the operation $*$ that $v = \sum_{g\in G} v(g)g$, where

$$v(g) = \sum_{r=1}^{n}((\alpha_1)_{ir}(\beta_2)_{rj})(g) + ((\beta_1)_{ir}(\alpha_2)_{rj})(g) + ((\beta_1)_{ir}(\beta_2)_{rj})(g).$$

We infer from (5.2) that

$$\begin{aligned}\sum_{r=1}^{n}((\alpha_1)_{ir}(\beta_2)_{rj})(g) &= \sum_{r=1}^{n}\sum_{h\in G}((\alpha_1)_{ir}(\beta_2)_{rj})(g)(h)h \\ &= \sum_{r=1}^{n}\sum_{h,t\in G} \alpha_1(t)_{ir}\beta_2(gt)(t^{-1}h)_{rj}h \\ &= \sum_{h\in G}\sum_{t\in G}\sum_{r=1}^{n}\alpha_1(t)_{ir}\beta_2(gt)(t^{-1}h)_{rj}h \\ &= \sum_{h\in G}(\alpha_1\beta_2)(g)(h)_{ij}h \\ &= (\alpha_1\beta_2)(g)_{ij}.\end{aligned}$$

Similarly, we have the equalities $\sum_{r=1}^{n}((\beta_1)_{ir}(\alpha_2)_{rj})(g) = (\beta_1\alpha_2)(g)_{ij}$ and also $\sum_{r=1}^{n}((\beta_1)_{ir}(\beta_2)_{rj})(g) = (\beta_1\beta_2)(g)_{ij}$. Comparing to (7.2), it follows that v is equal to the singular part of $F(x_1 * x_2)_{ij}$. Consequently, we deduce from (7.3) that for all $1 \le i, j \le n$, we have

$$(F(x_1)F(x_2))_{ij} = F(x_1 * x_2)_{ij}.$$

Hence, $F(x_1)F(x_2) = F(x_1 * x_2)$ and we can finally conclude that F is a ring isomorphism. The proof is thus complete. ∎

The main result of the section is the following dynamical characterization of the direct finiteness of the ring $M_n(D^1(k[G]))$.

Theorem 7.2 *Let G be an infinite group, and let k be a field. Then for every integer $n \ge 1$, the following are equivalent:*

(i) *Every stably injective $\tau \in \mathrm{LNUCA}_c(G, k^n)$ is surjective.*
(ii) *The ring $\mathrm{LNUCA}_c(G, k^n)$ is directly finite.*
(iii) *The ring $M_n(D^1(k[G]))$ is directly finite.*

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Proof The equivalence between (ii) and (iii) results directly from Proposition 7.1 and Theorem 6.2 which imply that $\mathrm{LNUCA}_c(G, k^n) \simeq M_n(D^1(k[G]))$.

Suppose that (i) holds and let $\tau, \sigma \in \mathrm{LNUCA}_c(G, k^n)$ be two linear NUCA such that $\tau \circ \sigma = \mathrm{Id}$. Then Theorem 4.3 implies that σ is stably injective. Consequently, we infer from (i) that σ is surjective. In particular, σ is bijective and thus so is τ. It follows from $\tau \circ \sigma = \mathrm{Id}$ that $\sigma \circ \tau = \mathrm{Id}$ as well. This shows that the ring $\mathrm{LNUCA}_c(G, k^n)$ is directly finite. Therefore, we have shown that (i) implies (ii).

