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Quasi-copure Submodules

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Abstract. All rings are commutative with identity, and all modules are unital. In this paper we introduce the concept of a quasi-copure submodule of a multiplication R-module M and will give some results about it. We give some properties of the tensor product of finitely generated faithful multiplication modules.

1 Introduction

Let R be a commutative ring with identity and let M be a unitary R-module. We will show that the set of quasi-copure submodules of multiplication modules on arithmetical rings is a lattice. An R-module M is called a *multiplication module* if for every submodule N of M, there exists an ideal I of R such that $N = IM = [N:M]M$ (see [6, 7, 11]). An R-module M is called a *cancellation module* if $IM = JM$ for some ideals I and J of R implies $I = J$. Equivalently, $[IM:M] = I$ for all ideals I of R. If M is a finitely generated faithful multiplication R-module, then M is a cancellation module (see [11, Corollary to Theorem 9]), from which one can easily verify that $[IN:M] = I[N:M]$ for all ideals I of R and all submodules N of M.

A ring R is said to be an *arithmetical ring* if, for all ideals I, J, and K of R, we have $I+(J\cap K) = (I+J)\cap(I+K)$. Obviously, Prüfer domains and, in particular, Dedekind domains are arithmetical. A module M is called *distributive* if one of the following two equivalent conditions holds:

(i) $N \cap (K + L) = (N \cap K) + (N \cap L)$ for all submodules N, L, K of M;
(ii) $N + (K \cap L) = (N + K) \cap (N + L)$ for all submodules N, L, K of M.

For any submodule N of an R-module M, we define $V(N)$ to be the set of all prime submodules of M containing N. For any family of submodules N_λ $(\lambda \in \Lambda)$ of M, $\bigcap_{\lambda\in\Lambda} V(N_\lambda) = V(\sum_{\lambda\in\Lambda} N_\lambda)$. The M-radical of a submodule N of an R-module M is the intersection of all prime submodules of M containing N, *i.e.*, $\operatorname{rad}(N) = \bigcap V(N)$. Of course, $V(M)$ is just the empty set and $V(0) = \operatorname{Spec}(M)$. Every finitely generated multiplication module on an arithmetical ring is distributive. By [5], a submodule N of M is called *copure* if for each ideal I of R, $[N:_M I] = N + [0:_M I]$. An R-module M is called *fully copure* if every submodule N of M is copure. We will denote the set of all copure prime submodules of M containing N by $CV(N)$. We will show that for submodules N and K of M, $CV(N) \cap CV(K) = CV(N + K)$. Moreover, if M is a multiplication module on an arithmetical ring R, then the intersection of a

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finite collection of copure submodules of M is also copure. If M is a finitely generated faithful multiplication module, then $CV(N) \cap CV(K) = CV(NK)$.

A submodule N of M is called a *pure submodule* in M if $IN = N \cap IM$ for every ideal I of R. Hence, an ideal I of a ring R is pure if for every ideal J of R, $JI = J \cap I$. Consequently, if I is pure, then $J = JI$ for every ideal $J \subseteq I$.

Let R be a domain, K the field of fractions of R, and M a torsion free R-module; then a nonzero ideal I of R is said to be *invertible* if $II^{-1} = R$, where $I^{-1} = \{x \in K : xI \subseteq R\}$. The associated ideal $\theta(M) = \sum_{m \in M}[Rm : M]$ and the trace ideal $\operatorname{Tr}(M) = \sum_{f \in \operatorname{Hom}(M,R)} f(M)$ of a module M play analogous but distinct roles in the study of multiplication and projective modules respectively.

