On the number of Independent Conditions involved in the vanishing of a Rectangular Array.

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1. The notation for a rectangular array can be extended so as to admit of arrays in which the number of rows exceeds the number of columns.

Let

$$\left\| \begin{matrix} a_{11}, & \ldots\ldots\ldots\ldots\ldots\ldots, & a_{1p} \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots \\ a_{q1}, & \ldots\ldots\ldots\ldots\ldots\ldots, & a_{qp} \end{matrix} \right\|_m$$

denote the aggregate of all determinants of the mth order which can be formed from the rectangular array of pq elements by deleting $p - m$ columns and $q - m$ rows.

We may also use the abbreviated notation

$$\| a_{qp} \|_m .$$

Further, let the equation

$$\| a_{qp} \|_m = 0 \qquad (1)$$

denote the aggregate of equations obtained by equating each of the determinants to zero. Equations of this form are of common occurrence in the analytical geometry of n dimensions, and we shall give examples from this field. (1) contains ${}_pC_m \cdot {}_qC_m$ separate equations, but not all of them are independent.

2. *The number of conditions involved in the equation* $\| a_{qp} \|_m = 0$ *is* $(p - m + 1)(q - m + 1)$.

Consider first the array

$$\| a_{mp} \|_m .$$

Form all the determinants which have the first $m - 1$ columns the same. The number of these is $p - m + 1$. Let the second subscript of the mth column be μ. Expand each of the determinants in terms of the co-factors of $a_{\nu\mu}$. The co-factor of $a_{\nu\mu_1}$ is the same as the co-factor of $a_{\nu\mu_2} = A_\nu$ say. Equating each of these determinants to zero we get $p - m + 1$ equations

$$\sum_{\nu=1}^{m} a_{\nu\mu} A_\nu = 0, \quad (\mu = m,\ m + 1,\ \ldots\ldots,\ p).$$

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Now take any other determinant of the array,

$$\begin{vmatrix} a_{1\mu_1}, & a_{1\mu_2}, & \ldots, & a_{1\mu_m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m\mu_1}, & a_{m\mu_2}, & \ldots, & a_{m\mu_m} \end{vmatrix}$$

Multiply the rows respectively by $A_1, A_2, \ldots, A_m$ and add each to the first row. The elements of the first row will then be

$$\sum_{\nu=1}^{m} a_{\nu\mu_\lambda} A_\nu, \quad (\lambda = 1, 2, \ldots, m).$$

If $\mu\lambda = 1, 2, \ldots$, or $m-1$, this is a determinant which vanishes identically since it has two columns the same, and if $\mu\lambda = m, m+1, \ldots$, or p it is a determinant which has already been equated to zero. Hence each element of the first row vanishes and the whole determinant vanishes. The vanishing of $p-m+1$ determinants of the array is therefore sufficient, and it is also necessary, for the vanishing of all the determinants, provided no relations exist between the elements.

Hence the number of conditions involved in the equation

$$\| a_{mp} \|_m = 0$$

is $p-m+1$.*

Now we can write down all the ${}_pC_m \cdot {}_qC_m$ determinants in the form of an array, such that all those in any row or column are obtained from the same rows and columns respectively of the original array, thus

$$\begin{array}{lll} (1, 1) & (1, 2) \ldots\ldots\ldots & (1, {}_pC_m) \\ (2, 1) & (2, 2) \ldots\ldots\ldots & (2, {}_pC_m) \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \\ ({}_qC_m, 1) & ({}_qC_m, 2) \ldots\ldots & ({}_qC_m, {}_pC_m). \end{array}$$

* This theorem appears to be well known. It is stated by Cayley, "Chapters in the analytical geometry of (n) dimensions," *Camb. Math. Jour.*, iv., p. 119. (1843.) Dr R. F. Muirhead has pointed out to me that the general theorem which heads this section was given in answer to his Question 13651 by Professor E. J. Nanson in the *Educational Times*; see *Reprint*, vol. lxix., p. 52, and lxxi., p. 121-122 (1898-99). The theorem is there proved by a somewhat different method, and the question of the choice of determinants to be originally equated to zero is also considered. *E.g.* if a whole column of elements vanish, all the ${}_{p-1}C_{m-1}$ determinants containing this column vanish identically, and no amount of these equated to zero will be a sufficient condition for the vanishing of the others; but if only $p-m$ of the others are equated to zero this is a sufficient condition for all vanishing.

Then if we make $p-m+1$ determinants in any row vanish, the others in that row will vanish; so that if we take $p-m+1$ columns and make $q-m+1$ determinants in each vanish, all the others will vanish.

Hence the number of conditions involved in equation (1) is $(p-m+1)(q-m+1)$.

Example: The conditions that s points $x_\nu^{(\mu)}(\mu=1, 2, \ldots, s)$ in space of n dimensions should all lie in the same homaloid of p dimensions ($s>p+1$) are expressed by

$$\left\| \begin{matrix} x_1', & \ldots\ldots\ldots\ldots, & x_n', & 1 \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{(s)}, & \ldots\ldots\ldots\ldots, & x_n^{(s)}, & 1 \end{matrix} \right\|_{p+2} = 0.$$

This represents $(n-p)(s-p-1)$ independent conditions.

3. If the rectangular array is formed from a symmetrical determinant, as frequently happens, the number of conditions is in general fewer.

