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EXACT SOLUTION TO THE CHOW–ROBBINS GAME FOR ALMOST ALL *n*, BY USING A CATALAN TRIANGLE

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Abstract

The payoff in the Chow–Robbins coin-tossing game is the proportion of heads when you stop. Stopping to maximize expectation was addressed by Chow and Robbins (1965), who proved there exist integers k_n such that it is optimal to stop at n tosses when heads minus tails is k_n. Finding k_n was unsolved except for finitely many cases by computer. We prove an $o(n^{-1/4})$ estimate of the stopping boundary of Dvoretsky (1967), which then proves $k_n = \left\lceil \alpha\sqrt{n} \; - 1/2 \; + \frac{(-2\zeta(-1/2))\sqrt{\alpha}}{\sqrt{\pi}} n^{-1/4} \right\rceil$ except for n in a set of density asymptotic to 0, at a power law rate. Here, α is the Shepp–Walker constant from the Brownian motion analog, and ζ is Riemann's zeta function. An $n^{-1/4}$ dependence was conjectured by Christensen and Fischer (2022). Our proof uses moments involving Catalan and Shapiro Catalan triangle numbers which appear in a tree resulting from backward induction, and a generalized backward induction principle. It was motivated by an idea of Häggström and Wästlund (2013) to use backward induction of upper and lower Value bounds from a horizon, which they used numerically to settle a few cases. Christensen and Fischer, with much better bounds, settled many more cases. We use Skorohod's embedding to get simple upper and lower bounds from the Brownian analog; our upper bound is the one found by Christensen and Fischer in another way. We use them first for yet many more examples and a conjecture, then algebraically in the tree, with feedback to get much sharper Value bounds near the border, and analytic results. Also, we give a formula that gives the exact optimal stop rule for all n up to about a third of a billion; it uses the analytic result plus terms arrived at empirically.

Keywords: Chow–Robbins game; S_n/n problem; random walk; Brownian motion; Wald identity; Catalan number; Shapiro Catalan triangle; backward induction; optimal stopping

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Secondary 60J65; 05A10; 05A19

1. Introduction

1.1. Background

The Chow–Robbins game (also known as the S_n/n problem) is a classical optimal stopping problem that can be stated in the form of a simple coin-tossing game, for which the payoff is

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the proportion of heads when you stop. The goal is to maximize your expected payoff. It seems to have been first posed by Breiman [1] in 1964, but was first analyzed by Chow and Robbins [4] in 1965. Let $S_n = \sum_{i=1}^{n} X_i$, where the X_i are independent ± 1-valued mean-zero random variables, representing heads or tails in tossing a fair coin; this is a symmetric random walk. The object is to find a stopping time τ which is optimal in the sense that $E\left[\frac{S_\tau}{\tau}\right] = \sup_T E\left[\frac{S_T}{T}\right]$, the sup taken over stopping times (positive integer-valued random variables which do not anticipate the future, assumed almost surely finite). Chow and Robbins proved the existence of integers $0 < k_1 \leq k_2 \leq \ldots$ such that the stopping time $\tau = \inf\{n{:}S_n \geq k_n\}$ is optimal; no formula was given for the k_n.

Next, in 1967, Dvoretzky [6] found a representation of an optimal stopping time in terms of a more general payoff, or Value, function. Define the Value starting from initial 'position' (u,n) under stopping time T as

$$V(u, n, T) = E\left[\frac{u + S_T}{n + T}\right],$$

where u is a real number, n is a non-negative integer, and T is a stopping time for the symmetric random walk; and define

$$V(u, n) = \sup_T V(u, n, T).$$

In fact, Dvoretzky allowed the X_i to be more generally independent and identically distributed (i.i.d.) of mean zero and finite variance, but our paper is only concerned with the coin-tossing case. We emphasize that u is allowed to be real, not just integers, unlike what Chow and Robbins considered in their proofs. This turns out to be quite significant. Dvoretzky proved that V is a continuous function of the first argument u, and the equation $V(\beta_n, n) = \beta_n/n$ uniquely defines a strictly increasing sequence of positive real numbers $0 < \beta_1 < \beta_2 < \ldots$ such that the stop rule $\tau(u, n) = \min\left\{j : u + S_j \geq \beta_{n+j}\right\}$ is optimal in the sense that $V(u, n) = V(u, n, \tau(u, n))$. In particular, $\tau(0, 0) = \min\left\{j : S_j \geq \beta_j\right\}$ is optimal for the Chow–Robbins game. For our development below, we work with the real numbers β_n rather than the integers k_n because we can approximate them by approximating the Value function in the equation they satisfy; we can use real analysis. Our approximation of the real numbers β_n will allow us to give an *exact* formula for k_n for all n, except for a set whose density asymptotically approaches zero rapidly (by a power law). Dvoretzky showed that $0.32 < \beta_n/\sqrt{n} < 4.06$ for sufficiently large n, and conjectured that $\beta_n/\sqrt{n}$ approaches a limit.

In 1969, Shepp [13], and independently Walker [14], found a simple exact optimal stop rule for the continuous-time Brownian motion analog, which allowed them to prove Dvoretzky's conjecture. Let $W(t)$ be standard Brownian motion (Wiener process), and following Shepp's notation, define

$$V_W(u, b, T) = E\left[\frac{u + W(T)}{b + T}\right], \quad V_W(u, b) = \sup_T V_W(u, b, T),$$

where u and b are real numbers with $b > 0$. T being a stopping time means it is a non-negative real-valued random variable that does not anticipate the future, and the sup is taken over stopping times for which the expectation exists. Let α be the unique real

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root of $\alpha = (1-\alpha^2)\int_0^\infty \exp\left(\lambda\alpha - \lambda^2/2\right) d\lambda$. Computation gives $\alpha = 0.83992\ldots$. Let $\tau_\alpha = \min\left\{t{:}u + W(t) \geq \alpha\sqrt{b+t}\right\}$. They proved τ_α is the almost surely unique optimal stopping time, so $V_W(u, b) = V_W(u, b, \tau_\alpha)$; and

$$V_W(u, b) = (1-\alpha^2)\int_0^\infty \exp\left(\lambda u - \lambda^2 b/2\right) d\lambda \text{ if } u \leq \alpha\sqrt{b}, \quad \text{else } V_W(u, b) = u/b. \quad (1.1)$$

In other words, starting at time b, it is optimal to stop when you hit the square root boundary $\alpha\sqrt{b+t}$. Using the invariance principle (see, for example, [1, p. 281]), Shepp [13, pp. 1005–1006] used the Brownian motion result to show that the optimal stopping boundary for the random walk game is asymptotic to $\alpha\sqrt{n}$; that is, $\lim_{n\to\infty}\beta_n/\sqrt{n} = \alpha$. But that does not give a way of knowing if it is optimal to stop at any specific position (u,n) in the Chow–Robbins game, when u is an integer. Medina and Zeilberger [10] discuss this distinction, pointing out that, at the time of their article (2009), not even k_8 was known (they refer to it as β_8, but we are adopting the notation of the original papers). They give some numerical data about early positions, and some good insight into the difficulty.

In 2013, Häggström and Wästlund [6] showed, with a clever idea and the help of computer calculations, how to finesse the difficulty discussed in [10], and actually decide in some 'early' positions whether or not stopping is optimal. Let d be the number of heads minus the number of tails after n flips. They expressed their results in terms of the number of heads, but we will give equivalent statements using n, to align with the usual notation. Also they expressed the stop rule in terms of n as a function of d, the reverse of the usual. We found it very useful in our numerical experiments below to also use this reverse formulation, so we describe it here. Using somewhat crude upper and lower bounds for the value at any position, and using backward induction from 'way out' (a horizon), they computed, for d between 1 and 25, numbers $n_s(d)$ and $n_g(d)$, with $n_s(d) < n_g(d)$, such that if $n \leq n_s(d)$ you should stop, and if $n \geq n_g(d)$ you should go on. The idea is that as you work backward from the horizon, those numbers should pull closer together. For d less than 12, and for several more ds between 13 and 25, for their horizon they found $n_g(d) = n_s(d) + 2$ (note that d and n have the same parity), in which case stopping if and only if $n \leq n_s(d)$ is the rigorous optimal stop rule for that d. Shepp's asymptotic value for the stopping rule put in this reverse formulation is $n_s(d) \cong d^2/\alpha^2$. This is not so accurate: if you look at just the few cases where Häggström and Wästlund actually find $n_s(d)$, you can already see that it is off about first order in d, and actually it appears that $n_s(d) \cong (d^2 + d)/\alpha^2$. Solving for d in terms of n, this is consistent with the suggestion by Lai, Yao, and Aitsahlia [9, p. 768], that the stopping boundary for d in terms of n, should be $\beta_n = \alpha\sqrt{n} - 1/2 + o(1)$.

But this limited amount of numerical data is not able to suggest anything more. By having much better upper and lower bounds, it is possible to get *much* more data. Christensen and Fischer [5] (2022) gave much better upper and lower bounds for the optimal stopping value V for the random walk, and used it to numerically settle very many more cases. They found the stop rule for n up to 489 241, which corresponds to d up to about 588. We used the same upper bound that they did (with a different proof), but with a different lower bound, and settled yet again very many more cases than in [5], to attempt to get more numerical insight, as described in Section 1.3. This eventually led to using our bounds to prove the theoretical results which are the subject of this paper.

1.2. Embedding the random walk in Brownian motion, and Value bounds

Our proofs of the upper and lower bounds on V use the classical embedding of the random walk in W using first-exit times, due to A. V. Skorohod (see, for example, [2, p. 293]), which make the results seem rather intuitive. The embedding idea is quite natural: simply sample the Brownian path each time it changes by ± 1, and you get a version of the symmetric random walk. Formally, the properties follow from the strong Markov property. Let $T_0 = 0$, $T_n = \min\{t > T_{n-1} : |W(t) - W(T_{n-1})| = 1\}$, $n = 1, 2, \ldots$. Then $W(T_n)$, $n = 0, 1, \ldots$, has the same distribution as the process S_n, $n = 0, 1, 2, \ldots$, and $(T_n - T_{n-1}, W(T_n) - W(T_{n-1}))$, $n = 1, 2, \ldots$, is an i.i.d. sequence, and $E[T_n] = n$. Since the exit boundaries $W(T_{n-1}) + 1$, $W(T_{n-1}) - 1$ are symmetric about $W(T_{n-1})$ in this case, it can be shown that the sequence $T_1, T_2, \ldots, T_n, \ldots$ is independent from $W(T_1), W(T_2), \ldots, W(T_n), \ldots$.

Lemma 1.1. (Christensen and Fischer [5, Theorem 1, p. 3]) $V(u, b) \leqslant V_W(u, b)$.

Proof. Their proof uses superharmonic functions, in a more general setting. We give a proof using the embedding idea, as a preliminary for using it in our lower bound proof. Let n^* be a stopping time for S_n. This induces a stopping time $T^* = T_{n^*}$ on W. So

$$V_W(u, b) \geq E\left[\frac{u + W(T^*)}{b + T^*}\right] = \sum_{n=0}^{\infty} E\left[\frac{u + W(T_n)}{b + T_n}\,\middle|\, T^* = T_n\right] P(n^* = n)$$
$$= \sum_{n=0}^{\infty} E\left[\frac{u + W(T_n)}{b + n}\,\frac{b + n}{b + T_n}\,\middle|\, T^* = T_n\right] P(n^* = n).$$

But T_n is independent of $W(T_n)$, and is also independent of $I_{T^* = T_n}$ since the latter is a function of $W(T_1), W(T_2), \ldots, W(T_n)$, since T^* is a stopping time. Thus $E\left[\frac{u+W(T^*)}{b+T^*}\right] = \sum_{n=0}^{\infty} E\left[\frac{b+n}{b+T_n}\right] E\left[\frac{u+W(T_n)}{b+n}\,\middle|\, T^* = T_n\right] P(n^* = n)$. By Jensen's inequality, $E\left[\frac{b+n}{b+T_n}\right] \geq \frac{b+n}{E[b+T_n]} = 1$, so

$$V_W(u, b) \geq \sum_{n=0}^{\infty} E\left[\frac{u + W(T_n)}{b + n}\,\middle|\, T^* = T_n\right] P(n^* = n)$$
$$= \sum_{n=0}^{\infty} E\left[\frac{u + S_n}{b + n}\,\middle|\, n^* = n\right] P(n^* = n)$$
$$= E\left[\frac{u + S_{n^*}}{b + n^*}\right] = V(u, b, n^*).$$

□

In retrospect, it is as one would think: the random walk is a just a sampling of the Brownian motion, so naturally it cannot do any better. There is the little matter of different denominators in the payoff, but Jensen's inequality goes the right way for that. For a lower bound, we have the following result.

Lemma 1.2.

$$V(u, b) \geq V_W(u, b)\left(1 - \frac{5}{12b}\left(1 + \frac{1}{\sqrt{b}}\right)\right), \quad b > 1600.$$

We will give a detailed proof of Lemma 1.2 in Appendix A. But the idea for our proof of this lemma is simple enough. For u below the Brownian boundary, run the Brownian motion

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until it hits the nearest integer to $\alpha\sqrt{b+t} \quad - u$. With that stop rule, $E\left[\frac{u+W(T)}{b+T}\right]$ is about the same as $V_W(u, b)$ because we are so close to the boundary; we will quantify this using a modified fundamental Wald identity of Shepp to obtain $E\left[\frac{u+W(T)}{b+T}\right] \geq V_W(u, b)\left(1 - \frac{1}{4b}\left(1 + \frac{1}{\sqrt{b}}\right)\right)$. But $W(T)$ is an integer, so in terms of the embedded random walk process $S_n = W(T_n)$, $E\left[\frac{u+W(T)}{b+T}\right] = \sum_{n=0}^{\infty} E\left[\frac{u+S_n}{b+T_n}\middle| n^* = n\right] P(n^* = n)$. Proceed as in the proof of Lemma 1.1. Jensen's inequality goes the wrong way this time; but knowing the moments of the random time differences of the embedding, we can show that $E\left[\frac{b+n}{b+T_n}\right] \leq \left(1 + \frac{1}{6b}\right)$, so $E\left[\frac{u+W(T)}{b+T}\right] \leq \left(1 + \frac{1}{6b}\right) \sum_{n=0}^{\infty} E\left[\frac{u+S_n}{b+n}\middle| n^* = n\right] P(n^* = n) = \left(1 + \frac{1}{6b}\right) V(u, b)$, implying the lemma.

How good are these bounds? The data show, and we will prove later, that if integer u happens to be just a hair more than $\alpha\sqrt{b} - 1/2$, then $V(u, b) = u/b$, but $V_W(u, b) \cong \left(1 + 0.25b^{-1}\right) u/b$, so the Brownian upper bound overshoots the true value by a relative error of $O(b^{-1})$ at some places, for arbitrarily large b. The same examples give $V(u, b) \cong V_W(u, b)\left(1 - 0.25b^{-1}\right)$, so we cannot expect a lower bound of the form $V(u, b) \cong V_W(u, b)\left(1 - cb^{-1}\right)$ to do better than this. Lemma 1.2 gets lower bound $V_W(u, b)\left(1 - 0.42b^{-1}\right)$. Theorem 5.1 later will give a greatly improved approximation to V when u is near the boundary, showing that V_W is off from the true V by essentially a relative error of $0.25b^{-1}$ at half-integer values below the boundary, and the true V is approximately piecewise linear in between, when the distance below the boundary is not more than about $b^{1/12}$. Theorem 5.1 does not give specific values for the constants (though that could be done with enough pain), and it was not used in our numerical work. Christensen and Fischer also give a lower bound, but our simple formula is convenient for our numerical work, and more importantly, for the later proof of the theoretical main results.

1.3. Numerical exploration and speculation

Using these good upper and lower bounds, we carried out the Häggström–Wästlund method numerically from a horizon of $n = 10^9$, and found $n_g(d) = n_s(d) + 2$ for d up to 7995, and almost all cases up to 20 000, so that the stop rule is settled for those. Stated another way, it settles all cases where $n < 7995^2/\alpha^2 \cong 9 \times 10^7$, and most cases where $n < 20\,000^2/\alpha^2 \cong 5.7 \times 10^8$, over half a billion, going considerably beyond Christensen and Fischer. Our computations did not really take much computer time, and we could have gone a lot further, but we got pleasantly sidetracked by discovering the theoretical arguments of this paper. Having the stop-go boundaries as a function of d rather than n, following Häggström and Wästlund, made the computer algorithm extremely efficient and suitable for dealing with very large numbers. The spreadsheet for the answer has only 20 000 rows, rather than a billion. Figure 1 is a graph of $n_g(d) - \left(d^2 + d\right)/\alpha^2$ for d up to 20 000, from the spreadsheet; for almost all of those cases, and for all cases up to $d = 7995$, $n_g(d) = n_s(d) + 2$. It appears thick since it is oscillating with amplitude about 1 around a square root curve.

We originally speculated that $n_s(d) \cong \left(d^2 + d\right)/\alpha^2 - \pi^{-1/2}\sqrt{d} \quad + \varepsilon$, with ε wiggling around zero with amplitude about 1. But this coefficient of $\sqrt{d}$ turns out to be wrong, by about 1%. Theorem 1.1 will prove that the correct coefficient is $-\pi^{-1/2}\left(-4\zeta(-1/2)/\alpha\right)$. Since $-4\zeta(-1/2)/\alpha = 0.990\ldots$, nature had a laugh at us for jumping to conclusions! Figure 2 is the detail for the 100 points at the large d end of the curve, showing the oscillation.