If M is projective, then $M = \operatorname{Tr}(M)M$, $\operatorname{ann}(M) = \operatorname{ann}(\operatorname{Tr}(M))$, and $\operatorname{Tr}(M)$ is a pure ideal of R (see [8, Proposition 3.30]). In particular, if M is a finitely generated faithful multiplication R-module (hence projective), then pure ideals are flat, and hence $\operatorname{Tr}(M)$ is flat. Let M be an R-module and N a submodule of M; then $\Gamma(N) = [N : M] \operatorname{Tr}(M)$. Obviously, $\Gamma(M) = \operatorname{Tr}(M)$. It is shown in [4, Theorem 3] that if N is a submodule of a faithful multiplication or locally cyclic projective module M, then $\operatorname{Tr}(\operatorname{rad} N) = \sqrt{\Gamma(N)} = \Gamma(\operatorname{rad} N)$.

2 Preliminary Notes

Definition 2.1 Let N be a submodule of an R-module M. We will denote the set of all copure prime submodules of M containing N by $CV(N)$:

$$CV(N) = \{P \in V(N) : P \text{ is copure.}\}$$

Definition 2.2 A submodule N of M is called *quasi-copure* (or *weak-copure*) if every proper prime submodule P containing N is a copure submodule of M. Equivalently, if $V(N) = CV(N)$, then N is a quasi-copure submodule of M.

Example 2.3 We consider $M = \mathbb{Z}_8 \oplus \mathbb{Z}_6$ as a $\mathbb{Z}$-module and $N = \langle \overline{2} \rangle \oplus \langle \overline{3} \rangle$ as a submodule of M. We show that N is not a copure submodule of M and also that $L = \mathbb{Z}_8 \oplus \langle \overline{3} \rangle$ and $K = \langle \overline{2} \rangle \oplus \mathbb{Z}_6$ are proper prime submodules of M contained N, where both are copure submodules of M; therefore, N is a quasi-copure submodule of M:

$$[N :_M 2\mathbb{Z}] = \{ (\overline{m}, \overline{n}) \in \mathbb{Z}_8 \oplus \mathbb{Z}_6 \mid 2\mathbb{Z}(\overline{m}, \overline{n}) \subseteq \langle \overline{2} \rangle \oplus \langle \overline{3} \rangle \} = \mathbb{Z}_8 \oplus \langle \overline{3} \rangle$$

$$\begin{aligned} N + [\{\overline{0}\} \oplus \{\overline{0}\} :_M 2\mathbb{Z}] &= \langle \overline{2} \rangle \oplus \langle \overline{3} \rangle + \{ (\overline{m}, \overline{n}) \in \mathbb{Z}_8 \oplus \mathbb{Z}_6 \mid 2\mathbb{Z}(\overline{m}, \overline{n}) \subseteq \langle \overline{0} \rangle \oplus \langle \overline{0} \rangle \} \\ &= \langle \overline{2} \rangle \oplus \langle \overline{3} \rangle + \langle \overline{4} \rangle \oplus \langle \overline{3} \rangle = \langle \overline{2} \rangle \oplus \langle \overline{3} \rangle. \end{aligned}$$

Therefore, N is not a copure submodule of M. We know that $L = \mathbb{Z}_8 \oplus \langle \overline{3} \rangle$ and $K = \langle \overline{2} \rangle \oplus \mathbb{Z}_6$ are proper prime submodules of M contained N.

Case 1: If $k = p > 3$ is a prime number, then

$$[L :_M p\mathbb{Z}] = \{ (\overline{m}, \overline{n}) \in \mathbb{Z}_8 \oplus \mathbb{Z}_6 \mid p\mathbb{Z}(\overline{m}, \overline{n}) \subseteq \mathbb{Z}_8 \oplus \langle \overline{3} \rangle \} = \mathbb{Z}_8 \oplus \langle \overline{3} \rangle$$

$$\begin{aligned} L + [\{\overline{0}\} \oplus \{\overline{0}\} :_M p\mathbb{Z}] &= \mathbb{Z}_8 \oplus \langle \overline{3} \rangle + \{ (\overline{m}, \overline{n}) \in \mathbb{Z}_8 \oplus \mathbb{Z}_6 \mid p\mathbb{Z}(\overline{m}, \overline{n}) \subseteq \langle \overline{0} \rangle \oplus \langle \overline{0} \rangle \} \\ &= \mathbb{Z}_8 \oplus \langle \overline{3} \rangle + \langle \overline{0} \rangle \oplus \langle \overline{0} \rangle = \mathbb{Z}_8 \oplus \langle \overline{3} \rangle. \end{aligned}$$

