6

Connecting Finite-Dimensional, Infinite-Dimensional and Higher Geometry

In this chapter we will highlight an interesting connection between finite- and infinite-dimensional differential geometry. To this end, we shall consider in §6.2 elements from 'higher geometry', so-called Lie groupoids. The moniker higher geometry stems from the fact that in the language of category theory, these objects form higher categories. We shall not explore higher categories or their connection to differential geometry in this book (but the reader might consult Meyer and Zhu (2015) or the general introduction by Baez (1997)). In previous chapters we have discussed how finite-dimensional manifolds and geometric structures give rise to infinite-dimensional structures such as Lie groups (e.g. the diffeomorphism and groups of gauge transformations) and Riemannian metrics (such as the L^2-metric from shape analysis). While we have seen that every manifold determines an (in general, infinite-dimensional) group of diffeomorphisms, we turn this observation now on its head and ask:

Can we recognise the underlying finite-dimensional geometric structure from the infinite-dimensional object?

6.1 Diffeomorphism Groups Determine Their Manifolds

Let us examine this question for the diffeomorphism group. We have already seen that for every compact manifold we can associate the infinite-dimensional diffeomorphism group. Conversely, Takens (1979) has shown that the diffeomorphism group identifies (up to diffeomorphism) the underlying manifold. Namely, we have the following theorem (which we cite here from the much more general statement of Filipkiewicz (1982)):

6.1 Theorem (Takens (1979)/Filipkiewicz (1982)/Banyaga (1988)/Rubin (1989)) *If M, N are smooth compact, connected manifolds such that*

https://doi.org/10.1017/9781009091251.007 Published online by Cambridge University Press

$\phi\colon \mathrm{Diff}(M) \to \mathrm{Diff}(N)$ *is a group isomorphism then there exists a diffeomorphism* $\phi\colon M \to N$ *such that* $\Phi(\gamma) = \phi \circ \gamma \circ \phi^{-1}$.

A full proof of Theorem 6.1 would lead us too far astray, but it is possible to highlight certain aspects of the proof which are of special interest to us with regard to the question of whether finite-dimensional objects can be recognised from their associated infinite-dimensional objects. Before we begin, let us recall two concepts for diffeomorphism groups.

6.2 Definition Let M be a smooth and compact manifold.

- For $x_0 \in M$, the *stabiliser* is defined as

$$S_{x_0} \mathrm{Diff}(M) := \{h \in \mathrm{Diff}(M) \mid h(x_0) = x_0\}.$$

- The group $\mathrm{Diff}(M)$ *acts n-transitive* on the manifold M for $n \in \mathbb{N}$ if for any two sets $\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\}$ of non-repeating points in M, there is $h \in \mathrm{Diff}(M)$ with $h(x_i) = y_i, i \in \{1, \ldots, n\}$.

We shall show how a group isomorphism of diffeomorphism groups mapping stabilisers to each other induces a diffeomorphism on the base manifold. The first step towards this is Banyaga (1988, Lemma 1) (also compare Rybicki, 1995).

6.3 Lemma *Let* M, N *be two connected smooth manifolds and* $\phi\colon \mathrm{Diff}(M) \to \mathrm{Diff}(N)$ *a group isomorphism such that the following holds:*

(a) *For* $K \in \{M, N\}$, $\mathrm{Diff}(K)$ *acts n-transitively for* $n \in \{1, 2\}$*, and*
(b) *for each* $x_0 \in M$ *there exists* $y_0 \in N$ *such that* $\phi(S_{x_0} \mathrm{Diff}(M)) = S_{y_0} \mathrm{Diff}(N)$.

Then there is a unique homeomorphism $\omega\colon M \to N$ *such that* $\phi(f) = \omega f \omega^{-1}$.

Proof *Step 1: Construction of the homeomorphism* ω. Fix a pair of points x_0 and y_0 as in condition (b). Since $\mathrm{Diff}(M)$ and $\mathrm{Diff}(N)$ are 1-transitive, we see that

$$\mathrm{ev}_{x_0}\colon \mathrm{Diff}(M) \to M, \quad h \mapsto h(x_0) \text{ and } \mathrm{ev}_{y_0}\colon \mathrm{Diff}(N) \to N, \quad g \mapsto g(y_0)$$

are surjective with $\mathrm{ev}_{x_0}^{-1}(x_0) = S_{x_0} \mathrm{Diff}(M)$ (and similarly for y_0). We can thus choose for every $x \in M$ an (in general) non-unique diffeomorphism $h_x \in \mathrm{Diff}(M)$ such that $h_x(x_0) = x$. Now if $\tilde{h}$ is another diffeomorphism such that $\tilde{h}(x_0) = x$ we see that $\tilde{h}^{-1} h_x(x_0) = x_0$, and so $\tilde{h}^{-1} h_x \in S_{x_0} \mathrm{Diff}(M)$. As by assumption, $\phi(S_{x_0} \mathrm{Diff}(M)) = S_{y_0} \mathrm{Diff}(N)$, this implies $\phi(\tilde{h}^{-1} h_x) =$

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$\phi(\tilde{h}^{-1})\phi(h_x) \in S_{y_0}\operatorname{Diff}(N)$. Indeed, the value is independent of the choice of h_x, whence we obtain a well-defined map

$$\omega\colon M \to N, \quad x \mapsto \operatorname{ev}_{x_0}(\phi(h_x)) = \phi(h_x)(y_0)$$

as this mapping does not depend on the choice of h_x. Again by the above, the map ω is a bijection which is even a homeomorphism (details will be checked in Exercise 6.1.1).