We now have, from theory, the correct coefficient for the $\sqrt{d}$ term, but there is still something suggested by the numerics that the theory has not yet reached. Assume $\varepsilon = O(1)$. Using

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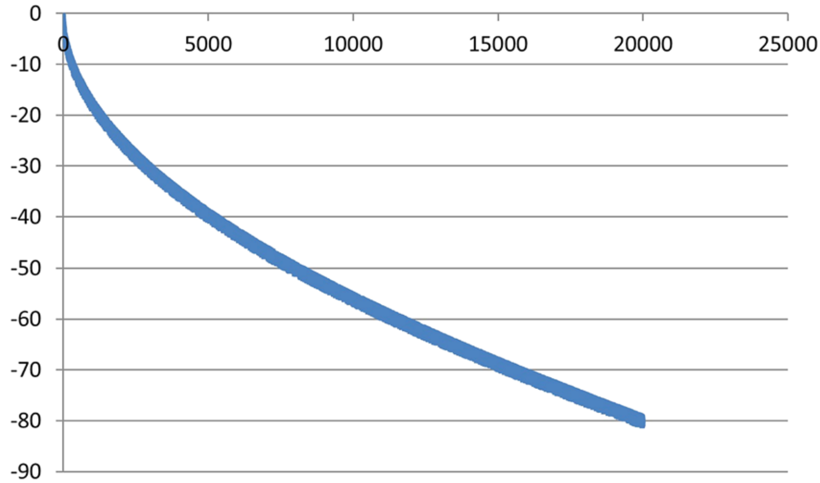

FIGURE 1. $n_g(d) - \left(d^2 + d\right)/\alpha^2$ versus d, $d \le 20\,000$.

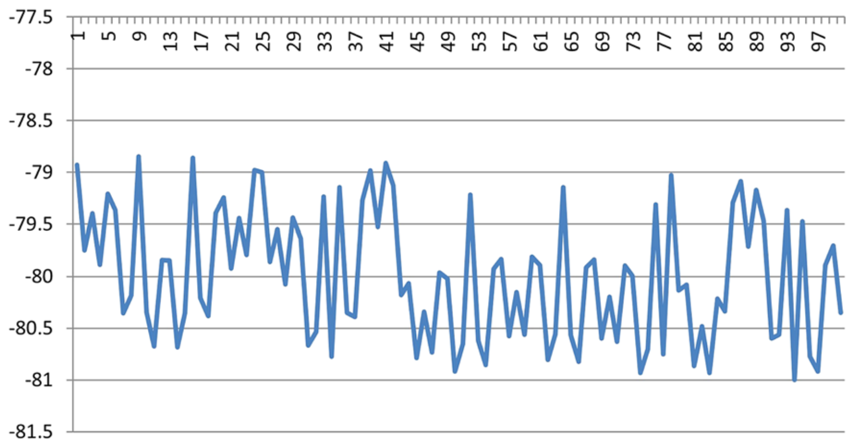

FIGURE 2. Last 100 points in Figure 1; horizontal coordinates shown are $d - 19\,900$.

the binomial expansion, we can solve for d in $n = \left(d^2 + d\right)/\alpha^2 - c\sqrt{d} + \varepsilon$, expressing the boundary in the more usual way with d as a function of n. With a little algebra one arrives at the following conjecture.

$$\text{Conjecture.} \quad \beta_n = \alpha\sqrt{n} - 1/2 + \frac{(-2\zeta(-1/2))\sqrt{\alpha}}{\sqrt{\pi}} n^{-1/4} + O(n^{-1/2})? \qquad (1.2)$$

Theorem 1.1 will prove that the coefficient of $n^{-1/4}$ is correct, but it will only get the error term to $O\left(n^{-7/24}\right)$. There is still theoretical work needed to catch up with the numerical speculation.

Christensen and Fischer [5] had conjectured an $n^{-1/4}$ dependence, based on numerical evidence. They found that for n up to 489 241, the Chow–Robbins boundary is $k_n =$

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$\left\lceil \alpha\sqrt{n} \;-1/2\;+\;\frac{1}{7.9+4.54n^{1/4}}\right\rceil$ except for eight stray ns in that range. Note that $\frac{1}{4.54}$ differs from $\frac{(-2\zeta(-1/2))\sqrt{\alpha}}{\sqrt{\pi}}$ by about 2.5%.

Added after review. From recent numerical work after this paper was already refereed, we discovered a simple formula that gives the stop rule for all cases up to a quite large number. Let

$p(d) = \left(d^2+d\right)/\alpha^2 - \pi^{-1/2}\left(-4\zeta(-1/2)/\alpha\right)\sqrt{d} - 0.064\,998\,6 - 7/(d+95)$. Then

$$n_s(d) = \text{nearest integer to } p(d) \text{ that has the same parity as } d$$

gives the exact optimal stop rule for all d from 2 to 15 363 (which covers all n up to over a third of a billion). We had thought that the oscillatory behavior meant that finding a simple exact formula was unlikely, but now it appears that it could be merely nearest-integer behavior around an analytic asymptotic formula. For now, we have no idea how to get any such formula; the correction terms above were purely *ad hoc*.

1.4. Statement of main theorems

Using the idea of Häggström and Wästlund to use backward induction from a horizon, but proceeding algebraically rather than numerically, we will be led to a tree with weights corresponding to Catalan numbers and Shapiro Catalan triangle numbers, and a generalized backward induction principle. Before starting the development, we will state up front the main theorems that eventually follow from it. The second one is a corollary of the first, and gives a formula for the exact optimal stopping rule for the original Chow–Robbins game for all n, except in a set whose density goes to zero at the rate $O(n^{-7/24})$. The longer the game goes on, the more likely it is that you will be able to stop optimally, and *know* that you did (from the description in Theorem 1.2). Actually, using the method of proof of Theorem 1.2, the estimate $\beta_n = \alpha\sqrt{n} - 1/2 + o(1)$ suggested in [9] already implies an exact stopping rule for all n except for a set of asymptotic density zero, although with no rate implied since there is no quantification of the $o(1)$ term. We thank a referee for pointing that out. As far as we know, it had not been noticed before.

The Riemann zeta function $\zeta(-1/2) = -0.207\,886\ldots$ appears because the analysis in Section 6 involves the asymptotic approximation of the sum of square roots of the first k integers, the generalized harmonic number $H_k^{(-1/2)}$.

Theorem 1.1.

$$\beta_n = \alpha\sqrt{n} \;-1/2\;+\;\frac{(-2\zeta(-1/2))\sqrt{\alpha}}{\sqrt{\pi}}n^{-1/4} + O\left(n^{-7/24}\right).$$

Theorem 1.2.

$$k_n = \left\lceil \alpha\sqrt{n} \;-1/2\;+\;\frac{(-2\zeta(-1/2))\sqrt{\alpha}}{\sqrt{\pi}}n^{-1/4}\right\rceil,$$

except for a set S of integers for which $|S\cap\{1,\ldots,n\}|/n = O(n^{-7/24})$. Specifically, there exists $A>0$ and n_0 such that for $n \geq n_0$, the given formula for k_n holds if

$$\left\lceil \alpha\sqrt{n} \;-1/2\;+\;\frac{(-2\zeta(-1/2))\sqrt{\alpha}}{\sqrt{\pi}}n^{-1/4} - An^{-7/24}\right\rceil = \left\lceil \alpha\sqrt{n} \;-1/2\;+\;\frac{(-2\zeta(-1/2))\sqrt{\alpha}}{\sqrt{\pi}}n^{-1/4} + An^{-7/24}\right\rceil.$$

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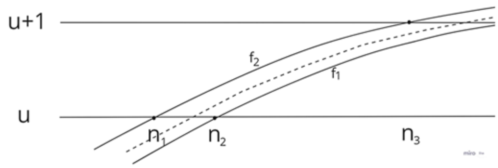

FIGURE 3. Graph of $f_1(n)$ and $f_2(n)$. β_n is somewhere between them, shown dotted.

Proof. Figure 3 shows how Theorem 1.2 follows easily from Theorem 1.1. Let

$$f_1(n) = \alpha\sqrt{n} \; - 1/2 \; + \; \frac{(-2\zeta(-1/2))\sqrt{\alpha}}{\sqrt{\pi}} n^{-1/4} - An^{-7/24},$$
$$f_2(n) = \alpha\sqrt{n} \; - 1/2 \; + \; \frac{(-2\zeta(-1/2))\sqrt{\alpha}}{\sqrt{\pi}} n^{-1/4} + An^{-7/24},$$

with A chosen according to Theorem 1.1 so that $f_1(n) < \beta_n < f_2(n)$ for all $n \geq n_0$. Let u be a positive integer, and let n_1, n_2, n_3 satisfy $f_2(n_1) = u, f_1(n_2) = u, f_2(n_3) = u + 1$, where for this purpose we extend the domain of f_1, f_2 so that n_1, n_2, n_3 are real, not necessarily integers. Assume also that u is large enough so that $n_1 \geq n_0$.

For $n_1 \leq n \leq n_3$, $\lceil f_1(n) \rceil = \lceil f_2(n) \rceil$ implies $n_2 \leq n \leq n_3$. For integers n such that $n_2 \leq n \leq n_3$, we have $u < \beta_n < u + 1$, so it is optimal to stop at $u + 1$, and for those integers $k_n = u + 1 = \left\lceil \alpha\sqrt{n} \; - 1/2 \; + \; \frac{(-2\zeta(-1/2))\sqrt{\alpha}}{\sqrt{\pi}} n^{-1/4} \right\rceil$. We are uncertain of the stop rule in the interval $[n_1, n_2]$, and S is contained in the union of those. By the mean value theorem, $f_2(n_2) - f_1(n_2) = f_2'(n*)(n_2 - n_1)$ for some $n_1 < n* < n_2$, and $f_2'(n*) \geq f_2'(n_2) = \alpha n_2^{-1/2}/2 - o(n^{-1/2}) > n_2^{-1/2}/2$ for n_2 large, so $n_2 - n_1 < An_2^{-7/24}/n_2^{-1/2} = O(n_2^{5/24}) = O(u^{5/12})$. For the intervals $[n_1, n_2]$ contained in $\{1, \ldots, n\}$, we have $u \leq \alpha\sqrt{n}$. So $|S \cap \{1, \ldots, n\}|/n = O(\sum_{u=1}^{\alpha\sqrt{n}} u^{5/12}/n) = O(n^{-7/24})$. □

The set where we are uncertain has a simple description as a union of intervals $[n_1, n_2]$ pictured above. The ith such interval is centered halfway between where f_1 and f_2 cross the horizontal line of height i, which is approximately at i^2/α^2. The space between the ith and $(i+1)$th interval is $O(i)$. The length of the ith interval is $O\left(i^{5/12}\right)$; if our speculation (1.2) were true, this length would be bounded. We have not given a specific numerical value for A, though it could be done. We resorted to expressing results in big-O notation, in spite of originally hoping not to do that, wanting results that are usable for computer exploration. But the calculations in later sections became too onerous.

The goal of the rest of this paper is to prove Theorem 1.1, by using a direct combinatorial assault going backwards in a tree. It is the Shapiro Catalan triangle properties that come to the rescue. Having gotten this far, we are more optimistic than Medina and Zeilberger [10] about whether it is possible to get a formula for k_n for all n. Conjecture (1.2) is possibly provable with refinements of the techniques in this paper, or something similar.

Added after review: And the recently discovered formula given above at the end of Section 1.3 makes us yet more optimistic about the existence of a simple formula that works for all cases.

2. Generalized backward induction and the Shapiro Catalan triangle; plan for the proof of Theorem 1.1

We start by patiently wading through some backward induction steps, rewarding us with a recognized pattern. We will want to decide whether to stop or continue when u is below and near the Brownian boundary. Since we will be using the Brownian motion value function heavily for everything that follows, we switch back to using (u,b) for position, as Shepp did in his wonderful paper [13] which inspired our theoretical work on bounds. We remark again that in our development from now on, u is real, not just an integer, even though the Chow–Robbins game itself has only integer values for positions. This is important because our analysis depends on approximating $V(u,b)$ which is continuous in u. The graphs in Section 1.4 suggest the advantage of extending to the real case to do the analysis.

The famous *backward induction principle* of optimal stopping (see, for example, [3]) applied to this simple random walk is

$$V(u, b) = \max\left\{\frac{u}{b}, \frac{1}{2}V(u+1, b+1) + \frac{1}{2}V(u-1, b+1)\right\}. \tag{2.1}$$

$$\text{Do not stop if } \frac{1}{2}V(u+1, b+1) + \frac{1}{2}V(u-1, b+1) > \frac{u}{b}; \quad \text{stop otherwise.}$$

This is the starting point for everything. Let us use it to get a preliminary result. Numerical evidence showed that if $\delta = \alpha\sqrt{b} - u$ is larger than 1/2 minus a hair (the hair being of the order of $b^{-1/4}$), you should not stop. Our bounds on V are not alone good enough to prove that without any backward steps, but let us see what we get that way. Using only our lower bound from Lemma 1.2 (assuming $b > 1600$), and a differential approximation $V_W(u, b) \geq \left(1 + \delta^2 b^{-1}\right) u/b$ (from (3.6) in the next section), we get $V(u, b) \geq V_W(u, b)\left(1 - 0.43b^{-1}\right) \geq \left(1 + \delta^2 b^{-1}\right)\left(1 - 0.43b^{-1}\right) u/b$, and this is greater than u/b if $\delta > 0.66$. Well, that is something: continue if $\delta > 0.66$. But just one step of backward induction with our lower bound will show how it begins to close in on 1/2. The distance of $u - 1$ from the Brownian boundary is $\alpha\sqrt{b+1} - (u-1) = \alpha\sqrt{b} + h - u + 1 = \delta + 1 + h$, where $0 < h < \alpha b^{-1/2}/2$. So

$$\begin{aligned}
\frac{1}{2}V(u+1, b+1) + \frac{1}{2}V(u-1, b+1) &\geqslant \frac{1}{2}\frac{u+1}{b+1} + \frac{1}{2}V_W(u-1, b+1)\left(1 - \frac{0.43}{b+1}\right)\\
&\geqslant \frac{1}{2}\frac{u+1}{b+1} + \frac{1}{2}\frac{u-1}{b+1}\left(1 + \frac{(1+\delta)^2}{b+1}\right)\left(1 - \frac{0.43}{b+1}\right)\\
&= \frac{u}{b} - \frac{u}{b(b+1)}\\
&\quad + \frac{1}{2}\frac{u\left(1 - \frac{1}{\alpha\sqrt{b}-\delta}\right)}{(b+1)^2}\left((1+\delta)^2 - 0.43 - \frac{0.43(1+\delta)^2}{b+1}\right).
\end{aligned}$$

For sufficiently large b, so that we can throw out small stuff, the condition for this to be greater than u/b is clearly $(1+\delta)^2 - 0.43 > 2$, or $\delta > 0.56$. But to be concrete, assume $b > 1600$ and $u = \alpha\sqrt{b} - \delta > \alpha\sqrt{b} - 0.66$ (we already know to continue if δ greater than 0.66). Then some arithmetic shows that $\delta > 0.58$ is sufficient. That is an improvement. And we could go more steps back and get a better go bound.

Similarly, we can use our upper bound and close in on 1/2 from above, and it is convenient to do one step of that, to get a preliminary stop bound, to avoid annoyances later. Stop if

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$(V(u+1,b+1)+V(u-1,b+1))/2 < u/b$. And $\alpha\sqrt{b+1} - (u-1) = \delta+1+h$ where h is as before. Looking ahead to (3.5) of the next section, $V_W(u,b) \le u/b + \alpha\delta^2 b^{-3/2}$. We have

$$\begin{aligned}\frac{1}{2}V(u+1,b+1)+\frac{1}{2}V(u-1,b+1) &\leqslant \frac{1}{2}\frac{u+1}{b+1}+\frac{1}{2}\left(\frac{u-1}{b+1}+\alpha\frac{(1+\delta+h)^2}{(b+1)^{3/2}}\right)\\ &= \frac{u}{b}-\frac{u}{b(b+1)}+\frac{1}{2}\alpha\frac{(1+\delta+h)^2}{(b+1)^{3/2}}\\ &= \frac{u}{b}-\frac{\alpha\sqrt{b}-\delta}{b(b+1)}+\frac{1}{2}\alpha\frac{(1+\delta+h)^2}{(b+1)^{3/2}}.\end{aligned}$$

For sufficiently large b, the condition for this to be less than u/b is clearly $(1+\delta)^2 < 2$, or $\delta < 0.414$. But again to be concrete, assuming $b > 1600$, $\delta < 0.38$ can be shown to be sufficient.

With just one step of backward induction, we have already narrowed the range for the stopping boundary and proved the following lemma.

Lemma 2.1. *For $b > 1600$, $\alpha\sqrt{b} - 0.58 < \beta_b < \alpha\sqrt{b} - 0.38$. Go if $\delta > 0.58$, stop if $\delta < 0.38$.*

This will be useful later, so it is noted. But the goal is to get to $\alpha\sqrt{b} - 1/2 + cb^{-1/4} + o(b^{-1/4})$, by continuing way down the backward induction tree.

We proceed to systematize going backward. Continued backward induction leads to the tree in Figure 4, where further branching is stopped at $u+1$, creating leaves of the tree at those nodes, which are shown boxed. The V value at a node of the tree is greater than or equal to the average of the V values at its two parents. Figure 4 is a picture of eight rows of the backward induction tree. The meaning of the coefficients (weights) will be explained shortly, though it is perhaps already obvious from the way backward induction works for this simple symmetric random walk.