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Case 2: Otherwise, we have that

$$
\begin{aligned}
[L :_M 2\mathbb{Z}] &= \{ (\overline{m}, \overline{n}) \in \mathbb{Z}_8 \oplus \mathbb{Z}_6 \mid 2\mathbb{Z}(\overline{m}, \overline{n}) \subseteq \mathbb{Z}_8 \oplus \langle \overline{3} \rangle \} = \mathbb{Z}_8 \oplus \langle \overline{3} \rangle \\
L + [\{\overline{0}\} \oplus \{\overline{0}\} :_M 2\mathbb{Z}] &= \mathbb{Z}_8 \oplus \langle \overline{3} \rangle + \{ (\overline{m}, \overline{n}) \in_{\mathbb{Z}} 8 \oplus \mathbb{Z}_6 \mid 2\mathbb{Z}(\overline{m}, \overline{n}) \subseteq \langle \overline{0} \rangle \oplus \langle \overline{0} \rangle \} \\
&= \mathbb{Z}_8 \oplus \langle \overline{3} \rangle + \langle \overline{4} \rangle \oplus \langle \overline{3} \rangle = \mathbb{Z}_8 \oplus \langle \overline{3} \rangle \\
[L :_M 3\mathbb{Z}] &= \{ (\overline{m}, \overline{n}) \in \mathbb{Z}_8 \oplus \mathbb{Z}_6 \mid 3\mathbb{Z}(\overline{m}, \overline{n}) \subseteq \mathbb{Z}_8 \oplus \langle \overline{3} \rangle \} = \mathbb{Z}_8 \oplus \mathbb{Z}_6 \\
L + [\{\overline{0}\} \oplus \{\overline{0}\} :_M 3\mathbb{Z}] &= \mathbb{Z}_8 \oplus \langle \overline{3} \rangle + \{ (\overline{m}, \overline{n}) \in \mathbb{Z}_8 \oplus \mathbb{Z}_6 \mid 3\mathbb{Z}(\overline{m}, \overline{n}) \subseteq \langle \overline{0} \rangle \oplus \langle \overline{0} \rangle \} \\
&= \mathbb{Z}_8 \oplus \langle \overline{3} \rangle + \langle \overline{0} \rangle \oplus \langle \overline{2} \rangle = \mathbb{Z}_8 \oplus \mathbb{Z}_6 \\
[L :_M 4\mathbb{Z}] &= \{ (\overline{m}, \overline{n}) \in \mathbb{Z}_8 \oplus \mathbb{Z}_6 \mid 4\mathbb{Z}(\overline{m}, \overline{n}) \subseteq \mathbb{Z}_8 \oplus \langle \overline{3} \rangle \} = \mathbb{Z}_8 \oplus \langle \overline{3} \rangle \\
L + [\{\overline{0}\} \oplus \{\overline{0}\} :_M 4\mathbb{Z}] &= \mathbb{Z}_8 \oplus \langle \overline{3} \rangle + \{ (\overline{m}, \overline{n}) \in \mathbb{Z}_8 \oplus \mathbb{Z}_6 \mid 4\mathbb{Z}(\overline{m}, \overline{n}) \subseteq \langle \overline{0} \rangle \oplus \langle \overline{0} \rangle \} \\
&= \mathbb{Z}_8 \oplus \langle \overline{3} \rangle + \langle \overline{2} \rangle \oplus \langle \overline{3} \rangle = \mathbb{Z}_8 \oplus \langle \overline{3} \rangle.
\end{aligned}
$$

Definition 2.4 Let R be a ring and M an R-module. An ideal I of R is called an *M-cancellation module* (resp., *M-weak cancellation module*) if for all submodules K and N of M, $IK = IN$ implies $K = N$ (resp. $K + [0 :_M I] = N + [0 :_M I]$). Equivalently, we have $[IN :_M I] = N$ (resp. $[IN :_M I] = N + [0 :_M I]$) for all submodules N of M (see [1]).