Let an array of p columns and q rows ($p>q$) be formed from the symmetrical determinant

$$\left| \begin{matrix} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_{12}, & a_{22}, & \ldots, & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{1n}, & a_{2n}, & \ldots, & a_{nn} \end{matrix} \right|$$

in such a way that the subscripts of the q rows are all included in the subscripts of the p columns. Such an array is

$$\left\| \begin{matrix} a_{11}, & a_{12}, & \ldots, & a_{1q}, & a_{1,q+1}, & \ldots, & a_{1p} \\ a_{12}, & a_{22}, & \ldots, & a_{2q}, & a_{2,q+1}, & \ldots, & a_{2p} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{1q}, & a_{2q}, & \ldots, & a_{qq}, & a_{q,q+1}, & \ldots, & a_{qp} \end{matrix} \right\|$$

The number of different elements here is $pq-\frac{1}{2}q(q-1)$. The number of different determinants of the mth order is ${}_pC_m \cdot {}_qC_m - \frac{1}{2}{}_qC_m({}_qC_m-1)$, since the only determinants which occur twice are those symmetrical about the diagonal of the symmetrical part.

We have to find the number of conditions involved in the equation

$$\| a_{qp} \|_m = 0$$

where $a_{\nu\mu}=a_{\mu\nu}$.

4. The number of conditions involved in the equation

$$\| a_{mp} \|_m = 0$$

is evidently the same as if all the elements were different, *i.e.*, $p - m + 1$.

5. Consider next the array

$$\| a_{qq} \|_m .$$

The number of distinct determinants of the mth order is $\frac{1}{2}{}_qC_m({}_qC_m + 1)$. We can arrange these in the form of a symmetrical determinant of order ${}_qC_m$, such that all the determinants in any row or column are formed out of the same rows and columns respectively of the array.

Now, taking any row of this determinant, put $q - m + 1$ of the determinant elements equal to zero; it follows, by 4, that the remaining determinants of the row vanish, and hence also all the determinants in the corresponding column. Next, in any other row put $q - m$ of the determinants equal to zero. This row has now $q - m + 1$ elements zero, hence the remaining elements vanish, as also all the elements in the corresponding column. Continuing this process with $q - m + 1$ rows we have all the determinants vanishing. Hence the number of independent conditions in the vanishing of the above array is

$$(q - m + 1) + (q - m) + \ldots + 2 + 1 = \tfrac{1}{2}(q - m + 1)(q - m + 2).$$

6. Now consider the original array of $pq - \frac{1}{2}q(q - 1)$ elements. The number of determinants of the mth order which can be formed out of it is ${}_pC_m \cdot {}_qC_m - \frac{1}{2}{}_qC_m({}^qC_m - 1)$. These can be taken as the elements of a symmetrical array ${}_pC_m$ by ${}_qC_m$.

Then, just as in 5, we see that by making

$$(p - m + 1) + (p - m) + \ldots + (p - q + 1)$$

of the determinants vanish the remaining ones will also vanish. Hence *the number of independent conditions in the equation*

$$\| a_{qp} \|_m = 0$$

where $a_{\nu\mu} = a_{\mu\nu}$ *and* $p > q$ *is*

$$\tfrac{1}{2}(q - m + 1)(2p - q - m + 2) = (p - m + 1)(q - m + 1) - \tfrac{1}{2}(q - m + 1)(q - m).$$

All these results can be conveniently summarised as follows :—

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If $f(p, q)$ is the number of different elements of an array, whether symmetrical or with all its elements different, the number of determinants of the mth order is $f({}_pC_m, {}_qC_m)$ and the number of independent conditions in the vanishing of all these determinants is $f(p-m+1,\ q-m+1)$. If the elements are all different, $f(p, q)=pq$; if the array is symmetrical $f(p, q)=pq-\frac{1}{2}q(q-1)$.

7. *Examples*: Given the quadric locus in space of n dimensions

$$\left(\begin{array}{ccc} a_{11}, & \ldots, & a_{1,n+1} \\ \ldots & \ldots & \ldots \\ a_{1,n+1}, & \ldots, & a_{n+1,n+1} \end{array}\right)\!\!\left(\!\!\!\!\!\!\!\!\quad\right. x_1, x_2, \ldots x_n, 1)^2=0,$$

the conditions that it breaks up into two $(n-1)$-dimensional homaloids are

$$\| a_{n+1,n+1} \|_3=0,$$

i.e., $\frac{1}{2}n(n-1)$ conditions.

If the homaloids are parallel, the conditions are

$$\left\| \begin{array}{ccc} a_{11}, & \ldots\ldots, & a_{1,n+1} \\ \ldots & \ldots & \ldots \\ a_{1n}, & \ldots\ldots, & a_{n,n+1} \end{array} \right\|_2=0,$$

i.e., $\frac{1}{2}(n-1)(n+2)$ conditions.

If they are coincident, the conditions are

$$\| a_{n+1,n+1} \|_2=0,$$

i.e., $\frac{1}{2}n(n+1)$ conditions.

The conditions that the locus is a cylinder, whose base is a quadric locus of $n-2$ dimensions, are

$$\left\| \begin{array}{ccc} a_{11}, & \ldots\ldots, & a_{1,n+1} \\ \ldots & \ldots & \ldots \\ a_{1n}, & \ldots\ldots, & a_{n,n+1} \end{array} \right\|_n=0,$$

i.e., 2 conditions, etc.

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