Step 2: ω induces ϕ. Let $y \in N$ and $h \in \operatorname{Diff}(N)$ with $h(y_0) = y$ and $x = \phi^{-1}(h)(x_0)$. By construction we have $\omega(x) = y$. If $f \in \operatorname{Diff}(M)$ we pick $g \in \operatorname{Diff}(M)$ with $g(x_0) = f(x)$. Then $f^{-1}(g(x_0)) = x = \phi^{-1}(h)(x_0)$ and thus $g^{-1}f\phi^{-1}(h) \in S_{x_0}\operatorname{Diff}(M)$. We deduce that $(\phi(g))^{-1} \circ \phi(f) \circ h \in S_{y_0}\operatorname{Diff}(N)$ or in other words $\phi(f) \circ h(y_0) = \phi(g)(y_0)$. Now $h(y_0) = y = \omega(x)$ and $\phi(g)(y_0) = \omega(f(x))$ (as $g(x_0) = f(x)$). We deduce that

$$\phi(f)(\omega(x)) = \omega(f(x)), \quad \phi(f) \circ \omega = \omega \circ f,$$

and using that ω is bijective, this yields $\phi(f) = \omega \circ f \circ \omega^{-1}$.

Step 3: ω is unique. Assume that there is another homeomorphism $\tilde{\omega}\colon M \to N$ inducing ϕ. Then

$$\omega \circ f \circ \omega^{-1} = \phi(f) = \tilde{\omega} \circ f \circ \tilde{\omega}, \quad \text{for all } f \in \operatorname{Diff}(M).$$

In other words we have for $\rho := \tilde{\omega}^{-1} \circ \omega$ that $\rho \circ f \circ \rho^{-1} = f$, for all $f \in \operatorname{Diff}(M)$. Arguing by contradiction we assume that $\rho \neq \mathrm{id}_M$. Then there exists $x \in M$ with $y = \rho(x) \neq x$. Pick $z \in M \setminus \{x, y\}$. Now $\operatorname{Diff}(M)$ is 2-transitive, whence there is $f \in \operatorname{Diff}(M)$ with $f(x) = x$ and $f(y) = z$. We see that

$$\rho \circ f \circ \rho^{-1}(y) = \rho(f(x)) = \rho(x) = y \neq z = f(y).$$

However, this contradicts $\rho \circ f \circ \rho^{-1} = f$, hence we must have $\tilde{\omega} = \omega$. □

Sketch of the proof of Theorem 6.1 Every group isomorphism $\phi\colon \operatorname{Diff}(M) \to \operatorname{Diff}(N)$ satisfies condition (b) in the statement of Lemma 6.3 (this is far from trivial; see Filipkiewicz, 1982). Moreover, the group of smooth diffeomorphisms acts n-transitively for every $n \in \mathbb{N}$ if $\dim M > 1$; see Michor and Vizman (1994). Thus we can apply Lemma 6.3 to obtain a homeomorphism $\omega\colon M \to N$.

We prove that ω is a diffeomorphism under the assumption that $\phi\colon \operatorname{Diff}(M) \to \operatorname{Diff}(N)$ is a Lie group isomorphism. Note that the statement of the theorem is much stronger as, a priori, ϕ need not even be continuous. However, in this case one needs a deep result on Lie group actions on manifolds; see Filipkiewicz (1982, Step 3 on p. 173). A posteriori this implies that any group isomorphism $\operatorname{Diff}(M) \to \operatorname{Diff}(N)$ is already a Lie group isomorphism.

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So let us assume that ϕ is a Lie group isomorphism and let us study the composition $\omega \circ \mathrm{ev}_{x_0} \colon \operatorname{Diff}(M) \to N$. By Step 2 of the proof of Lemma 6.3 we can rewrite this as

$$\omega \circ \mathrm{ev}_{x_0}(h) = \mathrm{ev}_{y_0}(\phi(h)) \quad (\text{and conversely } \omega^{-1} \circ \mathrm{ev}_{y_0} = \mathrm{ev}_{x_0} \circ \phi^{-1}).$$

Now Exercise 2.3.3(a) shows that ev_{x_0} and ev_{y_0} are smooth surjective submersions. Hence the smoothness of the right-hand side together with Exercise 1.7.5 shows that ω and ω^{-1} are smooth. □

We have seen that diffeomorphism groups determine (up to diffeomorphism) their underlying manifold uniquely. In the next section we shall discuss objects, so-called Lie groupoids, which can be used to describe many finite-dimensional geometric structures and which admit a similar connection to infinite-dimensional groups.

Exercises

6.1.1 We are working in the setting of Lemma 6.3 and let $\phi \colon \operatorname{Diff}(M) \to \operatorname{Diff}(N)$ be a group isomorphism which maps the stabiliser $S_{x_0} \operatorname{Diff}(M)$ to the stabiliser $S_{y_0} \operatorname{Diff}(N)$.

(a) Show that the mapping $\omega \colon M \to N,\ x \mapsto \phi(h_x)(y_0)$ is a bijection (where $h_x \in \operatorname{Diff}(M)$ with $h_x(x_0) = x$).