3 Main Results

Definition 3.1 Let M be a multiplication R-module and let $N = IM$ and $K = JM$ be submodules of M. The product of N and K is denoted by NK and is defined by IJM. Clearly, NK is a submodule of M and contained in $N \cap K$.

Lemma 3.2 *Let M be a multiplication R-module.*

(i) *If M be finitely generated faithful, then M is a cancellation module.*
(ii) *Every proper submodule of M is contained in a maximal submodule of M and P is a maximal submodule of M if and only if there exists a maximal ideal m of R such that $P = mM \neq M$.*

Proof (i) By [11, Corollary 1 to Theorem 9], M is a cancellation module, and therefore

$$IN = [IN : M]M = I[N : M]M \Rightarrow [IN : M] = I[N : M]$$

for all ideals I of R and all submodules N of M.

(ii) [7, Theorem 2.5] ∎

Theorem 3.3 *Let M be an R-module and let N and K be submodules of M.*

(i) *If $N \subseteq L \subset M$ and N is quasi-copure, then L is also quasi-copure. In particular, if one of the N or K are quasi-copure submodules, then $N + K$ is also a quasi-copure submodule of M.*
(ii) *Let M be a multiplication R-module on an arithmetical ring R. If N and K are copure submodules, then $N \cap K$ is also a copure submodule of M. Moreover, if $V(N)$ is a finite set and N quasi-copure, then* $\mathrm{rad}(N)$ *is copure.*

(iii) *If M is a multiplication module and N and K are quasi-copure submodules of M, then NK is also a quasi-copure submodule of M. Therefore, $CV(NK) = CV(N) \cap CV(K)$.*

Proof (i) Let $P \in CV(N)$, then $P \supset L \supseteq N$, since N is quasi-copure, hence P is copure. Therefore, $P \in CV(L)$. For the second part we set $L = N + K$, which contains N and K.

(ii) Every finitely generated multiplication module M on an arithmetical ring R is a distributive module. Since N and K are copure submodules of M, hence for every ideal I of R,

$$[N \cap K :_M I] = [N :_M I] \cap [K :_M I] = \big(N + [0 :_M I]\big) \cap \big(K + [0 :_M I]\big)$$
$$= N \cap K + [0 :_M I].$$

Therefore, $N \cap K$ is a copure submodule of M.

Since N is quasi-copure by definition, each $P \in V(N)$ is copure, and therefore $\operatorname{rad}(N) = \bigcap_{P \in V(N)} P$ is copure.

(iii) Let $P \in CV(N) \cap CV(K)$ and $P \in V(NK)$. By [7, Corollary 2.11], there exists a prime ideal $p \supseteq \operatorname{ann}(M)$, where $P = pM$ and $[P : M] = [pM : M] = p$. Let $N = IM$ and $K = JM$ for some ideals I and J of R; then $NK = IJM \subset pM$. Since M is a finitely generated faithful multiplication module, it is cancellation module [11, Corollary 1 to Theorem 9], hence $IJ \subset p$. Therefore, $I \subset p$ or $J \subset p$, and this implies that $N \subset P$ or $K \subset P$, respectively. In each of those two cases, P is copure, and hence $P \in CV(NK)$. It follows that NK is a quasi-copure submodule of M. Conversely, let $P \in CV(NK)$; then $P \supseteq NK$ and by [7, Theorem 3.16 and Corollary 3.17], $P \supseteq N$ or $P \supseteq K$. It follows that $P \in CV(N) \cup CV(K) \supseteq CV(N) \cap CV(K)$. ■