Suppose now that (ii) holds and let $\tau \in \mathrm{LNUCA}_c(G, k^n)$ be a stably injective linear NUCA. Then we deduce from Theorem 4.2 that τ is left-invertible, i.e., there exists $\sigma \in \mathrm{LNUCA}_c(G, k^n)$ such that $\sigma \circ \tau = \mathrm{Id}$. Hence, (ii) implies that $\tau \circ \sigma = \mathrm{Id}$ and it follows at once that τ is surjective. Therefore, we also have that (ii) implies (i). The proof is thus complete. ∎

Corollary 7.3 *Suppose that G is an infinite group. Then the following are equivalent:*

(a) *The group G is L^1-surjunctive.*
(b) *For every field k, the ring $D^1(k[G])$ is stably finite.*

Proof It is a direct consequence of Theorem 7.2. ∎

8 Stable finiteness of generalized group rings and L^1-surjunctive groups

Extending [36, Theorem B], we establish various characterizations of the stable finiteness of the ring $D^1(k[G])$ (for all field k) notably in terms of the finite L^1-surjunctivity of the group G. For ease of reading, we recall the statement of Theorem B in the Introduction.

Theorem 8.1 *For every infinite group G, the following are equivalent:*

(i) *G is L^1-surjunctive;*
(ii) *G is finitely L^1-surjunctive;*
(iii) *for every field k, the ring $D^1(k[G])$ is stably finite;*
(iv) *for every finite field k, the ring $D^1(k[G])$ is stably finite;*
(v) *G is dual L^1-surjunctive;*
(iv) *G is finitely dual L^1-surjunctive.*

Proof Let V be a finite-dimensional vector space, and let $\tau \in \mathrm{LNUCA}_c(G, V)$. Then we obtain a dual linear NUCA $\tau^* \in \mathrm{LNUCA}_c(G, V)$ whose dual is exactly τ, that is, $(\tau^*)^* = \tau$ (see [37]). We infer from [37, Theorem A] that τ is pre-injective if and only if τ^* is surjective and that τ^* is stably injective if and only if τ is stably post-surjective. Hence, we deduce immediately the equivalences (i) $\Longleftrightarrow$ (v) and (ii) $\Longleftrightarrow$ (vi).

The equivalence (i) $\Longleftrightarrow$ (ii) is the content of Corollary 7.3. Similarly, the exact same proof of Theorem 7.2 shows that (ii) $\Longleftrightarrow$ (iv). Finally, the equivalence (i) $\Longleftrightarrow$ (ii) results from Theorem 8.2. The proof is thus complete. ∎

Our next key result extends [36, Theorem A]. The proof follows quite closely the reduction strategy of the proof of [36, Theorem A] and [35, Theorem B] which is less involved in our linear case.

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Theorem 8.2 *Let G be a group, and let $n \geq 1$ be an integer. Then the following are equivalent:*

(i) *For every finite field k, all stably injective $\tau \in \mathrm{LNUCA}_c(G, k^n)$ are surjective.*
(ii) *For every field k, all stably injective $\tau \in \mathrm{LNUCA}_c(G, k^n)$ are surjective.*

Proof Since the case when G is finite is clear and (ii) $\Longrightarrow$ (i) trivially, we suppose in the rest of the proof that G is an infinite group which satisfies (i). Let V be a finite-dimensional vector space over a field k (not necessarily finite), and let $\tau \in \mathrm{LNUCA}_c(G, V)$. Suppose that τ is stably injective. Then, by definition, we can choose a finite subset $M \subset G$ with $1_G \in M = M^{-1}$ and a configuration $s \in \mathcal{L}(V^M, V)^G$ which is asymptotic to a constant configuration $c \in \mathcal{L}(V^M, V)^G$ such that $\tau = \sigma_s$ and $s_{G \setminus M} = c|_{G \setminus M}$. We infer from Theorem 4.2 that τ is left-invertible. Hence, we can find $\sigma \in \mathrm{LNUCA}_c(G, V)$ such that $\sigma \circ \tau = \mathrm{Id}$. Moreover, up to enlarging the finite set M, we can find $t \in \mathcal{L}(V^M, V)^G$ asymptotic to a constant configuration $d \in \mathcal{L}(V^M, V)^G$ such that $\sigma = \sigma_t$ and $t|_{G\setminus M} = d|_{G \setminus M}$.