(b) It is well known that $\operatorname{Diff}(M)$ satisfies the following condition: *For any non-empty connected $U \Subset M$ and $x \in U$, there exists $h \in \operatorname{Diff}(M) \setminus \{\mathrm{id}_M\}$ such that $\overline{\{y \in M \mid h(x) \neq x\}} \subseteq U$ and x is contained in the interior of $\overline{\{y \in M \mid h(x) \neq x\}}$.*
Let $f \in \operatorname{Diff}(M)$ and define $\operatorname{Fix}(f) := \{x \in M \mid f(x) = x\}$. Show that the set $\mathcal{B} := \{M \setminus \operatorname{Fix}(f) \mid f \in \operatorname{Diff}(M)\}$ is a basis for the topology on M.

(c) Show that $\operatorname{Fix}(\phi(g)) = \omega(\operatorname{Fix}(g))$ holds and conclude that ω is a homeomorphism.

6.2 Lie Groupoids and Their Bisections

In this section we consider a generalisation of Lie groups called Lie groupoids. These objects allow one to treat constructions in differential geometry as differentiable objects. For example, quotients of manifolds modulo Lie group actions may fail to be manifolds. However, one can encode them using suitable

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Lie groupoids. We motivate the construction with the following example of an ill-behaved quotient.

6.4 Example In Chapter 5, we studied shape spaces which arise as quotients of manifolds of mappings modulo an action of the diffeomorphism group. Namely, we considered the canonical action of the Lie group $\mathrm{Diff}(\mathbb{S}^1)$ on the open submanifold $\mathrm{Imm}(\mathbb{S}^1,\mathbb{R}^2) \subseteq^{\circ} C^\infty(\mathbb{S}^1,\mathbb{R}^2)$ (see Example 3.5 and Lemma 2.6) via precomposition

$$p\colon \mathrm{Imm}(\mathbb{S}^1,\mathbb{R}^2) \times \mathrm{Diff}(\mathbb{S}^1) \to \mathrm{Imm}(\mathbb{S}^1,\mathbb{R}^2), \quad (f,\varphi) \mapsto f \circ \varphi.$$

The shape space $\mathcal{S} := \mathrm{Imm}(\mathbb{S}^1,\mathbb{R}^2)/\,\mathrm{Diff}(\mathbb{S}^1)$ is then the quotient modulo the action. It inherits a natural topology which, however, does not turn $\mathcal{S}$ into a manifold. As the action p is not free, the quotient has singular points in which one fails to obtain charts. An example for such a point is the image of the immersion of the circle into $\mathbb{R}^2$ given by the map tracing out the circle in 'double speed':

$$c\colon \mathbb{S}^1 \to \mathbb{R}^2, \quad e^{i\theta} \mapsto e^{i2\theta}.$$

Since c traces the circle twice we see that $p(c,\varphi) = c \circ \varphi = c$ for the diffeomorphism $\varphi\colon \mathbb{S}^1 \to \mathbb{S}^1$, $e^{i\theta} \mapsto e^{i(\theta+\pi)}$, and we deduce that the immersion c has a non-trivial stabiliser under the action ρ. Thus, in particular, the quotient $\mathcal{S}$ is not a manifold (though every singularity is mild in the sense that it is generated by a finite group, i.e. one obtains an infinite-dimensional orbifold; see Michor, 2020, Section 7.3). As manifolds are the basic setting for differential geometry, one needs to pass to the subset generated by the free immersions (i.e. those immersions with trivial stabiliser). These indeed form a dense open subset which is a manifold.

The last example exhibits that quotients of Lie group actions will, in general, not be manifolds. While in this special example one could still say a lot about the structure of the quotient, it shows that quotient constructions with manifolds are, in general, very badly behaved (not only in infinite dimensions). Thus we would like to avoid quotients and obtain an object which contains the same information as the quotient: a (Lie) groupoid. There are many literature accounts for the basic theory of (finite-dimensional) Lie groupoids such as Mackenzie (2005) and Meinrencken (2017). While the finite-dimensional examples will be most important for us as they describe geometric constructions, the concept of a Lie groupoid can also be formulated in the infinite-dimensional context (as the notion of submersion makes sense in this setting).

6.5 Definition (Groupoid) Let G, M be two sets with surjective maps $\mathbf{s}, \mathbf{t}\colon G \to M$ (*source* and *target*) and a partial multiplication $\mathbf{m}\colon G \times G \supseteq (\mathbf{s}, \mathbf{t})^{-1}(M \times M) \to G$, $(a, b) \mapsto ab$ which satisfies:

(a) $\mathbf{s}(ab) = \mathbf{s}(b)$ and $\mathbf{t}(ab) = \mathbf{t}(a)$, and $(ab)c = a(bc)$;
(b) *identity section* $\mathbf{1}\colon M \to G$ with $\mathbf{1}(\mathbf{t}(g))g = g$ and $g\,\mathbf{1}(\mathbf{s}(g)) = g$ for all $g \in G$;
(c) *inverses* for all $g \in G$ there is $g^{-1} \in G$ with $g^{-1}g = \mathbf{1}(\mathbf{s}(g))$ and $gg^{-1} = \mathbf{1}(\mathbf{t}(g))$.