Corollary 3.4 *Let M be a nonzero multiplication R-module.*

(i) *If M is a faithful prime and N a copure submodule of M, then $N = IN$ for every nonzero ideal I of R.*
(ii) *If M be finitely generated and Q a quasi-copure primary submodule of M, then $\operatorname{rad}(Q)$ is a copure submodule of M.*
(iii) *For every two copure submodules N_1, N_2 of M, if $IN_1 = IN_2$, then $N_1 = N_2$.*
(iv) *If M is Noetherian and R an arithmetical ring, then for quasi-copure submodules N and K of M, $\operatorname{rad}(N \cap K)$ is copure.*

Proof (i) M is faithful, $\operatorname{ann}_R(M) = 0$, and M is prime, hence for each submodule N of M, $\operatorname{ann}_R(N) = \operatorname{ann}_R(M) = 0$; then $\operatorname{ann}_M(N) = \operatorname{ann}_R(N)M = 0$. Now M is a multiplication R-module therefore for each ideal I of R and every submodule L of M, $[L :_M I] = [L :_M IM]$. In particular, $\operatorname{ann}_M(I) = [0 :_M I] = \operatorname{ann}_M(IM) = 0$. Since N is copure, $[N :_M I] = N + [0 :_M I] = N$. It follows that $N = IN$.

(ii) Since Q is primary submodule, $\sqrt{[Q : M]}$ is a prime ideal containing $\operatorname{ann}(M)$. Therefore, by [9, Lemma 3 and Theorem 4], $\operatorname{rad}(Q) = \sqrt{[Q : M]}M$ is a prime submodule of M and Q is quasi-copure, hence $\operatorname{rad}(Q)$ is copure.

(iii) The proof follows from (i) immediately.

(iv) Since the radical of any submodule of a Noetherian multiplication module is a finite intersection of prime submodules, by Theorem 3.3(ii), $\operatorname{rad}(N)$ and $\operatorname{rad}(K)$

are copure submeodules of M. By [7, Theorems 1.6 and 2.12] it follows that $\operatorname{rad}(N) \cap \operatorname{rad}(K) = \operatorname{rad}(N \cap K)$, and by Theorem 3.3(ii) $\operatorname{rad}(N \cap K)$ is also copure. ■

Theorem 3.5 *Let (R, m) be a Noetherian local ring and M a cancellation multiplication R-module. If P is a copure maximal submodule of M, then for every ideal I of R, $\operatorname{ann}_M(I) \subseteq P$. Moreover, for every submodule N of M, $\operatorname{ann}_M(I) = \operatorname{ann}_M(N) \subseteq P$.*

Proof Since P is a copure submodule of M, for every ideal I of R,

$$P \subseteq [P :_M I] = P + [0 :_M I] \subseteq M$$

Therefore, by maximality of P, $P = P + [0 :_M I]$ or $P + [0 :_M I] = M$. Let $P + [0 :_M I] = M$; then $IP + I[0 :_M I] = IM$ hence $IP = IM$. Since M is cancellation, hence $[IN : M] = I[N : M]$ for all ideals I of R and all submodules N of M, and also $[P : M] = m$, therefore

$$Im = I[P : M] = [IP : M] = [IM : M] = I.$$

By Nakayama's lemma, since I is a finitely generated R-module and $m = Jac(R)$ and $I = mI$, we have $I = 0$; therefore, $P = P + [0 :_M I] = P + M = M$, which is a contradiction. It follows that $P = P + [0 :_M I]$ and so $\operatorname{ann}_M(I) \subseteq P$.