To explain the coefficients (the weights) displayed in Figure 4, consider the following succession of inequalities, using only the basic backward induction inequality (2.1):

$$V(u,b) \geqslant \frac{1}{2}V(u+1,b+1)+\frac{1}{2}V(u-1,b+1),$$

$$\frac{1}{2}V(u-1,b+1) \geqslant \frac{1}{4}V(u,b+2)+\frac{1}{4}V(u-2,b+2),$$

$$\begin{aligned}&\frac{1}{4}V(u,b+2)+\frac{1}{4}V(u-2,b+2)\\ &\quad\geqslant \frac{1}{8}V(u+1,b+3)+\frac{1}{8}V(u-1,b+3)+\frac{1}{8}V(u-1,b+3)+\frac{1}{8}V(u-3,b+3)\\ &\quad= \frac{1}{8}V(u+1,b+3)+\frac{2}{8}V(u-1,b+3)+\frac{1}{8}V(u-3,b+3),\end{aligned}$$

$$\begin{aligned}&\frac{2}{8}V(u-1,b+3)+\frac{1}{8}V(u-3,b+3)\\ &\quad\geqslant \frac{2}{16}V(u,b+4)+\frac{2}{16}V(u-2,b+4)+\frac{1}{16}V(u-2,b+4)+\frac{1}{6}V(u-4,b+4)\\ &\quad= \frac{2}{16}V(u,b+4)+\frac{3}{16}V(u-2,b+4)+\frac{1}{6}V(u-4,b+4),\end{aligned}$$

$\ldots.$

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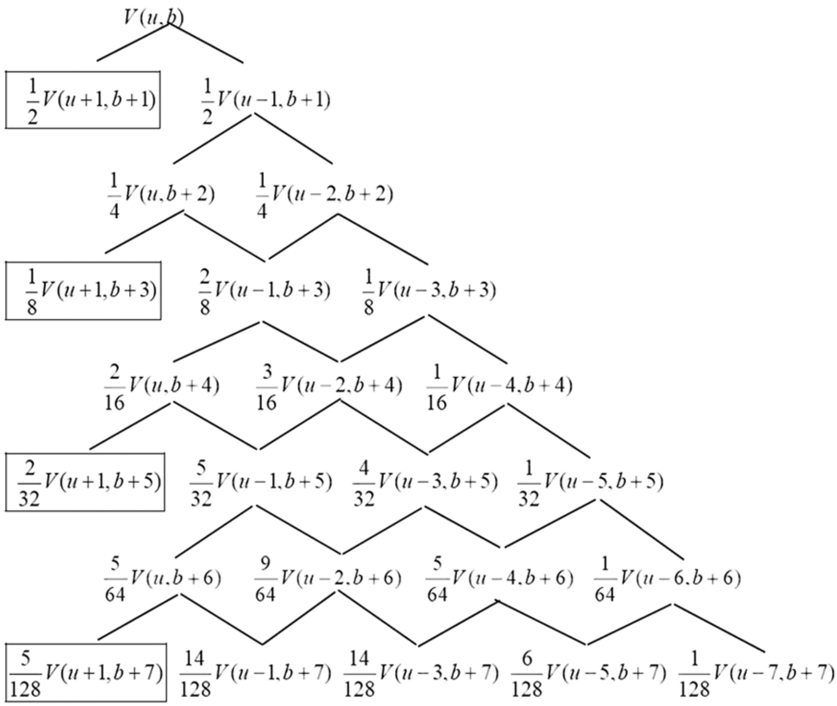

FIGURE 4. Backward induction tree, with leaves boxed.

That is how the weights are generated, recursively. Considering any row of the tree, $V(u,b)$ is greater than or equal to the sum of the weights times values at the leaves at or above that row, plus the weights times values at the non-leaf nodes at that row. The picture shows the weights through row 7, with the initial node at row 0. We were patient enough to carry out the trivial calculations by hand through seven rows, at which point we recognized the leaf weight numerators as being the Catalan numbers.

Now formalize the notation and the recursion. First, look at the nodes that are not leaves. With m indicating row number starting from 0 and j column number starting from 0 in the tree, let $T(m,j)$ be the coefficient of $2^{-m}V(u-j,\, b+m)$, $m \geqslant 0$, $0 \leqslant j \leqslant m$, and $T(m, j) = 0$ outside this range. The initial condition is $T(0, 0) = 1$. The recursion $T(m, j) = T(m-1, j-1) + T(m-1, j+1)$ follows purely from backward induction. Using this recursion produces the table in Figure 5, through $m = 7$.

In what follows we will only make use of the odd rows and columns of the tree in Figure 4, which correspond to the rows of the tree with leaves. Let $B(n, k) = T(2n-1, 2k-1)$, $n \geq 1$, $k \geq 1$; Figure 6 shows the first four rows of B. The recursion for T implies (in two steps) the following recursion for B: $B(n, k) = B(n-1, k-1) + 2B(n-1, k) + B(n-1, k+1)$, $n \geq 2$, $k \geq 2$, with initial conditions $B(1, 1) = 1$ and $B(1, k) = 0$, $k > 1$. This is the recursion and initial conditions for the Shapiro Catalan triangle [12], one of the many arrays referred to in the literature as a Catalan triangle, or unfortunately sometimes as ‘the’ Catalan triangle. The columns of B (omitting the leading zeros) appear as every other diagonal in another popular ‘Catalan’s triangle’, the one that appears, for example, in the Wikipedia article of that name, which gives references to the history. But we need to reference our tree by row and column

j \ m	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	1	0	1	0	0	0	0	0
3	0	2	0	1	0	0	0	0
4	2	0	3	0	1	0	0	0
5	0	5	0	4	0	1	0	0
6	5	0	9	0	5	0	1	0
7	0	14	0	14	0	6	0	1

FIGURE 5. $T(m,j)$, coefficient numerator in row m, col. j of tree in Figure 4.

k \ n	1	2	3	4
1	1	0	0	0
2	2	1	0	0
3	5	4	1	0
4	14	14	6	1

FIGURE 6. $B(n,k)$, the odd rows and columns of T; it is the Shapiro Catalan triangle.

indices, so B is the right one for us. Fortunately, Miana and Romero [11] have the row moments for B (see (4.2)), which is exactly what we need for our purposes.

We now have a formula for the coefficients of the non-leaves in the odd rows of our backward induction tree in terms of well-known numbers:

$$\text{coefficient of } V(u-2j+1,\, b+2m-1) = 2^{-2m+1}B(m,j), \quad m \geq 1, j \geq 1. \tag{2.2}$$

There are simple known formulas for the entries in B:

$$B(m,j) = \frac{j}{m}\binom{2m}{m-j} = \binom{2m-1}{m-j} - \binom{2m-1}{m-j-1}.$$

The first column is

$$B(m,1) = \frac{1}{m}\binom{2m}{m-1} = \frac{1}{m+1}\binom{2m}{m} = \binom{2m-1}{m-1} - \binom{2m-1}{m-2} = C_m, \quad m \geqslant 1,$$

recognized as the mth Catalan number. $C_0 = 1$ by definition.

For the leaves, let $L(n)$ be the coefficient of $2^{-2n+1}V(u+1, b+2n-1)$, $n \geq 1$, where $n = 1, 2, 3, 4, \ldots$ corresponds to rows $1, 3, 5, 7, \ldots$ of the tree of Figure 4. The first four values of $L(n)$ are 1,1,2,5, and from the way the tree is built, it is seen that $L(n) = T(2n-2, 0) = T(2n-3, 1)$, $n \geq 2$, with $L(1) = 1$. But $T(2n-3, 1) = B(n-1, 1) = C_{n-1}$ for $n \geq 2$. So, we have the following formula for the leaf weights:

$$\text{coefficient of } V(u+1,\, b+2n-1) = 2^{-2n+1}C_{n-1}, \quad n \geqslant 1. \tag{2.3}$$

Returning to the inequality we started with, for any row, $V(u,b)$ is greater than or equal to the sum of the weights times values at the leaves at or above that row, plus the weights times values at the non-leaf nodes in that row. Looking at only odd rows of the tree, for $n \geq 1$, let

$$S_L(n, u, b) := \sum_{m=0}^{n-1} 2^{-2m-1} C_m V(u+1, b+2m+1),$$

the sum of the leaves down through the row indexed by n; and

$$S_R(n, u, b) := 2^{-2n+1} \sum_{j=1}^{n} B(n, j) V(u-2j+1, b+2n-1),$$

the sum of the non-leaves across the row indexed by n (an odd row of the original tree). Define

$$TreeSum(n, u, b) := S_L(n, u, b) + S_R(n, u, b). \tag{2.4}$$

Thus for any $n \geq 1$, $V(u, b) \geq TreeSum(n, u, b)$. This inequality is the summary of the chain of inequalities from repeated backward induction.

One property of $TreeSum(n,u,b)$ is immediate from the basic backward induction principle (2.1): $TreeSum(n+1, u, b) \leq TreeSum(n, u, b)$, $n \geq 1$, which implies $TreeSum(n, u, b) \leq TreeSum(1, u, b) = (V(u+1, b+1) + V(u-1, b+1))/2$. Thus for any n, $TreeSum(n, u, b) > u/b \Rightarrow TreeSum(1, u, b) > u/b \Rightarrow V(u, b) > u/b$, so do not stop at (u,b). To prove the other direction, let $j \geq 0$, $k \geq 0$. $V(u, b) > u/b \Rightarrow V(u-j, b) > (u-j)/b$ is obvious. Then $V(u-j, b) > (u-j)/b \Rightarrow u-j < \beta_b \Rightarrow u-b < \beta_{b+k}$, from the monotonicity of Dvoretzky's stop rules [6], so $V(u-j, b+k) > (u-j)/(b+k)$. In other words, if you should not stop at (u,b), then you should not stop for a smaller u or a larger b. This implies that if $TreeSum(1, u, b) > u/b$, then at every non-leaf node of the backward induction tree, the average of the values of its two children is greater than the ratio, so $TreeSum(n, u, b) = TreeSum(1, u, b)$ and $TreeSum(n, u, b) > u/b$. This leads to the following lemma.

Lemma 2.2. (Extended backward induction principle). *Let* $n \geq 1$. *Then* $V(u, b) = \max\{u/b, TreeSum(n, u, b)\}$*; stop at (u,b) if* $TreeSum(n, u, b) \leq u/b$*, else continue.* $TreeSum(1, u, b) > u/b \Rightarrow TreeSum(1, u, b) = TreeSum(n, u, b)$.

Proof. From the previous paragraph, $TreeSum(n, u, b) > u/b \Rightarrow TreeSum(1, u, b) > u/b$ which implies $TreeSum(n, u, b) = TreeSum(1, u, b) = V(u, b)$. Now suppose $TreeSum(n, u, b) \leq u/b$. If $TreeSum(1, u, b) > u/b$, then by the previous paragraph, $TreeSum(n, u, b) = TreeSum(1, u, b) > u/b$, a contradiction. So $TreeSum(1, u, b) \leq u/b$, and $V(u, b) = u/b$ by definition, and you may stop. □

We can now explain the plan for the proof of Theorem 1.1, and give some indication of why it should work. The proof is accomplished in three stages.

Stage one, in Section 4, gets preliminary $O(b^{-1/4})$ bounds on the stop rule β_b. For this stage, we consider ns of the form $c\sqrt{b}$ and u near enough to the boundary, that is, $\delta = \alpha\sqrt{b} - u$ small enough, so that $V(u+1, b+2m+1) = (u+1)/(b+2m+1)$ exactly for $m < n$. With a binomial expansion of that ratio, the leaf sum can be approximated to any accuracy desired using simple formulas for $\sum_{m=0}^{n-1} 2^{-2m-1} m^k C_m$, which we use for $k = 0, 1, 2$. For the row sum, we use our simple upper and lower bounds on V in terms of V_W, and approximate V_W by a Taylor expansion about the boundary using four derivatives, and this leads to sums $2^{-2n+1} \sum_{j=1}^{n} j^k B(n, j)$, which have simple known formulas. What is the good of letting n grow big? The point is that the larger n, the more of the weight of the tree sum is on the leaves, for which the value is exactly known as long as n does not get out of range, and the less is on the

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row sums, where we are limited in accuracy by our approximate bounds. This is our algebraic manifestation of Häggström and Wästlund's idea to let the errors in the bounds wash out by moving the horizon back. In fact these sum formulas are all just simple expressions involving n and the central binomial $2^{-2n}\binom{2n}{n} \cong (\pi n)^{-1/2}$, and n will be of order $\sqrt{b}$, which hints at why $b^{-1/4}$ shows up in the answer. The first stage results in $O(b^{-1/4})$ upper and lower bounds on the stop rule; this is Lemma 4.4, which in fact was our original goal. But it is not able to get the exact coefficient of $b^{-1/4}$.

Stage two, in Section 5, feeds the result of stage one back into the Value approximations developed in stage one, to obtain a much sharper approximation for V near the boundary. It is perhaps the most conceptually tricky part of the proof, with a repeated feedback argument that will be better motivated when we get there. It shows that V is essentially piecewise linear near the boundary, below and tangent at integer points of δ, to the quadratic approximation to V_W near the boundary. This is Theorem 5.1, perhaps of independent interest in showing the manner in which the Brownian Value overestimates V.

Finally, stage three, in Section 6, uses this improved estimate of V to go a bit further down the tree, by estimating the leaf values $V(u+1, b+2m+1)$ for a range of m such that V is no longer just the ratio. By going just far enough down the tree, we are able to get the upper and lower bounds to come together, within an $o(b^{-1/4})$ error, finding the exact coefficient of $b^{-1/4}$ and proving Theorem 1.1. The plan is straightforward except perhaps for Section 5, and uses standard approximations, but lots of them, so it looks worse than it is when all the approximations and sums are written out.

3. Approximating the Brownian motion value V_W

From formula (1.1), one may differentiate under the integral sign to see that all the derivatives with respect to u are positive for $u \le \alpha\sqrt{b}$, so they all take their maximum value on the boundary; we use that several times below. To compute derivatives, and also for numerical work, it is best to write it in terms of standard functions. We have

$$\begin{aligned} V_W(u,b) &= (1-\alpha^2)\int_0^\infty \exp\left(\lambda u - \frac{\lambda^2 b}{2}\right) d\lambda \\ &= (1-\alpha^2) b^{-1/2} \exp\left(\frac{u^2}{2b}\right) \int_{-\infty}^{u/\sqrt{b}} \exp\left(-\frac{w^2}{2}\right) dw \\ &= (1-\alpha^2) b^{-1/2} G(u/\sqrt{b})/g(u/\sqrt{b}) \\ &= (1-\alpha^2) b^{-1/2} H(u/\sqrt{b}), \end{aligned}$$

where G and g are the cumulative distribution function (CDF) and probability density function (PDF) of the standard normal distribution, respectively, and we have defined $H(x) := G(x)/g(x)$. This is related to the Mills ratio by $H(x) = 1/g(x) - m(x)$, but we do not use that. On the boundary, $u_0 = \alpha\sqrt{b}$, $V_W(u_0, b) = u_0/b = \alpha b^{-1/2} = (1-\alpha^2)b^{-1/2}H(u_0/\sqrt{b}) = (1-\alpha^2)b^{-1/2}H(\alpha)$, so we get the equation that alpha satisfies as $\alpha = (1-\alpha^2)H(\alpha)$, which is useful for computing α using a library function for the Normal, or converted to an error function representation, to use that library function instead, if needed. We did that for the double-double precision numerical work for the extremely large number of positions we considered.

We will later use five derivatives of V_W. Use D_u for derivative operator with respect to the first argument. Then ${D_u}^n V_W(u, b) = (1-\alpha^2)b^{-(n+1)/2}H^{(n)}(u/\sqrt{b})$, so we need to find the successive derivatives of H. It satisfies $H'(x) = xH(x) + 1$, which makes this straightforward, and can easily be made systematic in terms of polynomials in x, similar to Hermite polynomials. One shows

$$H^{(n)} = P_n H + Q_n, \quad \text{with } P_{n+1} = P_n' + xP_n,\ Q_{n+1} = {Q_n}' + P_n,\ P_0 = 1,\ Q_0 = 0. \tag{3.1}$$

So ${D_u}^n V_W(u, b) = (1-\alpha^2)b^{-(n+1)/2}\left(P_n(u/\sqrt{b})H(u/\sqrt{b}) + Q_n(u/\sqrt{b})\right)$.

Remark. Just like for the Hermites, we have $P_n' = nP_{n-1}$ (an Appell sequence), so just like for the Hermites, there is a computationally practical recurrence, not involving derivatives: $P_{n+1}(x) = xP_n(x) + nP_{n-1}(x)$, the only difference from the Hermites being the positive sign. But we do not need to pursue this for our purposes, we just want a few derivatives.