We call G (or $\mathcal{G} = (G \rightrightarrows M)$) a *groupoid* and the set M is called the *set of units*. If G, M are smooth manifolds, such that the structure maps $\mathbf{s}, \mathbf{t}$ are smooth submersions and $\mathbf{m}, \mathbf{1}$ and the inversion map $\mathbf{i}\colon G \to G, \mathbf{i}(g) = g^{-1}$ are smooth maps, we say that $\mathcal{G} = (G \rightrightarrows M)$ is a *Lie groupoid*.

Standard Notation for Lie Groupoids Throughout this section (if nothing else is said), we write $\mathcal{G} = (G \rightrightarrows M)$ for a Lie groupoid with structure maps $\mathbf{1}$, $\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$, $\mathbf{i}$ as in the definition of a groupoid.

6.6 Remark In this book manifolds are required to be Hausdorff. Thus Definition 6.5 excludes by design Lie groupoids $\mathcal{G} = (G \rightrightarrows M)$ whose space of arrows G is not Hausdorff. A broad (and important) class of Lie groupoids with non-Hausdorff space of arrows are the so-called foliation groupoids, arising from the treatment of foliations in a groupoid framework; see Moerdijk and Mrčun (2003). In principle many of the results presented here are also valid in the non-Hausdorff setting; see, for example, Rybicki (2002).

A useful mental image to keep in mind is to picture the units of the groupoid as dots connected by arrows which represent the elements of the groupoid which are not units. This is illustrated in Figure 6.1. Obviously two arrows can then be composed only if one of them ends where the other starts. This picture also immediately shows in which way a groupoid generalises the concept of a group: It can possess more units and its elements are not necessarily composable.

6.7 Definition For $G \rightrightarrows M$ a groupoid, and $a \in M$ a unit, we consider the fibres $\mathbf{s}^{-1}(a)$ and $\mathbf{t}^{-1}(a)$ of all arrows starting, resp. ending at a. The intersection $G_a := \mathbf{s}^{-1}(a) \cap \mathbf{t}^{-1}(a)$ forms a group, called the *vertex group* at a of the groupoid.

If $(G \rightrightarrows M)$ is a Lie groupoid, $\mathbf{s}^{-1}(a)$ and $\mathbf{t}^{-1}(a)$ are submanifolds of G by Corollary 1.59. A natural question is then whether the vertex groups G_a inherit a Lie group structure. In general (for arbitrary infinite-dimensional Lie groupoids), this question is still open as the finite-dimensional argument

Figure 6.1 Picturing groups and groupoids. In the right picture we suppressed all arrows between the two nodes with looping arrows. As arrows tracing a path from one node to the other can always be composed, a picture of all groupoid elements would also need to represent these arrows.

(Mackenzie, 2005, Corollary 1.4.11) establishing the Lie group structure breaks down in infinite dimensions. However, it has recently been proven in Beltiţă et al. (2019) that if G, M are Banach manifolds, then the vertex groups of $G \rightrightarrows M$ are (Banach) Lie groups. Before we finally give examples for Lie groupoids, let us first define groups which will play roles similar to the diffeomorphism group in the previous section in relation to a manifold.

6.8 Definition The *group of bisections* $\mathrm{Bis}(\mathcal{G})$ of $\mathcal{G}$ is given as the set of smooth maps $\sigma\colon M \to G$ such that $\mathbf{s}\circ\sigma = \mathrm{id}_M$ and $\mathbf{t}\circ\sigma\colon M \to M$ is a diffeomorphism. The group structure is given by the product

$$(\sigma \star \tau)(x) := \sigma((\mathbf{t}\circ\tau)(x))\tau(x) \text{ for } x \in M. \tag{6.1}$$

The identity section $\mathbf{1}\colon M \to G$ is the neutral element and the inverse of σ is

$$\sigma^{-1}(x) := \mathbf{i}(\sigma((\mathbf{t}\circ\sigma)^{-1}(x))) \text{ for } x \in M. \tag{6.2}$$

The definition of bisection is not symmetric with respect to source and target. This lack of symmetry can be avoided by defining a bisection as a set (see Mackenzie, 2005, p. 23). However, this point of view does not fit well into the function space perspective we take. Thus we shall stick with the asymmetric definition.

Lie groupoids simultaneously generalise Lie groups and (differentiable) equivalence relations. To emphasise this, we recall the following standard examples (see Mackenzie, 2005).

6.9 Example In the following, we denote by $\{\bullet\}$ the one-point manifold.

(a) Let G be a Lie group. Then the Lie group structure yields a Lie groupoid $G \rightrightarrows \{\bullet\}$, that is, a Lie group is a Lie groupoid and conversely every Lie groupoid whose set of units contains only one element is a Lie group. In this case $\mathrm{Bis}(G \rightrightarrows \{\bullet\}) = G$.