Let $N = IM$ be a submodule of M. Since M is a multiplication R-module, for every ideal I of R and submodule K of M, $[K :_M I] = [K :_M N] = [K :_M IM]$. In particular, for $K = 0$, $\operatorname{ann}_M(I) = \operatorname{ann}_M(IM) = \operatorname{ann}_M(N) \subseteq P$. ■

Corollary 3.6 *Let M be an R-module and I an M-cancellation ideal of R. If P is a copure maximal submodule of M, then $\operatorname{ann}_M(I) \subseteq P$.*

Proof By the proof of Theorem 3.5 we have $IP = IM$, and since I is an M-cancellation ideal of R, $P = M$, which is a contradiction. Then $P = P + [0 :_M I]$, and therefore $\operatorname{ann}_M(I) \subseteq P$. Therefore,

$$\operatorname{ann}_M(I) \subseteq \bigcap_{P = \text{maximal copure}} P.$$

Moreover, if $\{I_\lambda\}_{\lambda \in \Lambda}$ is a collection of M-cancellation ideals of R, then

$$\bigcap_{\lambda \in \Lambda} \operatorname{ann}_M(I_\lambda) = \sum_{\lambda \in \Lambda} \operatorname{ann}_M(I_\lambda) \subseteq \bigcap_{P = \text{maximal copure}} P.$$ ■

Theorem 3.7 *Let M_1 and M_2 be finitely generated faithful multiplication R-modules. The following hold.*

(i) *If K and N are invertible in M_1 and M_2 respectively, or if one of K or N is flat, then $\Gamma(K \otimes N) \cong \Gamma(K)\Gamma(N)$. Moreover $\Gamma(K \otimes M_2) \cong \Gamma(K)\operatorname{Tr}(M_2)$.*
(ii) *If M_1 and M_2 are free R-modules, then $\operatorname{Tr}(\operatorname{rad}(K \otimes M_2)) \cong \Gamma(\operatorname{rad} K)\operatorname{Tr}(M_2)$.*

Proof (i) By [2, Theorem 2] $M_1 \otimes M_2$ is a finitely generated faithful multiplication R-module. If K is a nonzero submodule of multiplication R-module M_1 such that $[K : M_1]$ is an invertible ideal of R, then K is invertible in M_1. The converse is true if we assume further that M_1 is finitely generated and faithful (see [10, Lemmas 3.2 and 3.3]). Therefore, $[K : M_1]$ and $[N : M_2]$ are invertible ideals of R, and $\operatorname{Tr}(M_1)$ and $\operatorname{Tr}(M_2)$

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are flat ideals, hence $\mathrm{Tr}(M_1)\,\mathrm{Tr}(M_2) \cong \mathrm{Tr}(M_1)\otimes \mathrm{Tr}(M_2) \cong \mathrm{Tr}(M_1\otimes M_2)$. Also, since $M_1 \otimes M_2$ is projective, $\mathrm{ann}(M_1 \otimes M_2) = \mathrm{ann}(\mathrm{Tr}(M_1 \otimes M_2)) = 0$. It follows that

$$[K \otimes N : M_1 \otimes M_2] \cong [K : M_1] \otimes [N : M_2] \cong [K : M_1][N : M_2].$$

Therefore,

$$\begin{aligned}\Gamma(K \otimes N) &= [K \otimes N : M_1 \otimes M_2]\,\mathrm{Tr}(M_1 \otimes M_2)\\ &\cong [K : M_1][N : M_2]\,\mathrm{Tr}(M_1) \otimes \mathrm{Tr}(M_2)\\ &\cong [K : M_1][N : M_2]\,\mathrm{Tr}(M_1)\,\mathrm{Tr}(M_2)\\ &= [K : M_1]\,\mathrm{Tr}(M_1)[N : M_2]\,\mathrm{Tr}(M_2) = \Gamma(K)\Gamma(N).\end{aligned}$$

Also, if K or N is flat, then $[K : M_1]$ or $[N : M_2]$ is a flat ideal, and hence

$$[K : M_1] \otimes [N : M_2] \cong [K : M_1][N : M_2],$$

and the result is true. Since $M_1 \otimes M_2$ is a faithful multiplication R-module,

$$K \otimes N = \Gamma(K \otimes N)(M_1 \otimes M_2) \cong \Gamma(K)\Gamma(N)(M_1 \otimes M_2) \cong \Gamma(K \otimes N)(M_1 \otimes M_2).$$