On the boundary, $x = u_0/\sqrt{b} = \alpha\sqrt{b}/\sqrt{b} = \alpha$ and $H(\alpha) = \alpha(1-\alpha^2)^{-1}$, so

$$\begin{aligned} {D_u}^n V_W(u_0, b) &= (1-\alpha^2)b^{-n/2+1}\left(P_n(\alpha)H(\alpha) + Q_n(\alpha)\right) \\ &= b^{-(n+1)/2}\left(\alpha P_n(\alpha) + (1-\alpha^2)Q_n(\alpha)\right). \end{aligned}$$

To get the polynomials, turn the recursion crank in (3.1) five times, resulting in

$$\begin{aligned} &P_0 = 1,\ Q_0 = 0; \quad P_1 = x,\ Q_1 = 1; \quad P_2 = 1 + x^2,\ Q_2 = x; \quad P_3 = 3x + x^3,\ Q_3 = 2 + x^2; \\ &P_4 = 3 + 6x^2 + x^4,\ Q_4 = 5x + x^3; \quad P_5 = 15x + 10x^3 + x^5,\ Q_5 = 8 + 9x^2 + x^4. \end{aligned} \tag{3.2}$$

To prove Lemma 1.2, we use three derivatives of H, which from (3.1) and (3.2) are

$$\begin{aligned} H^{(1)}(x) &= xH(x) + 1, \\ H^{(2)}(x) &= (1 + x^2)H(x) + x; \\ H^{(3)}(x) &= (3x + x^3)H(x) + 2 + x^2. \end{aligned} \tag{3.3}$$

The first five derivatives of V_W on the boundary are

$$\begin{aligned} D_u V_W(u_0, b) &= b^{-1}, \\ {D_u}^2 V_W(u_0, b) &= b^{-3/2}2\alpha, \\ {D_u}^3 V_W(u_0, b) &= b^{-2}\left(2 + 2\alpha^2\right), \\ {D_u}^4 V_W(u_0, b) &= b^{-5/2}\left(8\alpha + 2\alpha^3\right), \\ {D_u}^5 V_W(u_0, b) &= b^{-3}\left(8 + 16\alpha^2 + 2\alpha^4\right). \end{aligned} \tag{3.4}$$

Let $\delta = u_0 - u = \alpha\sqrt{b} - u$. Using Taylor with just the first two derivatives,

$$\begin{aligned} V_W(u, b) &= V_W(u_0, b) + (u - u_0)D_u V_W(u_0, b) + (u - u_0)^2 {D_u}^2 V_W(u_*, b)/2 \\ &\le \alpha b^{-1/2} - \delta b^{-1} + \delta^2 {D_u}^2 V_W(u_0, b)/2 = ub^{-1} + \alpha\delta^2 b^{-3/2}. \end{aligned}$$

The inequality is because ${D_u}^2 V_W(u, b)$ is an increasing function (recall all derivatives are positive), so is largest on the boundary. In summary,

$$V_W(u, b) \leqslant ub^{-1} + \alpha\delta^2 b^{-3/2}. \tag{3.5}$$

Using the third derivative with a similar monotonicity argument gives a lower bound, the one that we already used above in proving Lemma 2.1:

$$\begin{aligned} V_W(u,b) &\geq ub^{-1} + \alpha\delta^2 b^{-3/2} - (1+\alpha^2)\delta^3 b^{-2}/3 = ub^{-1} + \alpha\sqrt{b}\delta^2 b^{-2} - (1+\alpha^2)\delta^3 b^{-2}/3 \\ &= ub^{-1} + (u+\delta)\delta^2 b^{-2} - (1+\alpha^2)\delta^3 b^{-2}/3 \quad (\text{recall } u+\delta = \alpha\sqrt{b}) \\ &= ub^{-1}\left(1+\delta^2 b^{-1}\right) + \delta^3 b^{-2}\left(1-(1+\alpha^2)/3\right) \geq ub^{-1}\left(1+\delta^2 b^{-1}\right). \end{aligned}$$

Thus

$$V_W(u,b) \geqslant ub^{-1}\left(1+\delta^2 b^{-1}\right). \tag{3.6}$$

Using four derivatives, the monotonicity argument yields the upper bound

$$V_W(u,b) \leqslant ub^{-1} + \alpha\delta^2 b^{-3/2} - (1+\alpha^2)\delta^3 b^{-2}/3 + (4\alpha+\alpha^3)\delta^4 b^{-5/2}/12. \tag{3.7}$$

With the fifth derivative term, one gets the lower bound

$$\begin{aligned} V_W(u,b) \geqslant ub^{-1} + \alpha\delta^2 b^{-3/2} - (1+\alpha^2)\delta^3 b^{-2}/3 + (4\alpha+\alpha^3)\delta^4 b^{-5/2}/12 \\ -(4+8\alpha^2+\alpha^4)\delta^5 b^{-3}/60. \end{aligned} \tag{3.8}$$

These last two inequalities are used in proving Lemma 4.2.

4. Computing bounds on tree sums

This section is based on a lot of computation. We have resorted to big-O notation rather than getting specific constants, for which we apologized earlier. We will use big-O notation in inequalities, since we deal with approximate upper and lower bounds. The meaning will probably be clear from the context, but to be certain, we define our notation. Let f,g be real-valued functions of real variable b, with $g(b) > 0$ for all sufficiently large b. Define $f(b) \leq O(g(b))$ to mean there exist b_0 and $K > 0$ such that $f(b) \leq Kg(b)$, $b \geq b_0$. Usually only $f(b) = O(g(b))$ is defined in texts, requiring $|f(b)| \leq Kg(b)$, $b \geq b_0$. That would not be convenient for our purposes. We will write expressions such as $f(b) \leq U(b) + O\left(b^{-2}\right)$, to mean there exist b_0 and $K > 0$ such that $f(b) \leq U(b) + Kb^{-2}$ for all $b \geq b_0$. That is the same as $f(b) - U(b) \leq O\left(b^{-2}\right)$ in our definition. U is an approximate upper bound for f, but $f(b)$ could be arbitrarily far below $U(b)$. This seems very natural for dealing with inequalities rather than equalities, but does not seem to be a standard notation. Similarly, still assuming $g(b) > 0$ for all sufficiently large b, define $f(b) \geq O\left(g(b)\right)$ to mean there exist b_0 and $K > 0$ such that $f(b) \geq -Kg(b)$, $b \geq b_0$. For example, $f(b) \geq U(b) + O\left(b^{-2}\right)$ means there exist b_0 and $K > 0$ such that $f(b) \geq U(b) - Kb^{-2}$ for all $b \geq b_0$.

Here is a way to describe the situation using set language:

$$O(g(b)) = \{h : \text{there exist } K > 0 \text{ and } b_0 \text{ such that } \ |h(b)| \leq Kg(b) \text{ for all } b \geq b_0\}.$$

Note $h \in O(g(b))$ implies $|h|$, $-|h|$, and $-h$ are in $O(g(b))$, and $O(g(b)) = -O(g(b))$. Now $f(b) \leq O(g(b))$ means there exists $h \in O(g(b))$ such that $f(b) \leq h(b)$ for all b; $f(b) = O(g(b))$ means there exists $h \in O(g(b))$ such that $f(b) = h(b)$ for all b; and $f(b) \geq O(g(b))$ means there exists $h \in O(g(b))$ such that $f(b) \geq h(b)$ for all b. That unifies the settings: for some $h \in O(g(b))$, the inequality or equality is true when $h(b)$ replaces $O(g(b))$.

Throughout this section, for integer $b > 1600$ and real number u, let $\delta = \alpha\sqrt{b} - u$. If $\delta < 0.38$ or $\delta > 0.58$, we already know whether to stop or keep going, by Lemma 2.1. But we will

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also want to estimate tree sums for a somewhat larger delta, to estimate leaf sums in proving Theorem 5.1. For this section, we assert that $\delta \le b^p$, where $0 \le p \le 1/10$. We also assert that $n = O(b^{1/2+p})$, where, as above, n indexes how far we go down the tree, looking at only the odd rows (the ones with leaves). We prove Lemma 4.3 with this generality. In our first application of Lemma 4.3, to get preliminary stop bounds, p will be 0. But in Section 5 we will use Lemma 4.3 with larger delta to get an improved estimate of Value, and then in Section 6 with larger n to finally prove Theorem 1.1.

Let $G_n := 2^{-2n}\binom{2n}{n}$, for which it is well known as a central binomial coefficient that $G_n \cong (\pi n)^{-1/2}$; to be more precise,$(\pi(n+1/2))^{-1/2} \le G_n \le (\pi n)^{-1/2}$. The formulas that will be needed for the moments of Catalan numbers and Shapiro Catalan triangle rows can be simply expressed in terms of G_n.

Lemma 4.1. (Catalan and Shapiro Catalan triangle moments.) *We have*

$$\sum_{j=0}^{n-1} C_j 2^{-2j-1} = 1 - G_n, \quad \sum_{j=0}^{n-1} C_j 2^{-2j-1} j = nG_n + G_n - 1, \tag{4.1}$$
$$\sum_{j=0}^{n-1} C_j 2^{-2j-1} j^2 = \frac{1}{3} n^2 G_n - \frac{4}{3} nG_n - G_n + 1;$$

and

$$2^{-2n+1} \sum_{j=1}^{n} B(n, j) = G_n, \quad 2^{-2n+1} \sum_{j=1}^{n} jB(n, j) = \frac{1}{2}, \tag{4.2}$$
$$2^{-2n+1} \sum_{j=1}^{n} j^2 B(n, j) = nG_n, \quad 2^{-2n+1} \sum_{j=1}^{n} j^3 B(n, j) = \frac{3n-1}{4},$$
$$2^{-2n+1} \sum_{j=1}^{n} j^4 B(n, j) = n(2n-1)G_n, \quad 2^{-2n+1} \sum_{j=1}^{n} j^5 B(n, j) = \frac{15n(n-1)+2}{8}.$$

Proof. The first statement in (4.1) is proved quickly by induction. The other two are easily proved by telescoping; we omit the details. All of (4.2) can be found in Miana and Romero [9, pp. 5–6]. □

When proving Lemma 4.4, we will be using the ratio for the value, on all the leaves, to compute the leaf sum. For getting an upper bound we will be restricting n to be small enough so that the ratio is in fact the correct value on all leaves, and for the lower bound we will be restricting n to be small enough so that the ratio (which is always a lower bound) will not be far off from the correct value. But in finally proving Theorem 1.1 in Section 6, n will be large enough so that the values at some of the leaves will be significantly larger than the ratio, and we have to take that into account. It will be convenient to split the leaf sum into two parts. For general u and b, define $V_E(u, b) = V(u, b) - u/b$, the amount the value exceeds the ratio. Let

$$S_{LR}(n, u, b) := \sum_{m=0}^{n-1} 2^{-2m-1} C_m \frac{u+1}{b+2m+1},$$

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the contribution from the ratio, and

$$S_{LE}(n,u,b) := \sum_{m=0}^{n-1} 2^{-2m-1} C_m V_E(u+1, b+2m+1),$$

from the excess over the ratio. So

$$S_L = S_{LR} + S_{LE}.$$

We have

$$\begin{aligned} S_{LR} &= \sum_{m=0}^{n-1} 2^{-2m-1} C_m \frac{u+1}{b+2m+1} = \frac{u+1}{b} \sum_{m=0}^{n-1} 2^{-2m-1} C_m \left(1 + \frac{2m+1}{b}\right)^{-1} \\ &= \frac{u+1}{b} \sum_{m=0}^{n-1} 2^{-2m-1} C_m \left(1 - \frac{2m+1}{b} + \gamma(m) \frac{(2m+1)^2}{b^2}\right), \end{aligned}$$

where $1 - \frac{2m+1}{b} \le \gamma(m) \le 1$. From (4.1),

$$\sum_{m=0}^{n-1} 2^{-2m-1} C_m (2m+1)^2 = \frac{4}{3} n^2 G_n - \frac{4}{3} n G_n - G_n + 1,$$

$$\sum_{m=0}^{n-1} 2^{-2m-1} C_m (2m+1) = 2nG_n - 1 + G_n,$$

$$\sum_{m=0}^{n-1} 2^{-2m-1} C_m = 1 - G_n.$$

Since $\gamma(m) \le 1$,

$$\begin{aligned} S_{LR} &\le \frac{u+1}{b}(1-G_n) - \frac{u+1}{b^2}\left(2nG_n - 1 + G_n - \frac{4}{3}\frac{n^2}{b} G_n + \frac{4}{3}\frac{n}{b} G_n - \frac{1}{b} + \frac{G_n}{b}\right) \\ &= \frac{u+1}{b}(1-G_n) - \frac{u+1}{b^2}\left(2nG_n - 1 + G_n - \frac{4}{3}\frac{n^2}{b} G_n + O(b^{-1/2})\right), \end{aligned}$$

where we used $n = O(b^{1/2+p})$, $\frac{n}{b} G_n = O(b^{-3/4+p/2}) = O(b^{-1/2})$ for our range of p.

For a lower bound, use $\gamma(m) \ge 1 - 2b^{-1/2+p}$. Note $b^{-1/2+p}(n^2 G_n / b) = O(b^{-3/4+5p/2}) = O(b^{-1/2})$, from which it follows that the above upper bound for S_{LR} is also a lower bound, to that order.

Anticipating combining S_{LR} with the row sum S_R and then getting the difference between the total and the ratio in terms of δ, write

$$S_{LR} = \frac{u}{b} + \frac{1-(u+1)G_n}{b} - \frac{\alpha}{b^{3/2}}\left(1 + \frac{1-\delta}{\alpha\sqrt{b}}\right)\left(2nG_n - 1 + G_n - \frac{4}{3}\frac{n^2}{b} G_n + O(b^{-1/2})\right).$$

Using $\delta = O(b^p)$, $n = O(p^{1/2+p})$ and $p \le 1/10$, one can show $\dfrac{\delta n^2 G_n}{b^{3/2}} = O(b^{-1/2})$. It follows that

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$$S_{LR} = \frac{u}{b} + \frac{1-(u+1)G_n}{b} - \frac{\alpha}{b^{3/2}}\left(2nG_n - 1 + G_n - \frac{4}{3}\frac{n^2}{b}G_n + \frac{2nG_n}{\alpha\sqrt{b}} - \delta\frac{2nG_n}{\alpha\sqrt{b}} + O(b^{-1/2+p})\right). \tag{4.3}$$

That takes care of the leaves; the row sum will be added to this. To get the row sum, we first replace V with V_W and define

$$S_{RW}(n, u, b) := 2^{-2n+1}\sum_{j=1}^{n} B(n, j)V_W(u-2j+1, b+2n-1),$$

and use the Taylor approximation from Section 3 to compute it. This will be an upper bound for S_R.

Lemma 4.2. (Row sum with V_W) *We have*

$$S_{RW}(n, u, b) = \frac{(u+1)G_n - 1}{b} + \frac{\alpha}{b^{3/2}}X,$$

where

$$X := 2nG_n - 2 + 2G_n + 2(\alpha + 1/\alpha)\frac{nG_n}{\sqrt{b}} - \frac{(4+\alpha^2)}{3}\frac{n^2G_n}{b} + 2\delta\left(1 - G_n - (\alpha+1/\alpha)\frac{nG_n}{\sqrt{b}}\right) + \delta^2G_n + O(b^{-1/2+2p}).$$

The proof is along the lines of the above calculation of S_{LR} except using (4.2) instead of (4.1). But it is very much longer, so we defer the proof to Appendix B.

To get a lower bound for S_R, using Lemma 1.2 we can multiply S_{RW} by

$$1 - \frac{5/12}{b+2n-1}\left(1 + \frac{1}{\sqrt{b+2n-1}}\right) = 1 - \frac{5}{12b} + O\left(b^{-3/2+p}\right).$$

Now

$$\frac{5}{12b}S_{RW} = \frac{5}{12b}\frac{uG_n+1}{b} + O(b^{-2}) = \frac{\alpha}{b^{3/2}}\frac{5G_n}{12} + O(b^{-2+p}),$$

so

$$S_{RW} - \frac{\alpha}{b^{3/2}}\frac{5G_n}{12} + O(b^{-2+p}) \le S_R.$$

We get some simplification when we add

$$S_{LR} + S_{RW} = \frac{u}{b} + \frac{\alpha}{b^{3/2}}Y,$$

where

$$Y := -1 + G_n + 2\alpha\frac{nG_n}{\sqrt{b}} - \frac{\alpha^2}{3}\frac{n^2G_n}{b} + 2\delta\left(1 - G_n - \alpha\frac{nG_n}{\sqrt{b}}\right) + \delta^2G_n + O(b^{-1/2+2p}).$$

Thus

$$\frac{\alpha}{b^{3/2}}Y' \le S_{LR} + S_R - \frac{u}{b} \le \frac{\alpha}{b^{3/2}}Y,$$

where $Y' = Y - \frac{5}{12}G_n$. We have proved the following result.