(b) Let $\pi\colon M \to N$ be a submersion. Then the fibre product of M with itself gives rise to a Lie groupoid $M \times_N M \rightrightarrows M$. Its source and target maps are given as $\mathbf{s} = \mathrm{pr}_2$ and $\mathbf{t} = \mathrm{pr}_1$. Multiplication is then given by concatenation $(m,n)\cdot(n,k) := (m,k)$. Using Lemma 1.60, it is not hard to see that this construction yields a Lie groupoid encoding the equivalence relation $x \sim y \Leftrightarrow \pi(x) = \pi(y)$. We mention two special cases of this construction:

- If $\pi = \mathrm{id}_M$, we obtain the *unit groupoid* $\mathfrak{u}(M) := (M \rightrightarrows M)$ (where all structure mappings are the identity). Clearly $\mathrm{Bis}(\mathfrak{u}(M)) = \{\mathrm{id}_M\}$.
- The map $\pi\colon M \to \{\bullet\}$ yields the *pair groupoid*, $\mathfrak{p}(M) = (M \times M \rightrightarrows M)$. We shall see in Exercise 6.2.2 that $\mathrm{Bis}(\mathfrak{p}(M)) \cong \mathrm{Diff}(M)$.

6.10 Example Consider a (left) Lie group action $\alpha\colon G \times M \to M$, $(g,m) \mapsto g\cdot m$. We form the *action groupoid* $\mathcal{A}_\alpha = (G \times M \rightrightarrows M)$, where $\mathbf{s}(g,m) := m$ and $\mathbf{t}(g,m) := \alpha(g,m)$. Now multiplication is defined as $(g,hm)\cdot(h,m) := (gh,m)$. The associated bisection group can be identified as

$$\mathrm{Bis}(\mathcal{A}_\alpha) = \{\sigma \in C^\infty(M,G) \mid m \mapsto \alpha(\sigma(m),m) \text{ is a diffeomorphism}\}. \quad (6.3)$$

Now $C^\infty(M,G) \to C^\infty(M,M)$, $f \mapsto \alpha_*(f \times \mathrm{id}_M)$ is smooth. Continuity of this map together with $\mathrm{Diff}(M) \Subset C^\infty(M,M)$ yields $\mathrm{Bis}(\mathcal{A}_\alpha) \Subset C^\infty(M,G)$. However, the bisections are not an open subgroup of the current group $C^\infty(M,G)$ (see §3.4) as the multiplication is $\sigma \star \tau(m) = \sigma(\tau(m)\cdot m)\cdot m$ instead of the pointwise product.

It is important to note that an action groupoid contains the same information as the group action and the quotient space. So instead of the ill-behaved quotient of the Lie group action

$$p\colon \mathrm{Imm}(\mathbb{S}^1,\mathbb{R}^2) \times \mathrm{Diff}(\mathbb{S}^1) \to \mathrm{Imm}(\mathbb{S}^1,\mathbb{R}^2)$$

from Example 6.4, one could instead work with the (infinite-dimensional) Lie groupoid $\mathcal{A}_p = (\mathrm{Imm}(\mathbb{S}^1,\mathbb{R}^2) \times \mathrm{Diff}(\mathbb{S}^1) \rightrightarrows \mathrm{Imm}(\mathbb{S}^1,\mathbb{R}^2))$ and carry out the geometric analysis on the groupoid instead of the quotient shape space. Since this quotient is of interest in shape analysis, Riemannian structures compatible with the Lie groupoid structure would then be needed to replace the metric on the quotient space. A suitable concept for such metrics has been worked out in del Hoyo and Fernandes (2018).

6.11 Remark Another interesting perspective on Lie groupoids is that they model symmetries which cannot be described by a global group action. A class of groupoids which fits well to this theme are the orbifold atlas groupoids (Moerdijk and Pronk, 1997). Recall that an *orbifold* is a manifold with mild singularities, that is, a Hausdorff space which is locally homeomorphic to a manifold modulo a finite group of diffeomorphisms. The key point here is that the local group acting is allowed to change. As a visual example consider the sphere $\mathbb{S}^2$ where the upper half is rotated around the north pole by a rotation group of order p, while the lower half is rotated around the south pole by a rotation group of order q and the results are glued together. Topologically the space is still $\mathbb{S}^2$ but the manifold structure breaks down at the two fixed points. It has been shown that these structures are equivalent to certain Lie groupoids. We refrain from discussing the rather technical details and refer instead to Moerdijk and Pronk (1997) as well as Moerdijk and Mrčun (2003) for a detailed account.

The concept of a Lie groupoid carries over without any changes to infinite-dimensional settings (using submersions as defined in §1.7). For example, the action Lie groupoid modelling (6.4) is infinite dimensional. Let us mention further examples: In Beltiţă et al. (2019) Lie groupoids modelled on Banach spaces were studied. These arise naturally in studying certain pseudo-inverses in C^*-algebras. As a more concrete example of a genuine infinite-dimensional Lie groupoid consider the following.

6.12 Example Let $\mathcal{G} = (G \rightrightarrows M)$ be a finite-dimensional Lie groupoid and K a compact manifold. Then the pushforwards of the groupoid operations yield a Lie groupoid, called the *current groupoid* $C^\infty(K,\mathcal{G}) := (C^\infty(K,G) \rightrightarrows C^\infty(K,M))$. It is an easy exercise (Exercise 6.2.3) to verify that the current groupoid is a Lie groupoid. The theory for such groupoids was developed in Amiri et al. (2020). There it was shown that current groupoids inherit many structural properties from the finite-dimensional target groupoids.

However, it should be noted that from the rich theory available for finite-dimensional Lie groupoids (Mackenzie, 2005) virtually nothing is known for Lie groupoids modelled on general locally convex spaces. It is, for example, unclear as to whether the vertex groups always inherit a Lie group structure from the ambient groupoid.