(ii) Since $M_1 \otimes M_2$ is a faithful multiplication free R-module, therefore for some ideal I of R, $K = IM_1$ and then

$$\begin{aligned}\mathrm{rad}(K \otimes M_2) &= \mathrm{rad}(IM_1 \otimes M_2) \cong \mathrm{rad}(I(M_1 \otimes M_2)) = \sqrt{I}(M_1 \otimes M_2)\\ &\cong \sqrt{I}M_1 \otimes M_2 = (\mathrm{rad}\,K) \otimes M_2\end{aligned}$$

By [4, Theorem 3], and (i) and also since $\mathrm{Tr}(M_2)$ is flat, it follows that

$$\begin{aligned}\mathrm{Tr}(\mathrm{rad}(K \otimes M_2)) &= \mathrm{Tr}(\mathrm{rad}\,K \otimes M_2) \cong \mathrm{Tr}(\mathrm{rad}\,K) \otimes \mathrm{Tr}(M_2)\\ &\cong \mathrm{Tr}(\mathrm{rad}\,K)\,\mathrm{Tr}(M_2) = \sqrt{\Gamma(K)}\,\mathrm{Tr}(M_2)\\ &= \Gamma(\mathrm{rad}\,K)\,\mathrm{Tr}(M_2) = \Gamma(\mathrm{rad}\,K)\Gamma(M_2).\end{aligned}$$ ■

Theorem 3.8 *Let M be a finitely generated faithful multiplication R-module and let N_λ $(\lambda \in \Lambda)$ be a finite collection of submodules of M, where for all $\lambda \neq \mu$, $N_\lambda + N_\mu$ is a multiplication module.*

(i) *If $N = \bigcap_{\lambda\in\Lambda} N_\lambda$, then for every pure ideal I of R, $\Gamma(IN) = I\Gamma(N) = I \cap \Gamma(N)$.*
(ii) *If K is a pure idempotent submodule of M, then $K = \Gamma(K)K$.*

Proof (i) By [3, Theorem 1], $IN = \bigcap_{\lambda\in\Lambda} IN_\lambda$ and by [4, Lemma 2],

$$\begin{aligned}\Gamma(IN) &= \Gamma\Big(I \bigcap_{\lambda\in\Lambda} N_\lambda\Big) = \Gamma\Big(\bigcap_{\lambda\in\Lambda} IN_\lambda\Big) = \bigcap_{\lambda\in\Lambda} \Gamma(IN_\lambda)\\ &= \bigcap_{\lambda\in\Lambda} [IN_\lambda : M]\,\mathrm{Tr}(M) = \bigcap_{\lambda\in\Lambda} I[N_\lambda : M]\,\mathrm{Tr}(M)\\ &= \bigcap_{\lambda\in\Lambda} I\Gamma(N_\lambda) = I \bigcap_{\lambda\in\Lambda} \Gamma(N_\lambda) = I\Gamma(N) = I \cap \Gamma(N).\end{aligned}$$

(ii) By [11, Theorem 11], if M is a finitely generated multiplication R-module such that $\mathrm{ann}(M) = Re$ for some idempotent e, then M is projective, and hence, finitely

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generated faithful multiplication modules are projective and $M = \mathrm{Tr}(M)M$. Since K is pure and idempotent,

$$\mathrm{Tr}(M)K = K \cap \mathrm{Tr}(M)M = K \cap M = K,$$
$$K = [K:M]K \Rightarrow \mathrm{Tr}(M)K = \mathrm{Tr}(M)[K:M]K = \Gamma(K)K.$$

It follows that $K = \Gamma(K)K = K \cap \Gamma(K)M$. ∎

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