Let us denote $\Gamma = \tau(V^G)$. As $\sigma_t \circ \sigma_s = \mathrm{Id}$, we deduce for all $g \in G$ that

$$f^+_{\{g\}, t(g)} \circ f^+_{gM, s|_{gM}} = \pi_{gM^2, \{g\}}, \tag{8.1}$$

where $\pi_{F,E} : V^F \to V^E$ denotes the canonical projection for all sets $E \subset F$. Consider the similar condition where we switch the role of s and t:

$$f^+_{\{g\}, s(g)} \circ f^+_{gM, t|_{gM}} = \pi_{gM^2, \{g\}}. \tag{8.2}$$

Since G is infinite, we can choose a finite subset $M^* \subset G$ such that $M^2 \subsetneq M^*$. Then observe that $\sigma_t \circ \sigma_s = \mathrm{Id}$, resp. $\sigma_s \circ \sigma_t = \mathrm{Id}$, if and only if (8.1), resp. (8.2), holds for all $g \in M^*$ (see [34, Lemma 2.2] for the case of CA). Hence, up to making the base change to k' (replacing V, $s(g)$, $t(g)$ resp. by $V \otimes_k k'$, $s(g) \otimes_k k'$, $t(g) \otimes_k k'$ etc.), where k' is an algebraically closed field which contains k, we can suppose without loss of generality that k is algebraically closed.

We obtain from [36, Lemma 2.1] a finitely generated $\mathbb{Z}$-algebra $R \subset k$ and R-modules of finite type V_R and

$$s_R, t_R \in \mathrm{Hom}_{R-mod}((V_R)^M, V_R)^G$$

such that for some fixed $g_0 \in M^* \setminus M^2$, $s_R(g) = s_R(g_0)$, $t_R(g) = t_R(g_0)$ for all $g \in G \setminus M^*$ and the following hold:

I. $V = V_R \otimes_R k$,
II. $s(g) = s_R(g) \otimes_R k$ and $t(g) = t_R(g) \otimes_R k$ for all $g \in M^*$,

where $\pi_R : (V_R)^{E_n M} \to (V_R)^{\{1_G\}}$ is the canonical projection. Essentially, we can take $R = \mathbb{Z}[\Omega]$, where $\Omega \subset k$ is a finite subset consisting of the entries of the matrices which represent the linear maps $s(g)$, $t(g)$ for all $g \in M^*$.

Let us denote $S = \mathrm{Spec}\, R$ which is a $\mathbb{Z}$-scheme of finite type. Then we infer from [36, Lemma 2.2] that the set of closed points of $V_R^{E_n}$ is given by $\Delta = \cup_{p \in \mathcal{P}, a \in S_p, d \in \mathbb{N}} H^{E_n}_{p,a,d}$. Here, $\mathcal{P}$ denotes the set of prime numbers. By $a \in S_p = S \otimes_{\mathbb{Z}} \mathbb{F}_p$, we mean a is a closed point of S_p. In particular, $\kappa(a)$ is a finite field. The set $H_{p,a,d}$ is defined by

$$H_{p,a,d} = \{x \in V_a : |\kappa(x)| = p^r, 1 \leq r \leq d\}, \tag{8.3}$$

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which is a finite linear subspace of the finite-dimensional $\kappa(a)$-vector space $V_a = V_R \otimes_R \kappa(a)$.

Let us fix $p \in \mathcal{P}$, $a \in S_p$, $d \in \mathbb{N}$ and consider the configurations of local defining maps $s_a, t_a \in \mathcal{L}(V_a^M, V_a)^G$ where for all element $g \in G$, we define $s_a(g) = s_R(g) \otimes_R \kappa(a)$ and $t_a(g) = t_R(g) \otimes_R \kappa(a)$. Observe that $s_a(g)(H^M_{p,a,d})$ and $t_a(g)(H^M_{p,a,d})$ are subsets of $H_{p,a,d}$ for all $g \in G$ (cf. e.g., [35, Lemma 3.1]). Consequently, we can define $s_{p,a,d}, t_{p,a,d} \in \mathcal{L}(H^M_{p,a,d}, H_{p,a,d})$ by setting $s_{p,a,d} = s_a|_{H_{p,a,d}}$ and $t_{p,a,d} = t_a|_{H_{p,a,d}}$ for all $g \in G$. Thus, we obtain well-defined linear NUCA $\sigma_{s_{p,a,d}}, \sigma_{t_{p,a,d}} : H^G_{p,a,d} \to H^G_{p,a,d}$.