6.13 Example Let (E,p,M,F) be a principal G-bundle, where E,M are Banach manifolds and G is a Banach Lie group (see Definition 3.52). Denote by $e \cdot g$ the right-G action on E and consider the diagonal G-action $(e,f)\cdot g := (e\cdot g, f\cdot g)$ on $E\times E$. Then the quotient $Q := (E\times E)/G$ is a manifold (with the unique structure turning the quotient map into a submersion). We obtain a Lie groupoid, called the *Gauge groupoid* $\mathrm{Gauge}(E) = (Q \rightrightarrows M)$, associated to the principal G-bundle. The groupoid is given by the following source, target and identity maps (see Exercise 6.2.5):

$$\mathbf{s}([e,f]) := p(f), \quad \mathbf{t}([e,f]) := p(e), \quad \mathbf{1}(m) := [u,u],$$

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where $u \in p^{-1}(x)$ is arbitrary. Then we use the difference map $\delta\colon E\times_M E \to G$, $[e\cdot g, e] \mapsto g$ to define the multiplication

$$\mathbf{m}([e,f],[\tilde{e},\tilde{f}]) := [e, \tilde{f}\delta(f,\tilde{e})], \text{ where } (f,\tilde{e}) \in E\times_M E.$$

So we can associate to every principal bundle (of Banach manifolds) a Lie groupoid. Conversely, one can show that the gauge groupoid uniquely identifies the principal bundle (see Mackenzie, 2005, Proposition 1.3.5). Hence a gauge groupoid contains the same information as a principal bundle. The bisection group $\mathrm{Bis}(\mathrm{Gauge}(E))$ is the automorphism group

$$\mathrm{Aut}(E,p,M) = \\ \{f \in \mathrm{Diff}(E) \mid p\circ f \in \mathrm{Diff}(M), f(v\cdot g) = f(v)\cdot g, \text{ for all } v \in E, g \in G\}.$$

This group is known to be an infinite-dimensional Lie group (see Abbati et al., 1989) which contains the group of gauge transformations from Definition 3.56 as a (proper) Lie subgroup.

We have now seen in several examples that Lie groupoids can be used to formulate concepts from finite-dimensional differential geometry such as Lie group actions and principal bundles. Moreover, they come with an associated group, the bisection group, which in some instances can be identified with infinite-dimensional Lie groups. The next proposition shows that this is no accident.

6.14 Proposition *Assume that for a Lie groupoid $\mathcal{G}$, G is finite dimensional and M is compact. Then* $\mathrm{Bis}(\mathcal{G})$ *is a Lie group and* $\mathbf{t}_*\colon \mathrm{Bis}(\mathcal{G}) \to \mathrm{Diff}(M)$, $\sigma \mapsto \mathbf{t}\circ\sigma$ *is a Lie group morphism.*

Proof Recall from Example 3.5 that $\mathrm{Diff}(M)$ is an open submanifold of $C^\infty(M,M)$. Further, the pushforward $\mathbf{t}_*\colon C^\infty(M,G) \to C^\infty(M,M)$, $f \mapsto \mathbf{t}\circ f$ is smooth by Corollary 2.19. Since $\mathbf{s}\colon G \to M$ is a submersion, the Stacey–Roberts lemma, 2.24, asserts that $\mathbf{s}_*\colon C^\infty(M,G) \to C^\infty(M,M)$ is a submersion, whence the restriction $\theta := \mathbf{s}_*|_{\mathbf{t}_*^{-1}(\mathrm{Diff}(M))}$ is a submersion. We deduce that $\theta^{-1}(\mathrm{id}_M) = \mathrm{Bis}(\mathcal{G})$ is a submanifold of $C^\infty(M,G)$. To see that this manifold structure turns $\mathrm{Bis}(\mathcal{G})$ into a Lie group, we rewrite the formulae (6.1) and (6.2) as follows:

$$\sigma\star\tau = \mathbf{m}_*(\mathrm{Comp}(\sigma,\mathbf{t}_*(\tau)),\tau) \qquad \sigma^{-1} = \mathbf{1}_*\circ\mathrm{Comp}(\sigma,\iota\circ\mathbf{t}_*(\sigma)),$$

where $\mathbf{m}$ is groupoid multiplication, $\mathbf{i}$ is groupoid inversion and ι the inversion in the Lie group $\mathrm{Diff}(M)$ (see Example 3.5). Since M is compact, pushforwards and the composition map are smooth by Proposition 2.23. In conclusion the group operations are smooth as composition of smooth mappings.

As $\mathrm{Bis}(\mathcal{G}) \subseteq C^\infty(M,G)$ is a submanifold, the smoothness of $\mathbf{t}_*$ on $\mathrm{Bis}(\mathcal{G})$ follows from the smoothness of pushforwards on manifolds of mappings; Corollary 2.19. To see that $\mathbf{t}_*$ is a group morphism, we observe that

$$(\mathbf{t}_*(\sigma \star \tau)(x) = \mathbf{t}(\sigma(\mathbf{t}(\tau(x)))\tau(x)) = \mathbf{t}(\sigma(\mathbf{t}(\tau(x)))) = (\mathbf{t}_*(\sigma) \circ \mathbf{t}_*(\tau))(x). \quad \square$$

6.15 Remark The assumptions on $\mathcal{G}$ in the formulation of Proposition 6.14 are superfluous. The same proof (see Amiri and Schmeding, 2019, Proposition 1.3) works for any finite-dimensional Lie groupoid (dropping the compactness assumption on M), while in Schmeding and Wockel (2015, Theorem A) a proof for compact M but infinite-dimensional G was given (thus dropping the assumption on G). The latter proof is believed to generalise to non-compact M (and infinite-dimensional G).