Lemma 4.3. (Bounds on tree sum.) *Let* $0 \le p \le 1/10$, $n = O(b^{1/2+p})$ *and* $\delta \le b^p$, *where* $\delta = \alpha\sqrt{b} - u$. *Then*

$$\frac{\alpha}{b^{3/2}}\left(C' + \delta B + \delta^2 A\right) \le TreeSum(n, u, b) - S_{LE} - \frac{u}{b} \le \frac{\alpha}{b^{3/2}}\left(C + \delta B + \delta^2 A\right),$$

where

$$C := -(1 - C_1 G_n) + O(b^{-1/2+2p}), \quad C' := -(1 - C_1' G_n) + O(b^{-1/2+2p}), \quad A := G_n,$$
$$B := 2(1 - B_1 G_n), \quad \text{with } C_1 := 1 + 2\frac{\alpha n}{\sqrt{b}} - \frac{1}{3}\frac{\alpha^2 n^2}{b}, C_1' := C_1 - 5/12, B_1 := 1 + \frac{\alpha n}{\sqrt{b}}.$$

We can use this immediately to get preliminary $O(b^{-1/4})$ bounds on the stop rule. If we do not go too far down the tree, S_{LE} will be zero. By Lemma 2.1, the value $V(u+1, b+2n-1)$ at the leaf will be just the ratio if $\alpha\sqrt{b+2n-1} - (u+1) < 0.38$. But $\alpha\sqrt{b+2n-1} - (u+1) = \alpha\sqrt{b}\left(1 + \frac{2n-1}{b}\right)^{1/2} - (\alpha\sqrt{b} - \delta + 1) < \alpha n/\sqrt{b} + \delta - 1 \le 0.38$ if $\alpha n/\sqrt{b} \le 0.8$ and $\delta \le 0.58$, and then $S_{LE} = 0$. We make note of this for use later:

$$\alpha n/\sqrt{b} \le 0.8 \text{ and } \delta \le 0.58 \text{ implies } S_{LE} = 0. \tag{4.4}$$

If $S_{LE} = 0$ and $C + \delta B + \delta^2 A \le 0$, the tree sum will not exceed the ratio, and if $C' + \delta B + \delta^2 A \ge 0$, the ratio will not exceed the tree sum. Let δ_0 and δ_0' be the positive numbers satisfying $C + \delta_0 B + \delta_0^2 A = 0$ and $C' + \delta_0' B + \delta_0'^2 A = 0$, respectively. From Lemma 2.2, it follows that $\alpha\sqrt{b} - \delta_0' \le \beta_b \le \alpha\sqrt{b} - \delta_0$. In (a) of the next lemma, we will estimate δ_0 and δ_0' using some convenient ns to get preliminary stop bounds from this. In addition, in (b) below we will estimate δ_0 and δ_0' for a value of n that will be used throughout Section 5 for convenience.

Lemma 4.4. (Preliminary bounds on stop rule)

$$\text{(a) } \alpha\sqrt{b} - 1/2 + \frac{0.231}{\sqrt{\pi}}b^{-1/4} + O(b^{-1/2}) \le \beta_b \le \alpha\sqrt{b} - 1/2 + \frac{0.429}{\sqrt{\pi}}b^{-1/4} + O(b^{-1/2}).$$

(b) Let $n = \lfloor\sqrt{b}/2\rfloor$, *and let* δ_0 *and* δ_0' *be the positive numbers satisfying* $C + \delta_0 B + \delta_0^2 A = 0$ *and* $C' + \delta_0' B + \delta_0'^2 A = 0$, *respectively, where A, B, C, and C' are as in Lemma 5.2. Then there exists* b_0 *such that*

$$1/2 - \frac{0.44}{\sqrt{\pi}}b^{-1/4} \le \delta_0 \le \delta_0{}' \le 1/2 - \frac{0.13}{\sqrt{\pi}}b^{-1/4}, \quad b \ge b_0.$$

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Proof. We will have $n = \Theta(b^{-1/2})$ in all cases here, so $A = G_n = O(n^{-1/2}) = O(b^{-1/4})$. For δ_0, approximate the solution to the quadratic equation

$$\begin{aligned}\delta_0 &= \frac{-B+\sqrt{B^2-4AC}}{2A} \\ &= \frac{-B+B\left(1-2ACB^{-2}-2A^2C^2B^{-4}+O(A^3)\right)}{2A} \\ &= -\frac{C}{B}-\frac{AC^2}{B^3}+O(b^{-1/2})\end{aligned}$$

since $A = O(b^{-1/4})$. Now

$$\begin{aligned}-\frac{C}{B}-\frac{AC^2}{B^3} &= \frac{1-C_1G_n+O(b^{-1/2+2p})}{2(1-B_1G_n)} - G_n\frac{\left(1-C_1G_n+O(b^{-1/2+2p})\right)^2}{8(1-B_1G_n)^3} \\ &= \frac{1}{2}(1-C_1G_n)(1+B_1G_n) - \frac{G_n}{8} + O(b^{-1/2+2p}) \\ &= \frac{1}{2}-\frac{1}{2}\left(C_1-B_1+\frac{1}{4}\right)G_n + O(b^{-1/2+2p}).\end{aligned}$$

This gives

$$\delta_0 = \frac{1}{2}-\frac{1}{2}\left(\frac{\alpha n}{\sqrt{b}}-\frac{1}{3}\frac{\alpha^2n^2}{b}+\frac{1}{4}\right)G_n + O(b^{-1/2+2p}). \tag{4.5}$$

Changing C to C' in the above calculation gives

$$\delta_0' = \frac{1}{2}-\frac{1}{2}\left(\frac{\alpha n}{\sqrt{b}}-\frac{1}{3}\frac{\alpha^2n^2}{b}-\frac{1}{6}\right)G_n + O(b^{-1/2+2p}). \tag{4.6}$$

To get an upper bound on the stop rule, we may assume $\delta \le 0.58$ by Lemma 2.1. Take $n = \lfloor 0.8\sqrt{b}/\alpha \rfloor$, so $\alpha n/\sqrt{b} \le 0.8$ and $S_{LE} = 0$, and $\alpha n/\sqrt{b} = 0.8 + O(b^{-1/2})$, and $p = 0$. Then

$$A = G_n = \frac{1}{\sqrt{\pi n}}\left(1+O(n^{-1})\right) = \sqrt{\frac{\alpha}{0.8}}\frac{b^{-1/4}}{\sqrt{\pi}}\left(1+O(b^{-1/2})\right).$$

By (4.5),

$$\begin{aligned}\delta_0 &= \frac{1}{2}-\frac{1}{2}\left(0.8-\frac{1}{3}0.64+\frac{1}{4}\right)G_n + O(b^{-1/2}) \\ &= \frac{1}{2} - .4184\sqrt{\frac{\alpha}{0.8}}\frac{b^{-1/4}}{\sqrt{\pi}} + O(b^{-1/2}) \\ &\ge \frac{1}{2} - \frac{0.429}{\sqrt{\pi}}b^{-1/4} + O(b^{-1/2}),\end{aligned}$$

which proves the upper bound.

For the lower bound, we decide to take $n = \lfloor 1.1\sqrt{b}/\alpha \rfloor$ so $\alpha n/\sqrt{b} = 1.1 + O(b^{-1/2})$, to get a slightly better result. For this n, it is not quite true that $S_{LE} = 0$, but the expression is still a

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lower bound. With that choice, (4.6) gives

$$\delta_0' = \frac{1}{2} - \frac{1}{2}\left(1.1 - \frac{1}{3}1.21 - \frac{1}{6}\right)\sqrt{\frac{\alpha}{1.1}}\frac{b^{-1/4}}{\sqrt{\pi}} + O(b^{-1/2})$$
$$\leq \frac{1}{2} - \frac{0.231}{\sqrt{\pi}}b^{-1/4} + O(b^{-1/2}),$$

completing the proof of (a).

For (b), we take $n = \lfloor\sqrt{b}/2\rfloor$, to be used in the next section with larger δ, where p is not zero. Then (4.5) and (4.6) with this n yield

$$1/2 - \frac{0.433}{\sqrt{\pi}}b^{-1/4} + O(b^{-1/2+2p}) \leq \delta_0 \leq \delta_0' \leq 1/2 - \frac{0.137}{\sqrt{\pi}}b^{-1/4} + O(b^{-1/2+2p}),$$

which for some b_0 implies the result in (b). □

5. Improving estimate of V near the boundary, using feedback

Our next goal is to use the above to get an improved estimate of V that is very accurate when not too far from the boundary. This will allow us to go further down the backward induction tree by estimating the leaf values $V(u+1, b+2m-1)$ for larger m, for which $u+1$ is more than 1/2 below the boundary and the value is no longer just the ratio. This is used in Section 6 to obtain the correct coefficient for the $b^{-1/4}$ term. We shall show V is approximately piecewise linear near the boundary: that is Theorem 5.1. As a first step, Lemma 4.4 showed that $V(u, b) = u/b$ if $\delta \leq 1/2 - 0.43b^{-1/4}/\sqrt{\pi}$ for b sufficiently large, already an improvement over our previous upper bound $V(u, b) \leq u/b + \alpha\delta^2 b^{-3/2}$ coming from the Brownian motion case. The Brownian upper bound is too big by about $\alpha b^{-3/2}/4$ when delta is near 1/2.

We introduce a bit more notation. In this section and the next, we will be doing some estimates, in a chained fashion, an unbounded number of times, which could cause a problem if just using big-O notation. The book [7] gives notation (attributed to de Bruijn) which will be convenient. We let $f(b) = L(g(b))$ mean that $|f(b)| \leq |g(b)|$ for all b. In set language, $L(g) = \{f\colon |f(b)| \leq |g(b)| \text{ for all } b\}$. Also, let $L^+(g) = \{f \in L(g){:} f(b) \geq 0 \text{ for all } b\}$. Similarly, we let $O^+(g)$ be the non-negative members of $O(g)$.

We assert that throughout Section 5, $n = \lfloor\sqrt{b}/2\rfloor$. Lemma 4.4 (b) showed that using this n, $1/2 - 0.44b^{-1/4}/\sqrt{\pi} \leq \delta_0 \leq \delta_0' \leq 1/2 - 0.13b^{-1/4}/\sqrt{\pi}$, $b \geq b_0$, where δ_0 satisfies $C + \delta_0 B + \delta_0^2 A = 0$, and δ_0' satisfies $C' + \delta_0' B + \delta_0'^2 A = 0$. A,B,C,C' are defined in Lemma 4.3. We assert $b_0 > 1600$. Now $n \geq \sqrt{b}/2 - 1 = \sqrt{b}/2\left(1 - 2/\sqrt{b}\right)$, so $G_n \leq 1/\sqrt{\pi n} \leq \sqrt{2.14/\pi}\,b^{-1/4} \leq 0.83b^{-1/4}$ for $b \geq b_0$, a bound that will be used several times in the following. In going further away from boundary, we will use the following lemma. Recall that $V_E(u, b) = V(u, b) - u/b$.

Lemma 5.1. *For $1/2 \leq \delta \leq b^p$, with $p \leq 1/10$, $n = \lfloor\sqrt{b}/2\rfloor$, and $b \geq b_0$,*

$$V_E(u, b) = \frac{\alpha}{b^{3/2}}\left(2(\delta - 1/2) + L\left(\max\{\delta^2, 1\}b^{-1/4}\right)\right) + S_{LE}(n, u, b).$$

Proof. Let $1/2 \leq \delta \leq b^p$. The Value is the same as the tree sum because $\delta \geq \delta_0'$, so u is below the stop boundary. $\delta_0' = 1/2 - \gamma' b^{-1/4}$ for some $0.13/\sqrt{\pi} \leq \gamma' \leq 0.44/\sqrt{\pi}$. Let $x = \delta - \delta_0' =$

$\delta - 1/2 + \gamma' b^{-1/4}$. From Lemma 4.3 in the lower bound case,

$$\begin{aligned} V_E(u, b) - S_{LE} &\geq \alpha b^{-3/2}\left(C' + B\delta + A\delta^2\right) \\ &= \alpha b^{-3/2}\left(C' + B\delta_0' + A{\delta_0'}^2 + Bx + Ax^2 + 2Ax\delta_0'\right) \\ &= \alpha b^{-3/2}\left(Bx + Ax^2 + 2Ax\delta_0'\right). \end{aligned}$$

Now $B = 2\,(1 - B_1 G_n) = 2 - 2\left(1 + \alpha n/\sqrt{b}\right) G_n \geq 2 - (2+\alpha)\, G_n$ and $A = G_n$, so $Bx + Ax^2 + 2Ax\delta_0' \geq 2(\delta - 1/2) + 2\gamma' b^{-1/4} + \left(-\,(2+\alpha)\,x + x^2 + 2x\delta_0'\right) G_n = 2(\delta - 1/2) + 2\gamma' b^{-1/4} + x\left(x - \alpha - 1 - 2\gamma'\right) G_n$. But $x\left(x - \alpha - 1 - 2\gamma'\right)$ has a minimum of $-\left(\alpha + 1 + 2\gamma'\right)^2/4 > -1$, so $x\left(x - \alpha - 1 - 2\gamma'\right) G_n > -b^{-1/4}$, so $V_E(u, b) - S_{LE} \geq \alpha b^{-3/2}\left(2(\delta - 1/2) - b^{-1/4}\right)$.

Next, let $x = \delta - \delta_0 = \delta - 1/2 + \gamma b^{-1/4}$, where $0.13/\sqrt{\pi} \leq \gamma \leq 0.44/\sqrt{\pi}$. From Lemma 4.3 in the upper bound case, similar to lower bound case, we get

$$V_E(u, b) - S_{LE} \leq \alpha b^{-3/2}(Bx + Ax^2 + 2Ax\delta_0).$$

Now $B \leq 2 - 2G_n$, so

$$\begin{aligned} Bx + Ax^2 + 2Ax\delta_0 &\leq 2(\delta - 1/2) + 2\gamma b^{-1/4} - 2xG_n + (x^2 + x)G_n \\ &= 2(\delta - 1/2) + 2\gamma b^{-1/4} + (x^2 - x)G_n. \end{aligned}$$

If $\delta \leq 1$, then $x \leq 1$, so $2\gamma b^{-1/4} + (x^2 - x)G_n \leq 2\gamma b^{-1/4} < b^{-1/4}$. If $\delta > 1$, then $x^2 - x = (\delta - \delta_0)^2 - (\delta - \delta_0) = \delta^2 - \delta - \delta_0(2\delta - \delta_0 - 1) \leq \delta^2 - 1$. So $2\gamma b^{-1/4} + (x^2 - x)G_n \leq \left(.88/\sqrt{\pi}\right) b^{-1/4} + (\delta^2 - 1)\left(\sqrt{2.14}/\sqrt{\pi}\right) b^{-1/4} \leq \delta^2 b^{-1/4}$. □

Now we look at S_{LE}, first considering a few small ranges of delta, to establish a pattern. If $0 \leq \delta \leq 1/2$, then from (4.4), $S_{LE} = 0$. If $\delta \leq \delta_0$, $V(u, b) = u/b$. Now assume $\delta_0 \leq \delta \leq 1/2$. Let $x = \delta - \delta_0$, so $x \leq \gamma b^{-1/4}$. From Lemma 4.3, proceeding as in the above proof, $V_E(u, b) \leq \alpha b^{-3/2}\left(C + B\delta + A\delta^2\right) = \alpha b^{-3/2}\left(Bx + Ax^2 + 2Ax\delta_0\right)$. With this x, it is easy to see that $Bx + Ax^2 + 2Ax\delta_0 \leq b^{-1/4}$; we do not care to be any more precise than that. We get

$$0 \leq \delta \leq 1/2 \text{ implies } V_E(u, b) = \frac{\alpha}{b^{3/2}}\left(L^+(b^{-1/4})\right), \quad b \geq b_0. \tag{5.1}$$

When $1/2 \leq \delta$, the value $V(u,b)$ is the tree sum, bigger than the ratio. The leaf sum part involves $V(u+1, b+2m+1)$, where $m \leq n-1$. The distance of $u+1$ below the boundary is $d = \max\left\{\alpha\sqrt{b+2m+1} - (u+1), 0\right\}$. But $\alpha\sqrt{b+2m+1} - (u+1) = \alpha\sqrt{b}(1 + (2m+1)/b)^{1/2} - \alpha\sqrt{b} + \delta - 1$. Standard estimates give $1 + mb^{-1} \leq (1 + (2m+1)/b)^{1/2} \leq 1 + mb^{-1} + b^{-1}/2$, so $\alpha m b^{-1/2} \leq \alpha\sqrt{b}(1 + (2m+1)/b)^{1/2} - \alpha\sqrt{b} \leq \alpha m b^{-1/2} + \alpha b^{-1/2}/2$. Thus for $m \leq n-1$,

$$d = \left(\alpha\sqrt{b+2m+1} - (u+1)\right)^+ = \left(\frac{\alpha m}{\sqrt{b}} + \delta - 1 + L^+\left(\alpha b^{-1/2}/2\right)\right)^+. \tag{5.2}$$

We will use this in several places below.

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Consider $1/2 \le \delta \le 1$. Since $\delta \le 1$, $d \le \alpha m/\sqrt{b} + L^+\left(\alpha b^{-1/2}/2\right)$. But $m \le \sqrt{b}/2$, so $d \le \alpha/2 + L^+\left(\alpha b^{-1/2}/2\right)$, and computation shows this is less than $1/2 - 0.44b^{-1/4}/\sqrt{\pi}$ if $b \ge b_0$, assumed. So $V_E(u+1, b+2m+1) = 0$ (that is, the value is just the ratio), and $S_{LE} = 0$. Then Lemma 5.1 gives

$$1/2 \le \delta \le 1 \text{ implies } V_E(u, b) = \frac{\alpha}{b^{3/2}}\left(2(\delta - 1/2) + L(b^{-1/4})\right), b \ge b_0. \tag{5.3}$$

Our improved estimate so far for $V_E(u, b)$, $0 \le \delta \le 1$, is piecewise linear plus an $O(b^{-1/4})$ correction; zero for delta from 0 to 1/2, then slope $2\alpha b^{-3/2}$ from 1/2 to 1. The piecewise-linear part is accurate to order $b^{-1/4}$, relative to $b^{-3/2}$. Compare this to our previous upper bound estimate from the Brownian value, $V_E \le V_W(u, b) - ub^{-1} \le \alpha b^{-3/2}\delta^2$, and about equal to that for delta small compared to $\sqrt{b}$; that is, quadratic for delta not too big. Our piecewise-linear function matches the quadratic one at $\delta = 0$ and $\delta = 1$, and is tangent to the parabola at those points. At $\delta = 1/2$, the piecewise-linear function is below the parabola by 1/4. That is where the upper bound from the Brownian is worst.