From (8.1), it is clear from our construction that for all $g \in M^*$, we have

$$f^+_{\{g\},t_{p,a,d}(g)} \circ f^+_{gM,s_{p,a,d}|_{gM}} = \pi^{p,a,d}_{gM^2,\{g\}}, \tag{8.4}$$

where $\pi^{p,a,d}_{F,E} : H^F_{p,a,d} \to H^E_{p,a,d}$ denotes the canonical projection for all sets $E \subset F$. It follows that $\sigma_{t_{p,a,d}} \circ \sigma_{s_{p,a,d}} = \mathrm{Id}$. In particular, $\sigma_{s_{p,a,d}}$ is left-invertible and we deduce from Theorem 4.3 that $\sigma_{s_{p,a,d}}$ is stably injective. Since (i) holds by hypothesis and $H_{p,a,d}$ is a finite $\kappa(a)$-vector space, $\sigma_{s_{p,a,d}}$ is surjective. It follows at once that $\sigma_{s_{p,a,d}} \circ \sigma_{t_{p,a,d}} = \mathrm{Id}$.

Therefore, we deduce that for every $g \in M^*$, the equality

$$f^+_{\{g\},s_R(g)} \circ f^+_{gM,t_R|_{gM}} = \pi^R_{gM^2,\{g\}}, \tag{8.5}$$

where $\pi^R_{gM^2,\{g\}} : V_R^{gM^2} \to V_R^{\{g\}}$, holds over the set $\Delta = \cup_{p\in\mathcal{P},s\in S_p,d\in\mathbb{N}} H^{gM^2}_{p,s,d}$ of all closed points of $(V_R)^{gM^2}$. Since V_R is a Jacobson scheme (cf., e.g., [35, Section 3]), an argument using the equalizer as in [8, Lemma 7.2] shows that $f^+_{\{g\},s_R(g)} \circ f^+_{gM,t_R|_{gM}} = \pi^R_{gM^2,\{g\}}$ as R-morphisms $V_R^{gM^2} \to V_R^{\{g\}}$. Consequently, we obtain the relation (8.2) for all $g \in M^*$ by making the base change (8.5)$\otimes_R k$. It follows that $\sigma_s \circ \sigma_t = \mathrm{Id}$ and we can finally conclude that $\tau = \sigma_s$ is surjective. Therefore, we also have (i) $\Longrightarrow$ (ii) and the proof is complete. ∎

9 Applications

For the proof of Theorem C, we first establish the following extension of [33, Theorem B] and [37, Theorem D] to cover the case of initially subamenable group universes and finite vector space alphabets:

Theorem 9.1 *Every initially amenable group is finitely L^1-surjunctive.*

Proof Let G be an initially subamenable group, and let V be a finite vector space. Suppose that $\tau \in \mathrm{LNUCA}_c(G, V)$ is stably injective. Then we can infer without difficulty from [33, Theorem A] or [37, Theorem B] that there exist a large enough finite subset $M \subset G$ and two configurations $s, t \in \mathcal{L}(V^M, V)^G$ and another configuration $c \in \mathcal{L}(V^M, V)^G$ such that $\tau = \sigma_s$, $s|_{G\setminus M} = t|_{G\setminus M} = c|_{G\setminus M}$, and $\sigma_t \circ \sigma_s = \mathrm{Id}$. Up to enlarging M, we can suppose without loss of generality that $1_G \in M$ and $M = M^{-1}$.