6.16 Remark The Lie group structure of the bisections turns $\mathrm{Bis}(\mathfrak{p}(M)) \cong \mathrm{Diff}(M)$ into an isomorphism of Lie groups. Note however, that Proposition 6.14 cannot replace the classical construction of the Lie group structure on $\mathrm{Diff}(M)$ as we exploited this structure already in the proof of the proposition.

6.17 Remark The kernel of the Lie group morphism $\mathbf{t}_*\colon \mathrm{Bis}(\mathcal{G}) \to \mathrm{Diff}(M)$ is the *group of vertical bisections* $\mathrm{vBis}(\mathcal{G})$. Under certain assumptions on the Lie groupoid, it was shown in Schmeding (2020) that the vertical bisections form an infinite-dimensional Lie group.

If $\mathcal{G}$ is a gauge groupoid of some principal bundle, the vertical bisections coincide with the group of gauge transformations of the bundle. Moreover, in this case, the Lie group structure of the vertical bisections coincides with the Lie group structure on the group of gauge transformations; Remark 3.59.

Note that similar to the diffeomorphism group acting via evaluation on the underlying manifold, there is a canonical smooth action of the bisection group on the manifold of arrows of the groupoid.

6.18 Lemma *The evaluation map induces a Lie group action:*

$$\gamma\colon \mathrm{Bis}(\mathcal{G}) \times G \to G, \quad (\sigma, g) \mapsto \sigma(\mathbf{t}(g)) \cdot g.$$

Proof Setting in the definition, it is immediately clear that γ is a group action. Now rewrite γ as $\gamma(\sigma, g) = \mathbf{m}(\mathrm{ev}(\sigma, \mathbf{t}(g)), g), \quad \sigma \in \mathrm{Bis}(\mathcal{G}), g \in G$. Exploiting the smoothness of the evaluation map, Lemma 2.16, we see that the action is smooth. $\square$

With the help of the action one can identify the Lie algebra of the diffeomorphism group (see Exercise 6.2.6). We will focus here on global aspects of the theory and thus do not go into the details of the construction. However, it should be remarked that the Lie algebra of the bisection group is closely connected to the infinitesimal level of Lie groupoid theory. To make sense of this, let us mention that every Lie groupoid admits an infinitesimal object called a Lie algebroid (its role is similar to that of a Lie algebra associated to a Lie group). A Lie algebroid is a vector bundle together with certain additional structures. Equivalently, a Lie algebroid can be formulated as a special type of Lie algebra, called Lie–Rinehart algebra. In the present case, the Lie–Rinehart algebra turns out to be the Lie algebra of the bisection group. This is left as Exercise 6.2.7 and we refer to Schmeding and Wockel (2015) as well as Mackenzie (2005) for more information.

Exercises

6.2.1 Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid. Show that:

- (a) the domain of the multiplication $\mathbf{m}$ is a smooth manifold (whence it makes sense to require it to be smooth in the definition of a Lie groupoid);
- (b) the unit map $\mathbf{1}\colon M \to G$ is a section of $\mathbf{s}$ and $\mathbf{t}$ and as a consequence, $\mathbf{1}$ is a smooth embedding (i.e. an immersion which is a homeomorphism onto its image);
- (c) if only $\mathbf{s}$ is a submersion, so is $\mathbf{t}$ (vice versa if $\mathbf{t}$ is a submersion, so is $\mathbf{s}$). Hence the submersion requirements in the definition of a Lie groupoid can be weakened.

6.2.2 Let M be a manifold and $\alpha\colon G \times M \to M$ a Lie group action.

- (a) Show that the bisection group of the pair groupoid $\mathfrak{p}(M)$ is isomorphic (as a group) to the diffeomorphism group $\mathrm{Diff}(M)$.
- (b) Assume, in addition, that M is compact. Show that the group isomorphism from (a) becomes a Lie group isomorphism where the Lie group structure of $\mathrm{Bis}(\mathcal{G})$ is as in Proposition 6.14 and the one on $\mathrm{Diff}(M)$ as in Example 3.5.
- (c) Work out the bisection group $\mathrm{Bis}(\mathcal{A}_\alpha)$ of the associated action groupoid. When is this group isomorphic to the current group $C^\infty(M,G)$?

6.2.3 Use the Stacey–Roberts Lemma, 2.24, to prove that the current groupoid $C^\infty(K,\mathcal{G})$ is a Lie groupoid for a finite-dimensional Lie groupoid $\mathcal{G}$.

6.2.4 Let $G \rightrightarrows M$ be a Lie groupoid such that G, M are manifolds modelled on Banach spaces. Show that the multiplication map $\mathbf{m}\colon G \times_M G \to G$ is a submersion. Deduce that the multiplication in every Banach Lie group is a submersion.