Now we show that same linear piece continues up to 3/2. Let $1 \le \delta \le 3/2$. The distance of $u+1$ from the boundary $\alpha\sqrt{b+2m+1}$ is, from (5.2), $d = \alpha m b^{-1/2} + \delta - 1 + L^+\left(\alpha b^{-1/2}/2\right) \le \alpha/2 + 1/2 + L^+\left(\alpha b^{-1/2}/2\right) \le 1$, for $b \ge b_0$. If also $d \ge 1/2$, we can feed this into (5.3) with d in the role of δ and $b+2m+1$ in the role of b. This is the key feedback idea that will be used in proving Theorem 5.1. If $d < 1/2$, (5.3) still holds if we replace $d - 1/2$ by $(d-1/2)^+$. So $V_E(u+1, b+2m+1) = \alpha(b+2m+1)^{-3/2}\left(2(d-1/2)^+ + L(b^{-1/4})\right)$. Note $d - 1/2 \le \alpha m b^{-1/2} + \alpha b^{-1/2}/2$. Using (4.1),

$$\begin{aligned} S_{LE} &\le \sum_{m=0}^{n-1} 2^{-2m-1} C_m \alpha (b+2m+1)^{-3/2}\left(2\alpha m b^{-1/2} + \alpha b^{-1/2} + b^{-1/4}\right) \\ &\le \alpha b^{-3/2}\left(nG_n 2\alpha b^{-1/2} + \alpha b^{-1/2} + b^{-1/4}\right) \le \alpha b^{-3/2}\left(2b^{-1/4}\right), \quad \text{for } b \ge b_0. \end{aligned}$$

Using Lemma 5.1 and adding in this estimate of S_{LE} gives the same answer as (5.3), except for increasing the error bound, which then covers (5.3) as well:

$$1/2 \le \delta \le 3/2 \text{ implies } V_E(u, b) = \frac{\alpha}{b^{3/2}}\left(2(\delta - 1/2) + L(5b^{-1/4})\right), \quad b \ge b_0. \tag{5.4}$$

Now that we see that the piecewise-linear function below and tangent to the quadratic at integers gets us this far, we guess that this pattern continues (up to some error), and Theorem 5.1 will prove this. By stepping along one 1/2 unit at a time, and feeding the result back into the previous step the way we did to get to 3/2, we will get our piecewise-linear estimate that will be good enough for our purpose.

Theorem 5.1. (Value near the boundary.) *Assume $b \ge b_0$. Then*

$$\begin{aligned} &V_E(u, b) = \frac{\alpha}{b^{3/2}}\left\{2j(\delta - 1/2) - j(j-1) + L\left(M_j b^{-1/4}\right)\right\}, \quad j - 1/2 \le \delta \le j + 1/2, \\ &0 \le j \le b^{1/10}, \end{aligned}$$

where j is integral, $M_0 = 1$, and $M_j \le 5j^3, j \ge 1$.

Figure 7 is a graph of the piecewise-linear function $2j(\delta - 1/2) - j(j-1)$ for $0 \le \delta \le 2.5$, compared to δ^2. which is shown dashed. The j changes at the half-integer points. It is tangent to

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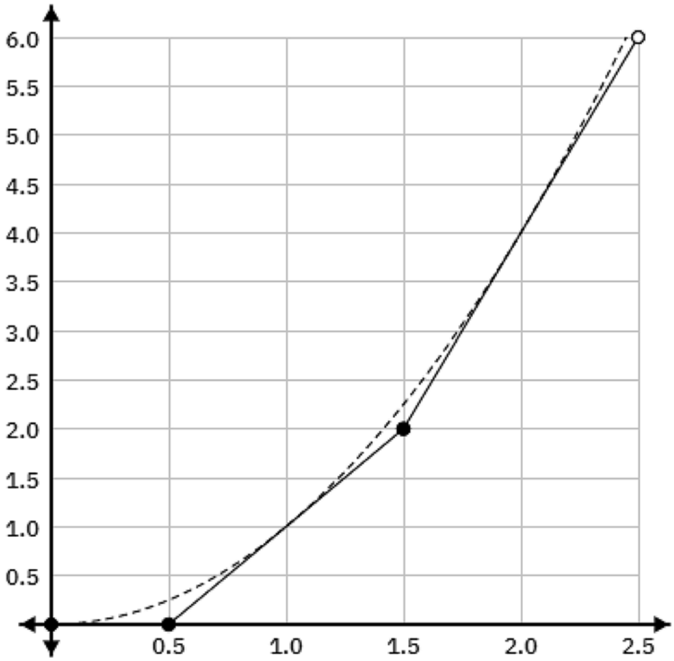

FIGURE 7. Graph of piecewise-linear approximation to scaled excess value, $\alpha^{-1}b^{3/2}V_E$, as a function of distance δ from boundary. The dashed curve is δ^2, the quadratic approximation to the scaled excess value that comes from the Brownian upper bound.

the parabola at integers, and 1/4 below the parabola at half-integers, with a straight-line graph between half-integer points.

The induction proof of this theorem uses the same feedback logic above that established the case $j = 1$ at (5.4), but it is long and tedious, so we put it in Appendix C.

6. Proof of Theorem 1.1

In this section, let $n_0 = \lfloor \sqrt{b}/\alpha \rfloor$, so $\alpha n_0/\sqrt{b} = 1 - L^+(\alpha b^{-1/2})$ and $\alpha/\sqrt{b} = 1/n_0 - L^+(\alpha b^{-1})$; we will see that replacing $\sqrt{b}/\alpha$ by its integer part is not going to matter, to the order of interest. Let $0 < p \leq 1/10$, to be decided later (in the end, it will be 1/12). Let $J + 1 = \lfloor b^p \rfloor$, so $J \sim b^p$. In this section, we are going to let $n = (J + 1)n_0$, so n will go to infinity faster than $\sqrt{b}$, but only slightly. This larger n will cause some of the leaf values to be more than just the ratio. By dividing n into J stretches of size n_0, we will get a linear approximation to the extra part of the leaf values, $V_E(u + 1, b + 2m + 1)$, on each stretch, via Theorem 5.1; that is Lemma 6.1.

Assume throughout this section that $b \geq b_0$ and $\delta = \alpha\sqrt{b} - u$. In this section, where we seek the stop rule, we only consider δ in the range $1/2 - 0.44\pi^{-1/2}b^{-1/4} \leq \delta \leq 1/2 - 0.13\pi^{-1/2}b^{-1/4}$, because we already know that the exact stop value occurs for some delta in this range. So $\delta = 1/2 - \gamma b^{-1/4}$, for some $0.13\pi^{-1/2} \leq \gamma \leq 0.44\pi^{-1/2}$.

We will estimate $V_E(u + 1, b + 2m + 1)$ by using results from Section 5, when $u + 1$ is further away from $\alpha\sqrt{b + 2m + 1}$, but in this section we will use d instead of δ to denote that distance, since δ is fixed throughout this section to be $\alpha\sqrt{b} - u$. To avoid confusion, note that

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we will be using results from Section 5 with d in the role of δ there, and $b+2m+1$ in the role of b.

By letting J go to infinity with b, at just the right rate, we can make the upper and lower bounds from Lemma 4.3 come together, to order $o(b^{-1/4})$, as we will see later.

Lemma 6.1. *There exists K such that for $\delta = 1/2 - L^+(b^{-1/4})$ and for $j = 1, \ldots, J$,*

$$V_E(u+1, b+2m+1) = \frac{\alpha}{b^{3/2}}\left\{2j\frac{m}{n_0} - j(j+1) + L\left(Kj^3 b^{-1/4}\right)\right\}, \quad jn_0 \le m \le (j+1)n_0.$$

Proof. Let $jn_0 \le m \le (j+1)n_0$. Then $d = \alpha\sqrt{b+2m+1} - (u+1) = \alpha\sqrt{b}(b+2m+1)^{1/2} - (\alpha\sqrt{b} - \delta + 1)$. This can be estimated using the binomial expansion, similar to what was done to get (5.2), but now m is bounded below by n_0 and above by $b^p n_0$, so the result is different. One can show $1 + m/b - b^{-1+2p}/(2\alpha^2) \le (1+(2m+1)/b)^{1/2} \le 1 + m/b$, so $d \le \alpha m/\sqrt{b} + \delta - 1 \le m/n_0 - 1/2 - \gamma b^{-1/4}$; and $d \ge \alpha m/\sqrt{b} - b^{-1/2+2p}/(2\alpha) + \delta - 1 \ge m/n_0 - m\alpha/b - b^{-1/2+2p}/(2\alpha) - 1/2 - \gamma b^{-1/4} \ge m/n_0 - 1/2 - b^{-1/4}$. Putting it all together, $d = m/n_0 - 1/2 - \psi b^{-1/4}$, for some $0 \le \psi \le 1$, is good enough. Note $j - 1/2 \le m/n_0 - 1/2 \le j + 1/2$.

(i) If $j - 1/2 \le d = m/n_0 - 1/2 - \psi b^{-1/4} \le j + 1/2$, then Theorem 5.1 gives

$$V_E(u+1, b+2m+1) = \alpha(b+2m+1)^{-3/2}\left\{\begin{matrix} 2j\left(m/n_0 - 1/2 - \psi b^{-1/4} - 1/2\right) \\ -j(j-1) + L\left(5j^3 b^{-1/4}\right)\end{matrix}\right\}.$$

Approximating $\alpha(b+2m+1)^{-3/2}$ and with an argument similar to earlier ones,

$$V_E(u+1, b+2m+1) = \alpha b^{-3/2}\left\{2jm/n_0 - j(j+1) + L\left((5j^3 + 2j^2 + 2j)b^{-1/4}\right)\right\}.$$

(ii) Suppose, however, that $d = m/n_0 - 1/2 - \psi b^{-1/4} \le j - 1/2$: the $\psi b^{-1/4}$ term bumped us down into the next interval below. But just barely: $m/n_0 - 1/2 \ge j - 1/2$, and $m/n_0 - 1/2 - \psi b^{-1/4} \le j - 1/2$, so $m/n_0 = j + L^+(\psi b^{-1/4})$. Using Theorem 5.1, but with $j-1$ in place of j,

$$V_E(u+1, b+2m+1) = \alpha(b+2m+1)^{-3/2}\left\{\begin{matrix} 2(j-1)\left(m/n_0 - 1/2 - \psi b^{-1/4} - 1/2\right) \\ -(j-1)(j-2) + L\left(5(j-1)^3 b^{-1/4}\right)\end{matrix}\right\}.$$

Using $m/n_0 = j + L^+(\psi b^{-1/4})$ and arguments similar to previous ones,

$$V_E(u+1, b+2m+1) = \alpha b^{-3/2}\left\{\begin{matrix} 2jm/n_0 - j(j+1) \\ +L\left(\left(5(j-1)^3 + 2j^2 + 2j\right) b^{-1/4}\right)\end{matrix}\right\}.$$

Cases (i) and (ii) are covered by $V_E(u+1, b+2m+1) = \alpha b^{-3/2}\left\{2jm/n_0 - j(j+1) + L\left(9j^3 b^{-1/4}\right)\right\}$, so $K = 9$ is good enough. □

To prove the main theorem, we have to put in the extra contribution S_{LE} to the leaf sum that comes from the leaf values that exceed the ratio, when going down to $n = (J+1)n_0$.

Lemma 6.2.

$$S_{LE} = \frac{\alpha}{b^{3/2}}\left\{C_{LE}G_n + O\left(J^{5/2} b^{-1/2}\right)\right\},$$

where

$$C_{LE} := \frac{1}{3}J^2 - \frac{1}{3}J - 4\zeta(-1/2)\sqrt{J+1} + O(1).$$

Proof. Let $S_{LE}(j)$ denote the contribution to S_{LE} from summing $2^{-2m-1}C_m V_E(u+1, b+2m+1)$ over m in the range $jn_0 \le m < (j+1)n_0$. Let $W(j) = S_{LE}(j)b^{3/2}/\alpha$. $W(0) = 0$. Now let $j \ge 1$. By Lemmas 6.1 and 4.1,

$$\begin{aligned} W(j) &= \frac{2j}{n_0}\sum_{m=jn_0}^{(j+1)n_0-1} m2^{-2m-1}C_m - \left(j(j+1) + L\left(Kj^3b^{-1/4}\right)\right)\sum_{m=jn_0}^{(j+1)n_0-1} 2^{-2m-1}C_m \\ &= \frac{2j}{n_0}\left((j+1)n_0G_{(j+1)n_0} - jn_0G_{jn_0}\right) - \left(\frac{2j}{n_0} + j(j+1) + L\left(Kj^3b^{-1/4}\right)\right)\left(G_{jn_0} - G_{(j+1)n_0}\right) \\ &= 3j(j+1)G_{(j+1)n_0} - j(3j+1)G_{jn_0} + L\left((K+1)j^3b^{-1/4}\right)\left(G_{jn_0} - G_{(j+1)n_0}\right). \end{aligned}$$

Now

$$G_{jn_0} - G_{(j+1)n_0} \le \frac{1}{\sqrt{\pi}}\left(\frac{1}{\sqrt{n_0 j}} - \frac{1}{\sqrt{n_0(j+1)+1/2}}\right) \le \frac{j^{-3/2}}{2\sqrt{\pi n_0}} \le \frac{j^{-3/2}b^{-1/4}}{2\sqrt{\pi}},$$

and

$$\frac{1}{\sqrt{k\pi}}\left(1 - \frac{1}{4k}\right) \le G_k \le \frac{1}{\sqrt{k\pi}} \implies \frac{G_{jn_0}}{G_{(J+1)n_0}} = \frac{\sqrt{J+1}}{\sqrt{j}}\left(1 + L\left(\frac{1}{4jn_0}\right)\right).$$

Using the latter to express G_{jn_0} and $G_{(j+1)n_0}$ in terms of $G_{(J+1)n_0}$,

$$\begin{aligned} W(j) &= \begin{pmatrix} 3j\sqrt{j+1}\left(1 + L\left((4(j+1)n_0)^{-1}\right)\right) \\ -\sqrt{j}(3j+1)\left(1 + L\left((4jn_0)^{-1}\right)\right) \end{pmatrix}\sqrt{J+1}G_{(J+1)n_0} + L\left((K+1)j^{3/2}b^{-1/2}\right) \\ &= \left(3j\sqrt{(j+1)} - 3(j-1)\sqrt{j} - 4\sqrt{j}\right)\sqrt{J+1}G_n + L\left((K+2)j^{3/2}b^{-1/2}\right). \end{aligned}$$

We wrote it that way so that telescoping occurs for the first two terms when summing

$$\begin{aligned} \sum_{j=1}^{J} W(j) &= \left(\sum_{j=1}^{J}\left(3j\sqrt{(j+1)} - 3(j-1)\sqrt{j}\right) - 4\sum_{j=1}^{J}\sqrt{j}\right)\sqrt{J+1}G_n + O\left(J^{5/2}b^{-1/2}\right) \\ &= \left(3J\sqrt{(J+1)} - 4\sum_{j=1}^{J}\sqrt{j}\right)\sqrt{J+1}G_n + O\left(J^{5/2}b^{-1/2}\right). \end{aligned}$$

At this point we are through summing over an unbounded range, so we replaced the big-L with big-O notation, with no harm. There is a well-known asymptotic formula for the sum of square roots as a generalized harmonic number: $H_J^{(-1/2)} = \sum_{j=1}^{J}\sqrt{j} = 2J^{3/2}/3 + J^{1/2}/2 + \zeta(-1/2) + O\left(J^{-1/2}\right)$ [7, p. 594]. And $\sqrt{J+1} = J^{1/2} + J^{-1/2}/2 + O\left(J^{-3/2}\right)$, so $4H_J^{(-1/2)}\sqrt{J+1} = 8J^2/3 + 10J/3 + 4\zeta(-1/2)\sqrt{J+1} + O(1)$, and $3J(J+1) - 4H_J^{(-1/2)}\sqrt{J+1} = J^2/3 - J/3 - 4\zeta(-1/2)\sqrt{J+1} + O(1)$. So $S_{LE} = \alpha b^{-3/2}\left\{\left(J^2/3 - J/3 - 4\zeta(-1/2)\sqrt{J+1} + O(1)\right)G_n + O\left(J^{5/2}b^{-1/2}\right)\right\}$. □

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We can now conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. Go back to Lemma 4.3, to get the bounds on the tree sum using this n, adding in the S_{LE} term from Lemma 6.2. Our assumption about J implies $n = O(b^{1/2+p})$ with $p \le 1/10$, so Lemma 4.3 applies. For the upper bound, $TreeSum(n, u, b) - u/b \le \alpha b^{-3/2}\left\{C + \delta B + \delta^2 A\right\} + S_{LE} = \alpha b^{-3/2}\left\{C^* + \delta B + \delta^2 A\right\}$, where $C^* = -(1-(C_1 + C_{LE})G_n) + O(J^{5/2} b^{-1/2})$. Proceeding as done in the proof of Lemma 4.4,

$$\delta_0 = \frac{-B + B\left(1 - 2AC^*B^{-2} - 2A^2C^{*2}B^{-4} - O(A^3)\right)}{2A} = -\frac{C^*}{B} - \frac{G_n C^{*2}}{B^3} + O(b^{-1/2-p}),$$

since $A = G_n = O(b^{-1/4-p/2})$. As in the earlier proof,

$$-\frac{C^*}{B} - \frac{G_n C^{*2}}{B^3} = \frac{1}{2} - \frac{1}{2}\left(C_1 + C_{LE} - B_1 + \frac{1}{4}\right) G_n + O(J^{5/2} b^{-1/2}).$$

But $\dfrac{\alpha n}{\sqrt{b}} = J + 1 + O\left((J+1)b^{-1/2}\right)$, so $C_1 + C_{LE} - B_1 + \dfrac{1}{4} = C_{LE} + \dfrac{\alpha n}{\sqrt{b}} - \dfrac{\alpha^2 n^2}{3b} + \dfrac{1}{4} = \dfrac{1}{3}J^2 - \dfrac{1}{3}J - 4\zeta(-1/2)\sqrt{J+1} - \dfrac{1}{3}J^2 + \dfrac{1}{3}J + O(1) = -4\zeta(-1/2)\sqrt{J+1} + O(1)$. The higher-power terms miraculously canceled, exposing the dominant zeta term. So

$$\begin{aligned}
\delta_0 &= \frac{1}{2} - \frac{1}{2}\left(-4\zeta(-1/2)\sqrt{J+1} + O(1)\right)\frac{1}{\sqrt{\pi(J+1)n_0}} + O\left(J^{5/2}b^{-1/2}\right) \\
&= \frac{1}{2} - \frac{1}{2}\left(-4\zeta(-1/2) + O(J^{-1/2})\right)\frac{\sqrt{\alpha}}{\sqrt{\pi}} b^{-1/4} + O(J^{5/2}b^{-1/2}) \\
&= \frac{1}{2} - \frac{(-2\zeta(-1/2))\sqrt{\alpha}}{\sqrt{\pi}} b^{-1/4} + O(J^{-1/2}b^{-1/4}) + O(J^{5/2}b^{-1/2}).
\end{aligned}$$

Since $J \sim b^p$, the best we can do with this is to let $p = 1/12$. We get

$$\delta_0 = \frac{1}{2} - \frac{(-2\zeta(-1/2))\sqrt{\alpha}}{\sqrt{\pi}} b^{-1/4} + O(b^{-7/24}).$$

For the lower bound, to get δ_0', the only change is that C_1 is replaced by $C_1' = C_1 - 5/12$, which does not affect the calculation to order $O(b^{-7/24})$. This completes the proof of Theorem 1.1. □

As a final remark, we could say that we expected those higher terms to cancel, based on the idea that going further down the tree leads to less weight on the row and more weight on the leaves, where, at least for a while, the errors are quite small thanks to Theorem 5.1. But honestly, when that happened with just the right choice of p, we thanked Tyche rather than crediting our insight, since we did not really know if it would happen before doing it.