If $G = M^4$ then G is finite and the theorem is trivial since every injective endomorphism of V^G is surjective. Consider the case $M^4 \subsetneq G$. Let $E \subset G$ be any finite subset which contains strictly M^4, that is, $M^4 \subsetneq E$. Since the group G is initially subamenable, we can find an amenable group H and an injective map $\varphi : E \to H$

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such that $\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in E$ such that $gh \in E$. In particular, $\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in M$. Since $M = M^{-1}$ and $1_G \in M$, we deduce that $1_H \in \varphi(M)$ and $\varphi(M) = \varphi(M)^{-1}$.

Up to replacing H by the subgroup generated by $\varphi(E)$, we can suppose that H is generated by $\varphi(E)$. As $\sigma_t \circ \sigma_s = \mathrm{Id}$, we deduce for all $g \in E$ that

$$f^{+}_{\{g\},t(g)} \circ f^{+}_{gM,s|_{gM}} = \pi_{gM^2,\{g\}}, \tag{9.1}$$

where we denote by $\pi_{F,Q} : V^F \to V^Q$ the canonical linear projection for all sets $Q \subset F$. The bijection $\varphi|_E : E \to \varphi(E)$ induces, in particular, an isomorphism $\phi : V^{\varphi(M)} \to V^M$. Let us fix $g_0 \in E\backslash M$. The patterns $s|_E$, $t|_E$ in turn induce the configurations $\tilde{s}, \tilde{t} \in \mathcal{L}(V^{\varphi(M)}, V)^H$ defined by $\tilde{s}(h) = s(h) \circ \phi$, $\tilde{t}(h) = t(h) \circ \phi$ for all $h \in \varphi(E)$ and $\tilde{s}(h) = s(g_0) \circ \phi$, $\tilde{t}(h) = t(g_0) \circ \phi$ for all $h \in H\backslash\varphi(E)$.

Since φ is injective, it follows from (9.1) that for all $h \in \varphi(E)$, we have

$$f^{+}_{\{h\},\tilde{t}(h)} \circ f^{+}_{h\varphi(M),\tilde{s}|_{h\varphi(M)}} = \pi_{h\varphi(M^2),\{h\}}. \tag{9.2}$$

Consequently, we deduce that $\sigma_{\tilde{t}} \circ \sigma_{\tilde{s}} = \mathrm{Id}$. In particular, $\sigma_{\tilde{s}}$ is injective. Since H is amenable and $\tilde{s}$ is asymptotically constant by construction, we infer from [33, Theorem B(i)] that $\tilde{s}$ is surjective. Hence, it follows from $\sigma_{\tilde{t}} \circ \sigma_{\tilde{s}} = \mathrm{Id}$ that $\sigma_{\tilde{s}} \circ \sigma_{\tilde{t}} = \mathrm{Id}$. We deduce that for every $h \in \varphi(E)$, we have

$$f^{+}_{\{h\},\tilde{s}(h)} \circ f^{+}_{h\varphi(M),\tilde{t}|_{h\varphi(M)}} = \pi_{h\varphi(M^2),\{h\}}. \tag{9.3}$$

Therefore, via the injection φ, we obtain for all $g \in E$ that

$$f^{+}_{\{g\},s(g)} \circ f^{+}_{gM,t|_{gM}} = \pi_{gM^2,\{g\}}. \tag{9.4}$$

By the choice of E and φ, we can thus conclude that $\sigma_s \circ \sigma_t = \mathrm{Id}$ which implies in particular that σ_s is surjective. The proof is thus complete. ∎

Observe that by a similar argument, [33, Theorem B(i)] also holds for initially subamenable group universes. As an immediate consequence of Theorems B and 9.1, we obtain the proof of Theorem C in the Introduction as follows:

Proof of Theorem C Thanks to Theorem B, we infer, respectively, from Theorem 9.1 and [33, Theorem B(ii)] that all initially amenable groups and all residually finite groups are L^1-surjunctive. We can thus conclude the proof of the theorem since dual L^1-surjunctivity is equivalent to L^1-surjunctivity also by Theorem B. ∎

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