Appendix A. Proof of Lemma 1.2

We will need the following estimate of the expected reciprocal which goes the other way.

Lemma A.1.

$$E\left[\frac{b+n}{b+T_n}\right] \le 1 + \frac{1}{6b}, \quad \text{where } T_n \text{ is as in Section 1.2.}$$

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Proof. We use the central moments $E\left[(T_1-1)^2\right]=2/3$, $E\left[(T_1-1)^3\right]=16/15$; this is from the Laplace transform $f(t)=E[e^{-tT_1}]=\left(\cosh\sqrt{2t}\right)^{-1}$, $t>0$ [2, p. 289]. Now

$$\frac{b+n}{b+T_n}=\frac{b+n}{b+n+T_n-n}=\left(1+\frac{T_n-n}{b+n}\right)^{-1}=1-\frac{T_n-n}{b+n}+\left(\frac{T_n-n}{b+n}\right)^2\left(1+\frac{T_n-n}{b+n}\right)^{-1}$$
$$=1-\frac{T_n-n}{b+n}+\frac{(T_n-n)^2}{(b+n)^2}-\frac{(T_n-n)^3}{(b+n)^2\,(b+n+T_n-n)}.$$

Note that $T_n-n=\sum_{j=1}^{n}(T_j-T_{j-1}-1)$is the sum of n i.i.d. mean-zero random variables, so $E\left[(T_n-n)^2\right]=\sum_{j=1}^{n}E\left[\left(T_j-T_{j-1}-1\right)^2\right]=nE\left[(T_1-1)^2\right]=2n/3$, and similarly, $E\left[(T_n-n)^3\right]=nE\left[(T_1-1)^3\right]=16n/15$. In both cases, the cross terms disappear because of independence and mean zero. Note the function $f(x)=x^3/(c+x)$ is convex for $x\geq-3c/2$, and $T_n-n\geq-3(b+n)/2$ is always true because $T_n\geq 0$. By Jensen,

$$E\left[\frac{(T_n-n)^3}{(b+n+T_n-n)}\right]\geq\frac{E\left[(T_n-n)^3\right]}{(b+n+E\left[T_n-n\right])}=\frac{n}{b+n}>0.$$

Thus

$$E\left[\frac{b+n}{b+T_n}\right]\leqslant 1-\frac{E\left[T_n-n\right]}{b+n}+\frac{E\left[(T_n-n)^2\right]}{(b+n)^2}=1+\frac{(2/3)n}{(b+n)^2}.$$

It is easy to show that $n(b+n)^{-2}\leq\frac{1}{4b}$, so

$$E\left[\frac{b+n}{b+T_n}\right]\leq 1+\frac{1}{6b}.$$

□

We need to estimate the loss in Value for the Brownian motion case if we get as close to the optimal boundary as possible while restricted to only stopping at integer steps from the start of the Brownian motion. Let b be an integer, and $u<\alpha\sqrt{b}$. Let $f(t)=\lfloor\alpha\sqrt{b+t}\quad-u+\quad 1/2\rfloor$. Let T be the first time t that $W(t)=f(t)$ (set $T=\infty$ if there is no such t). Let $F(t)=P[T\leq t]$. We will follow Shepp [13], using his Wald fundamental identity argument, except with this f.

Lemma A.2. *For* $\lambda\geqslant 0$, $\int_0^\infty dF(t)\exp\left\{\lambda\left(\left\lfloor\alpha\sqrt{b+t}\quad-u+\quad 1/2\right\rfloor\right)-\lambda^2 t/2\right\}=1$.

Proof. This follows immediately from [13, Theorem 2, p. 996 and immediately below]. That theorem stipulated that f be continuous, but the continuity was used only to justify the implication $T\geq t\Rightarrow W(t)\leq f(t)$, and this is true when f is monotone non-decreasing, without requiring continuity. □

Proof of Lemma 1.2. As in [13], multiply both sides in Lemma A.2 by $\exp\left(\lambda u-\lambda^2 b/2\right)$ and integrate over λ from 0 to ∞, getting

$$I:=\int_0^\infty dF(t)\int_0^\infty\exp\left\{\lambda\left(\left\lfloor\alpha\sqrt{b+t}-u+1/2\right\rfloor+u\right)-\lambda^2(b+t)/2\right\}d\lambda$$
$$=\int_0^\infty\exp\left(\lambda u-\lambda^2 b/2\right)d\lambda=V_W(u,b)/(1-\alpha^2).$$

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Define $r(t) = \lfloor \alpha\sqrt{b+t} - u + 1/2 \rfloor - \alpha\sqrt{b+t} + u$, so $\lfloor \alpha\sqrt{b+t} - u + 1/2 \rfloor = \alpha\sqrt{b+t} - u + r(t)$, where $-1/2 \le r(t) < 1/2$. Let $\varepsilon = \varepsilon(t) = r(t)(b+t)^{-1/2}$, so

$$I = \int_0^\infty dF(t) \int_0^\infty \exp\left\{\lambda\sqrt{b+t}\,(\alpha+\varepsilon) - \lambda^2(b+t)/2\right\} d\lambda.$$

Making the change of variable $w = \lambda\sqrt{b+t} - (\alpha+\varepsilon)$ to complete the square in the integral yields

$$\begin{aligned} I &= \int_0^\infty dF(t)(b+t)^{-1/2} \exp\left((\alpha+\varepsilon)^2/2\right) \int_{-(\alpha+\varepsilon)}^\infty \exp\left(-w^2/2\right) dw \\ &= \int_0^\infty dF(t)(b+t)^{-1/2} G(\alpha+\varepsilon)/g(\alpha+\varepsilon) = \int_0^\infty dF(t)(b+t)^{-1/2} H(\alpha+\varepsilon), \end{aligned}$$

where G and g are respectively the CDF and PDF of the standard normal, and $H = G/g$ was defined in Section 3. To summarize so far,

$$V_W(u,b) = (1-\alpha^2) \int_0^\infty dF(t)(b+t)^{-1/2} H(\alpha+\varepsilon). \tag{A.1}$$

This is a perturbation of what it would be for the Brownian optimal boundary, for $\varepsilon = 0$. To approximate the perturbation, the derivatives of H given in Section 3 will be useful. $H(\alpha+\varepsilon) = H(\alpha) + \varepsilon H'(\alpha) + \varepsilon^2 H''(\alpha)/2 + \varepsilon^3 H'''(z)/6$ for some z between α and $\alpha+\varepsilon$ (note ε can be negative). From (3.3), and recalling $H(\alpha) = \alpha/(1-\alpha^2)$, using the terms through the second derivative gives

$$\begin{aligned} &H(\alpha) + \varepsilon H'(\alpha) + \varepsilon^2 H''(\alpha)/2 \\ &\qquad = H(\alpha) + \varepsilon\,(\alpha H(\alpha) + 1) + \varepsilon^2 \left(\left(1+\alpha^2\right) H(\alpha) + \alpha\right)/2 \\ &\qquad = H(\alpha)\left\{1 + \varepsilon\left(\alpha + (1-\alpha^2)/\alpha\right) + \varepsilon^2\left(\left(1+\alpha^2\right) + (1-\alpha^2)\right)/2\right\} \\ &\qquad = H(\alpha)\left\{1 + \varepsilon/\alpha + \varepsilon^2\right\}. \end{aligned}$$

For the third derivative term, we just want to bound it. Section 3 noted all derivatives of H are positive, so for $\varepsilon \le 0$, $\varepsilon^3 H'''(z) \le 0$. For $\varepsilon > 0$, $\varepsilon^3 H'''(z) \le \varepsilon^3 H'''(\alpha+\varepsilon)$, and also, since H' is increasing, $H(\alpha+\varepsilon) \le H(\alpha) + \varepsilon H'(\alpha+\varepsilon) = H(\alpha) + \varepsilon\,((\alpha+\varepsilon)H(\alpha+\varepsilon) + \alpha + \varepsilon)$. Solving, $H(\alpha+\varepsilon) \le H(\alpha)\left(\alpha + \varepsilon(1-\alpha^2)\right)(1-\varepsilon(\alpha+\varepsilon)) \le 1.0152 H(\alpha)$ assuming $b > 1600$ so that $\varepsilon = r(t)(b+t)^{-1/2} \le 1/80$. Then from (3.3),

$$H'''(\alpha+\varepsilon) \le \left\{\left(3(\alpha+\varepsilon) + (\alpha+\varepsilon)^3\right)(1.0152) + \left(2 + (\alpha+\varepsilon)^2\right)/H(\alpha)\right\} H(\alpha) \le 4.2 H(\alpha).$$

Thus

$$\begin{aligned} H(\alpha+\varepsilon) &= H(\alpha) + \varepsilon H'(\alpha) + \frac{\varepsilon^2}{2} H''(\alpha) + \frac{\varepsilon^3}{6} H'''(z) \le H(\alpha)\left\{1 + \frac{\varepsilon}{\alpha} + \varepsilon^2 + .7|\varepsilon|^3\right\} \\ &\le \frac{\alpha}{1-\alpha^2}\left\{\frac{\alpha\sqrt{b+t} + r(t)}{\alpha\sqrt{b+t}} + \frac{1}{4(b+t)} + \frac{.7}{8(b+t)^{3/2}}\right\} \end{aligned}$$

$$\le \frac{\alpha}{1-\alpha^2}\frac{\alpha\sqrt{b+t}+r(t)}{\alpha\sqrt{b+t}}\left\{1+\frac{\alpha\sqrt{b+t}}{4(b+t)\left(\alpha\sqrt{b+t}-1/2\right)}\left(1+\frac{.35}{(b+t)^{1/2}}\right)\right\}$$
$$\le \frac{\sqrt{b+t}}{1-\alpha^2}\frac{W(t)+u}{b+t}\left\{1+\frac{1}{4b}\left(1-\frac{1}{2\alpha\sqrt{b}}\right)^{-1}\left(1+\frac{.35}{\sqrt{b}}\right)\right\}$$
$$\le \frac{\sqrt{b+t}}{1-\alpha^2}\frac{W(t)+u}{b+t}\left\{1+\frac{1}{4b}\left(1+\frac{1}{b^{1/2}}\right)\right\}, \text{ using } b>1600 \text{ in the last step.}$$

Referring back to (A.1),

$$\begin{aligned} V_W(u, b) &= (1-\alpha^2)\int_0^\infty dF(t)(b+t)^{-1/2}H(\alpha+\varepsilon) \\ &\le \int_0^\infty dF(t)\frac{W(t)+u}{b+t}\left\{1+\frac{1}{4b}\left(1+\frac{1}{\sqrt{b}}\right)\right\} \\ &= E\left[\frac{W(T)+u}{b+T}\right]\left\{1+\frac{1}{4b}\left(1+\frac{1}{\sqrt{b}}\right)\right\}, \end{aligned}$$

or

$$E\left[\frac{W(T)+u}{b+T}\right]\left\{1+\frac{1}{4b}\left(1+\frac{1}{\sqrt{b}}\right)\right\} \geqslant V_W(u, b). \tag{A.2}$$

Now embed the random walk in the Brownian motion as in Section 1.2, with $S_n = W(T_n)$; whenever $W(t)$ is an integer, $W(t) = W(T_n)$ for some n. For stop rule T, $W(T)$ only takes integer values. Let n^* be the first n such that $W(T) = W(T_n)$.

$$\begin{aligned} E\left[\frac{W(T)+u}{b+T}\right] &= \sum_{n=0}^{\infty} E\left[\frac{u+S_n}{b+n}\frac{b+n}{b+T_n}\,\middle|\, n^*=n\right]P(n^*=n) \\ &= \sum_{n=0}^{\infty} E\left[\frac{b+n}{b+T_n}\right]E\left[\frac{u+S_n}{b+n}\,\middle|\, n^*=n\right]P(n^*=n) \le \left(1+\frac{1}{6b}\right)V(u, b), \end{aligned}$$

using Lemma A.1. Combining with (A.2),

$$V(u, b) \ge V_W(u, b)\left(1-\frac{1}{4b}\left(1+\frac{1}{\sqrt{b}}\right)\right)\left(1-\frac{1}{6b}\right) \ge V_W(u, b)\left(1-\frac{5}{12b}\left(1+\frac{1}{\sqrt{b}}\right)\right).$$

□

REMARK. Numerical data suggests $V(u, b) \ge V_W(u, b)\left(1-\frac{1}{4b}\left(1+\frac{1}{\sqrt{b}}\right)\right)$.

Now the inequality $n(b+n)^{-2} \leqslant 1/(4b)$ used in proving Lemma A.1 is a gross overestimate when u is near the boundary $\alpha\sqrt{b}$, because in that case the time to crossing will probably be for n small compared to b. On the other hand, when u is far below $\alpha\sqrt{b}$, our replacing $(b+t)^{-1}/4$ by $b^{-1}/4$ in steps leading up to (A.2) is a significant overestimate, because t will be typically of order b in order to get to the boundary. But we will not pursue these things, because improving 5/12 to 1/4 in Lemma 1.2 would only slightly improve the lower estimate in Lemma 4.3, and the real goal is to prove Theorem 1.1.

Appendix B. Proof of Lemma 4.2

Proof of Lemma 4.2. $V(u-2j+1, b+2n-1) \le V_W(u-2j+1, b+2n-1)$, and the displacement of $u-2j+1$ from the Brownian boundary is

$$\begin{aligned} d &= u-2j+1-\alpha\sqrt{b+2n-1} \\ &= u-2j+1-\alpha\sqrt{b}\left(1+\frac{2n-1}{2b}\left(1-\tau(n)\frac{2n-1}{b}\right)\right) \\ &= -\left[2j-(1-\delta-\psi(n))\right], \end{aligned}$$

where $\psi(n)=\alpha\dfrac{n-1/2}{\sqrt{b}}\left(1-\tau(n)\dfrac{2n-1}{b}\right)$, $0<\tau(n)<1/4$. Now $\psi(n)\le \alpha n/\sqrt{b}$, and $\psi(n)=\alpha n/\sqrt{b}-O(b^{-1/2+2p})$. Thus, from (3.7),

$$\begin{aligned} V_W(u-2j+1, b+2n-1) \le \frac{u-2j+1}{b+2n-1} + \alpha\frac{\left[2j-(1-\delta-\psi(n))\right]^2}{(b+2n-1)^{3/2}} \\ -\frac{\left(1+\alpha^2\right)}{3}\frac{\left[2j-(1-\delta-\psi(n))\right]^3}{(b+2n-1)^2}+\frac{\alpha\left(4+\alpha^2\right)}{12}\frac{\left[2j-(1-\delta-\psi(n))\right]^4}{(b+2n-1)^{5/2}}. \end{aligned} \tag{B.1}$$

By (3.8), we get a lower bound for $V_W(u-2j+1, b+2n-1)$ by subtracting

$$\frac{\left(4+8\alpha^2+\alpha^4\right)}{60}\frac{\left[2j-(1-\delta-\psi(n))\right]^5}{(b+2n-1)^3}$$

from this. To get S_{RW}, these terms are to be weighted by $2^{-2n+1}B(n,j)$ and summed for j from 1 through n. But $2^{-2n+1}\sum_{j=1}^{n} j^5 B(n,j)=O\left(n^2\right)$, so this last term would add $O\left(n^2 b^{-3}\right)=O\left(b^{-2+2p}\right)$, so the upper and lower bounds of the sum are the same to that order, and only involve the terms up through the fourth power of j. We break the sum into four pieces, $s_1, \ldots, s_4$, corresponding to the four terms of the Taylor approximation on the right side of (B.1). The sums are done using (4.2) throughout. Some of the detail in obtaining the O bounds from the asserted bounds on n and δ is left to the reader. Let

$$\begin{aligned} s_1 &= 2^{-2n+1}\sum_{j=1}^{n} B(n,j)\frac{u-2j+1}{b+2n-1} \\ &= \frac{u+1}{b+2n-1}2^{-2n+1}\sum_{j=1}^{n}B(n,j)-\frac{2}{b+2n-1}2^{-2n+1}\sum_{j=1}^{n}jB(n,j)=\frac{(u+1)G_n-1}{b+2n-1} \\ &= \frac{(u+1)G_n-1}{b}\left(1-\frac{2n-1}{b}+\frac{(2n-1)^2}{b^2}+O(n^3b^{-3})\right) \\ &= \frac{(u+1)G_n-1}{b}-\frac{(u+1)G_n}{b^2}\left(2n-1-\frac{4n^2}{b}\right)+\frac{2n}{b^2}+O(b^{-2+p}). \end{aligned}$$

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The first term will conveniently cancel when we add to the leaf sum, so we leave that as is, but rewrite the second with δ rather than u as the variable:

$$s_1 - \frac{(u+1)G_n - 1}{b} = -\frac{\alpha\sqrt{b}+1-\delta}{b^2}\left(2nG_n - G_n - 4\frac{n^2G_n}{b}\right) + \frac{2n}{b^2} + O(b^{-2+p})$$
$$= -\frac{\alpha}{b^{3/2}}\left(1+\frac{1-\delta}{\alpha\sqrt{b}}\right)\left(2nG_n - G_n - 4\frac{n^2G_n}{b}\right) + \frac{\alpha}{b^{3/2}}\frac{2n}{\alpha\sqrt{b}} + O(b^{-2+p}).$$

Simplifying,

$$s_1 - \frac{(u+1)G_n - 1}{b} = \frac{\alpha}{b^{3/2}}\left\{\begin{array}{c} -2nG_n + G_n + 2\dfrac{n}{\alpha\sqrt{b}} - 2\dfrac{nG_n}{\alpha\sqrt{b}} + 4\dfrac{n^2G_n}{b} \\ +2\delta\left(\dfrac{nG_n}{\alpha\sqrt{b}}\right) + O(b^{-1/2+p}) \end{array}\right\}. \tag{B.2}$$

Let $h = 1-\delta-\psi(n)$, to simplify the writing for the next three terms. Let

$$s_2 = \frac{\alpha}{(b+2n-1)^{3/2}} 2^{-2n+1}\sum_{j=1}^{n} B(n,j)\big[2j-(1-\delta-\psi(n))\big]^2$$
$$= \frac{\alpha}{b^{3/2}}\left(1 - \frac{3}{2}\frac{(2n-1)}{b} + O(b^{-1+2p})\right) 2^{-2n+1}\sum_{j=1}^{n} B(n,j)\left[4j^2 - 4jh + h^2\right]$$
$$= \frac{\alpha}{b^{3/2}}\left(1 - \frac{3n}{b} + O(b^{-1+2p})\right)\left[4nG_n - 2h + G_nh^2\right].$$

This can be arranged to give (recall $\psi(n) = \alpha n/\sqrt{b} - O(b^{-1/2+2p})$)

$$s_2 = \frac{\alpha}{b^{3/2}}\left\{\begin{array}{c} 4nG_n - 2 + 2\alpha\dfrac{n}{\sqrt{b}} + G_n - \left(12-\alpha^2\right)\dfrac{n^2G_n}{b} \\ -2\alpha\dfrac{nG_n}{\sqrt{b}} + 2\delta\left(1 - G_n + \alpha\dfrac{nG_n}{\sqrt{b}}\right) + \delta^2G_n + O(b^{-1/2+2p}) \end{array}\right\}. \tag{B.3}$$

For the next two terms, note $h = O\left(b^p\right)$ to help simplify. Let

$$s_3 = -\frac{(1+\alpha^2)}{3(b+2n-1)^2} 2^{-2n+1}\sum_{j=1}^{n} B(n,j)\big[2j-(1-\delta-\psi(n))\big]^3$$
$$= -\frac{(1+\alpha^2)}{3b^2}\left(1 - \frac{4n}{b} + O(b^{-1+2p})\right) 2^{-2n+1}\sum_{j=1}^{n} B(n,j)\left[8j^3 - 12j^2h + 6jh^2 - h^3\right]$$
$$= -\frac{\alpha}{b^{3/2}}\frac{(1+\alpha^2)}{3\alpha\sqrt{b}}\left(1 + O(b^{-1/2+p})\right)\left[6n - 2 - 12nG_nh + 3h^2 - G_nh^3\right].$$

The h^2 and h^3 terms contribute only $O(b^{-2+2p})$, so this becomes

$$s_3 = \frac{\alpha}{b^{3/2}}\frac{(1+\alpha^2)}{\alpha}\left\{-\frac{2n}{\sqrt{b}} + \frac{4nG_n}{\sqrt{b}} - \frac{4\alpha n^2G_n}{b} - \delta\left(\frac{4nG_n}{\sqrt{b}}\right) + O(b^{-1/2+2p})\right\}. \tag{B.4}$$

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Let

$$s_4 = \frac{\alpha(4+\alpha^2)}{12(b+2n-1)^{5/2}} 2^{-2n+1} \sum_{j=1}^{n} B(n,j)\big[2j-(1-\delta-\psi(n))\big]^4$$
$$= \frac{\alpha(4+\alpha^2)}{12b^{5/2}} \left(1+O(b^{-1/2+p})\right) 2^{-2n+1} \sum_{j=1}^{n} B(n,j) \begin{bmatrix} 16j^4 - 32j^3h \\ +24j^2h^2 - 8jh^3 + h^4 \end{bmatrix}$$
$$= \frac{\alpha}{b^{3/2}} \frac{(4+\alpha^2)}{12b} \left(1+O(b^{-1/2+p})\right) \begin{bmatrix} 16n(2n-1)G_n - 8(3n-1)h \\ +24nG_nh^2 - 4h^3 + G_nh^4 \end{bmatrix}.$$

This reduces to

$$s_4 = \frac{\alpha}{b^{3/2}} \left\{ \frac{8(4+\alpha^2)}{3} \frac{n^2G_n}{b} + O(b^{-1/2+2p}) \right\}. \tag{B.5}$$

Combining (B.2) through (B.5),

$$S_{RW} - \frac{(u+1)G_n - 1}{b} = \frac{\alpha}{b^{3/2}} \left\{ \begin{array}{c} 2nG_n - 2 + 2G_n + 2(\alpha+1/\alpha)\dfrac{nG_n}{\sqrt{b}} - \dfrac{(4+\alpha^2)}{3}\dfrac{n^2G_n}{b} \\ +2\delta\left(1 - G_n - (\alpha+1/\alpha)\dfrac{nG_n}{\sqrt{b}}\right) + \delta^2 G_n + O(b^{-1/2+2p}) \end{array} \right\}.$$

□

Appendix C. Proof of Theorem 5.1

Proof of Theorem 5.1. The proof is by induction. The case $j=0$ is (5.1). The base case for the induction proof is $j=1$, which was done at (5.4). Assume true for j, show true for $j+1$. That is, assume formula is true for $j-1/2 \le \delta \le j+1/2$. Continue to use $n = \lfloor \sqrt{b}/2 \rfloor$. Now consider $j+1/2 \le \delta \le j+3/2$. Break the proof into cases.

Case (a): $j+1/2 \le \delta \le j+1$. We need to estimate $V(u+1, b+2m+1)$ for $0 \le m < n$. From (5.2), the distance of $u+1$ from boundary is $d = \alpha m b^{-1/2} + \delta - 1 + L^+\left(\alpha b^{-1/2}/2\right)$, so $\alpha m b^{-1/2} + j - 1/2 \le \alpha m b^{-1/2} + \delta - 1 + L^+\left(\alpha b^{-1/2}/2\right) \le \alpha m b^{-1/2} + j + L^+\left(\alpha b^{-1/2}/2\right)$. Thus for $0 \le m < n \le \sqrt{b}/2$, we have $j - 1/2 \le d \le j + \alpha/2 + L^+\left(\alpha b^{-1/2}/2\right) < j+1/2$ since $b \ge b_0$, so the induction hypothesis applies with d in the place of δ, and $b+2m+1$ in the place of b. So $V_E(u+1, b+2m+1) = \alpha(b+2m+1)^{-3/2}\left\{2j\,(d-1/2) - j(j-1) + L(M_j b^{-1/4})\right\}$. But $\alpha(b+2m+1)^{-3/2} = \alpha b^{-3/2}\left(1 - L^+(1.5b^{-1/2})\right)$, and $2j\,(d-1/2) - j(j-1) \le j^2 + j$. Using $j \le b^{1/10}$ and $b \ge b_0$, $V_E(u+1, b+2m+1) = \alpha b^{-3/2} \left\{2j\left(\alpha m b^{-1/2} + \delta - 3/2\right) - j(j-1) + L\left((M_j+2)b^{-1/4}\right)\right\}$. So $\alpha^{-1} b^{3/2} S_{LE} = \sum_{m=0}^{n-1} 2^{-2m-1} C_m \left\{\left(2j\left(\alpha m b^{-1/2} + \delta - 3/2\right) - j(j-1) + L\left((M_j+2)b^{-1/4}\right)\right)\right\} = (nG_n - 1 + G_n)\, 2j\alpha b^{-1/2} + (2j(\delta-3/2) - j(j-1))\,(1-G_n) + L\left((M_j+2)b^{-1/4}\right)$. But $0 \le (nG_n - 1 + G_n)\, 2j\alpha b^{-1/2} \le \alpha j b^{-1/4}$, and $0 \ge -\left(2j(\delta-3/2) - j(j-1)\right) G_n \ge -j^2 b^{-1/4}$, and $j^2 \ge \alpha j$, so $\alpha^{-1} b^{3/2} S_{LE} = (2j(\delta-3/2) - j(j-1)) + L\left(\left(M_j + j^2 + 2\right) b^{-1/4}\right)$. From Lemma 5.1, to get $\alpha^{-1} b^{3/2} V_E$, add this to $2(\delta - 1/2) + L(\delta^2 b^{-1/4})$. Noting $\delta \le j+1$, this gives

$$\frac{b^{3/2}}{\alpha} V_E(u,b) = (2(j+1)(\delta-1/2) - j(j+1)) + L\left((M_j + 2j^2 + 2j + 3)b^{-1/4}\right). \tag{C.1}$$

Case (b): $j+1 \le \delta \le j+3/2$. Let $m_0 = \min\left\{\left\lfloor \frac{j+3/2-\delta}{\alpha}\sqrt{b}\right\rfloor, n\right\}$. Break m into ranges.

Range (i): For $0 \le m \le m_0 - 1$, $\alpha m b^{-1/2} \le \alpha m_0 b^{-1/2} - \alpha b^{-1/2} \le j + 3/2 - \delta - \alpha b^{-1/2}$, so $d = \alpha m b^{-1/2} + \delta - 1 + L^+\left(\alpha b^{-1/2}/2\right) \le j + 3/2 - \delta - \alpha b^{-1/2} + \delta - 1 + L^+\left(\alpha b^{-1/2}/2\right) \le j + 1/2$, so the induction hypothesis applies with d in place of δ and $b + 2m + 1$ in place of b. The same argument that was used in (a) then shows $V_E(u+1, b+2m+1) = \alpha b^{-3/2}\left\{2j\left(\alpha m b^{-1/2} + \delta - 3/2\right) - j(j-1) + L\left((M_j+2)b^{-1/4}\right)\right\}$.

Range (ii): Suppose $m_0 + 1 \le m < n$ (this could be empty). Then $m \ge m_0 + 1 \ge \frac{j+3/2-\delta}{\alpha}\sqrt{b}$, so $d = \alpha m b^{-1/2} + \delta - 1 + L^+\left(\alpha b^{-1/2}/2\right) \ge j + 3/2 - \delta + \delta - 1 + L^+\left(\alpha b^{-1/2}/2\right) \ge j + 1/2$. And $\alpha m b^{-1/2} + \delta - 1 + L^+\left(\alpha b^{-1/2}/2\right) \le \alpha/2 + j + 1/2 + L^+\left(\alpha b^{-1/2}/2\right) \le j+1$ for $b \ge b_0$. Thus (C.1) applies with d in place of δ and $b + 2m + 1$ in place of b. Using the asserted bounds and arguments similar to those in used in (a) leads to $V_E(u+1, b+2m+1) = \alpha b^{-3/2}\left\{2(j+1)\left(\alpha m b^{-1/2} + \delta - 3/2\right) - j(j+1) + L\left(\left(M_j + 2j^2 + 2j + 6\right)b^{-1/4}\right)\right\}$.

Range (iii): For $m = m_0$, either $d \le 1/2$ or $d \ge 1/2$, so one of the two formulas applies. But $m_0 = \frac{j+3/2-\delta}{\alpha}\sqrt{b} - f, 0 \le f < 1$, so $\alpha m_0 b^{-1/2} - 3/2 + \delta = j - \alpha f b^{-1/2}$, so $2j\left(\alpha m_0 b^{-1/2} + \delta - 3/2\right) - j(j-1) = 2(j+1)\left(\alpha m_0 b^{-1/2} + \delta - 3/2\right) - j(j+1) + 2\alpha f b^{-1/2}$, which differ by only $2\alpha f b^{-1/2}$. We can use the formula from (ii) for $m = m_0$ regardless, and the result is the same, since $(M_j+2)b^{-1/4} + 2\alpha f b^{-1/2} = L\left(\left(M_j + 2j^2 + 2j + 6\right)b^{-1/4}\right)$.

Summing over the formulas for ranges (i)–(iii),

$$\alpha^{-1}b^{3/2}S_{LE} = \sum_{m=0}^{m_0-1} 2^{-2m-1}C_m \begin{pmatrix} 2j\left(\alpha m b^{-1/2} + \delta - 3/2\right) - j(j-1) \\ +L\left((M_j+2)b^{-1/4}\right) \end{pmatrix} + \sum_{m=m_0}^{n-1} 2^{-2m-1}C_m \begin{pmatrix} 2(j+1)\left(\alpha m b^{-1/2} + \delta - 3/2\right) - j(j+1) \\ +L\left(\left(M_j + 2j^2 + 2j + 6\right)b^{-1/4}\right) \end{pmatrix}.$$

The error terms can be gotten out of the sum since $\sum_{m=0}^{n} 2^{-2m-1}C_m < 1$. Algebra gives $\alpha^{-1}b^{3/2}S_{LE} = 2\alpha b^{-1/2}\left(j\left(nG_n - 1 + G_n\right)\sum_{m=m_0}^{n-1} 2^{-2m-1}C_m m + 2\left(\delta - j - 3/2\right)\left(G_{m_0} - G_n\right)\right) + \left(2j\left(\delta - 1/2\right) - j(j+1)\right)\left(1 - G_n\right) + L\left(\left(M_j + 2j^2 + 2j + 6\right)b^{-1/4}\right)$. Estimating terms,

$$2j\alpha b^{-1/2}\left(nG_n - 1 + G_n\right) \le 2j\alpha(2/n)nG_n \le j\alpha\sqrt{2.14/\pi}\,b^{-1/4},$$

$$\left(2j\left(\delta - 1/2\right) - j(j+1)\right)G_n \le (j^2+j)G_n \le (j^2+j)\sqrt{2.14/\pi}\,b^{-1/4},$$

$$2\alpha b^{-1/2}\sum_{m=m_0}^{n-1} 2^{-2m-1}C_m m \le 2\alpha b^{-1/2}(nG_n) \le \alpha\sqrt{2/\pi}\,b^{-1/4},$$

$$2\left(j + 3/2 - \delta\right)\left(G_{m_0} - G_n\right) \le 2\alpha b^{-1/2}\left(m_0 + 1\right)G_{m_0} \le 2\alpha b^{-1/2}nG_n + 2\alpha b^{-1/2}$$

(because mG_m is an increasing function), so $2\left(j + 3/2 - \delta\right)\left(G_{m_0} - G_n\right) \le \alpha\sqrt{2/\pi}\,b^{-1/4} + 2\alpha b^{-1/2} = L\left(b^{-1/4}\right)$. Using these estimates,

$$S_{LE} = \alpha b^{-3/2}\left\{2j\left(\delta - 1/2\right) - j(j+1) + L\left(\left(M_j + 3j^2 + 4j + 8\right)b^{-1/4}\right)\right\}.$$

Add this to $\alpha b^{-3/2}\left(2(\delta - 1/2) + L(\delta^2 b^{-1/4})\right)$ from Lemma 5.1, using $\delta \le j + 3/2$, to get

$V_E(u, b) = \frac{\alpha}{b^{3/2}}\left\{(2(j+1)(\delta - 1/2) - j(j+1)) + L\left(\left(M_j + 4j^2 + 7j + 11\right)b^{-1/4}\right)\right\}$, valid over the range $j + 1/2 \le \delta \le j + 3/2$, and the induction proof is complete, upon letting

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$M_{j+1} = M_j + 4j^2 + 7j + 11$. This recursion, with $M_1 = 5$, is easily seen to imply $M_j \leq 5j^3$. Of course a much smaller upper bound on M_j is possible. □

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