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Hint: Since we are in the Banach setting, a submersion is a mapping which admits smooth local sections (see Exercise 1.7.5).

6.2.5 Let (E,p,M,F) be a principal G-bundle, where E,M are Banach manifolds and G is a Banach Lie group. We check that the associated gauge groupoid is a Lie groupoid. Show that:

(a) One can construct a manifold structure on the quotient $(E \times E)/G$ turning the quotient map $E \times E \to (E \times E)/G$ into a submersion.
Hint: Cover M by domains of sections of the submersion p and use the sections to construct charts for the manifold. For the submersion use Exercise 6.2.4.

(b) The structure maps $\mathbf{s},\mathbf{t},\mathbf{1},\mathbf{m}$ are smooth and that $\mathbf{s},\mathbf{t}$ are submersion. Conclude that the gauge groupoid is a Lie groupoid.
Hint: Assume p is a surjective submersion and q a smooth map between Banach manifolds. Then if $q \circ p$ is a submersion so is q (Margalef-Roig and Domínguez, 1992, Proposition 4.1.5).

6.2.6 Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid. Show that by applying the tangent functor T to every manifold and structure map of $\mathcal{G}$, one obtains a Lie groupoid $T\mathcal{G}$. One calls $T\mathcal{G}$ the *tangent (prolongation) groupoid* of $\mathcal{G}$.

6.2.7 In this exercise we identify the Lie algebra of the bisections $\mathrm{Bis}(\mathcal{G})$ as a Lie algebra of sections of a certain vector bundle. Note that this is precisely the algebra of sections induced by the Lie algebroid $\mathbf{L}(\mathcal{G})$ associated to $\mathcal{G}$. The Lie algebroid is the infinitesimal object associated to $\mathcal{G}$ (similar to the Lie algebra associated to a Lie group). As we have no need for a discussion of Lie algebroids, we will not discuss it but refer instead to Mackenzie (2005, 3.5).

(a) Exploit that $\mathbf{s}$ is a submersion, and use submersion charts to show that

$$T^{\mathbf{s}}G := \bigcup_{g \in G} T_g\, \mathbf{s}^{-1}(\mathbf{s}\, g) = \bigcup_{g \in G} \ker T_g\, \mathbf{s}$$

is a submanifold of TG and even a subvector bundle of TG.

(b) Use Exercise 1.7.3 to show that

$$\begin{aligned} T_{\mathbf{1}}\, \mathrm{Bis}(\mathcal{G}) &= \ker T_{\mathbf{1}}\, \mathbf{s}_* \\ &= \{ f \in C^\infty(M,TG) \mid f(m) \in T_{\mathbf{1}(m)}G \text{ for all } m \in M \} \end{aligned}$$

and deduce that $T_{\mathbf{1}}\, \mathrm{Bis}(\mathcal{G}) \cong \Gamma(\mathbf{1}^*\, T^{\mathbf{s}}G)$ are vector spaces (where the right hand denotes sections of the pullback bundle $T^{\mathbf{s}}G$).

(c) Observe that $\mathbf{m}(g,\cdot)\colon \mathbf{s}^{-1}(\mathbf{s}\,g) \to \mathbf{s}(g),\ h \mapsto gh$ is smooth for every $g \in G$ and show that every $X \in \Gamma(\mathbf{1}^* T^{\mathbf{s}} G)$ extends to a vector field on G via the formula

$$\overrightarrow{X}(g) := T(\mathbf{m}(g,\cdot))(X(\mathbf{t}(g))).$$

Prove that $X = \overrightarrow{X} \circ \mathbf{1}$, whence the linear map $\Gamma(\mathbf{1}^* T^{\mathbf{s}} G) \to \mathcal{V}(G),\ X \mapsto \overrightarrow{X}$ must be injective and we can define a Lie bracket on $\Gamma(\mathbf{1}^* T^{\mathbf{s}} G)$ via $[X,Y] := [\overrightarrow{X},\overrightarrow{Y}] \circ \mathbf{1}$ (where the Lie bracket on the right is the Lie bracket of vector fields).

(d) Adapt the proof identifying the Lie bracket of the algebra for the diffeomorphism group to bisection groups, that is, show that if X^R is a right-invariant vector field on $\operatorname{Bis}(\mathcal{G})$ then the vector field $X^R \times 0_G \in \mathcal{V}(\operatorname{Bis}(\mathcal{G}) \times G)$ is γ-related to $\overrightarrow{X(\mathbf{1})}$. Deduce from this that the Lie bracket can be identified with the negative of the bracket from (c).

6.2.8 Let $\mathcal{G}_1, \mathcal{G}_2$ be Lie groupoids. A *morphism of Lie groupoids* is a pair of smooth maps $F\colon G_1 \to G_2$ and $f\colon M_1 \to M_2$ such that $\mathbf{s}_2 \circ F = f \circ \mathbf{s}_1$, $\mathbf{t}_2 \circ F = f \circ \mathbf{t}_1$ and $F(gh) = F(g)F(h)$ (whenever, $g,h \in G_1$ are composable). If $f = \mathrm{id}_{M_1}$, we say F is a *morphism over the identity*. Show that every morphism F over the identity induces a Lie group morphism $F_*\colon \operatorname{Bis}(\mathcal{G}_1) \to \operatorname{Bis}(\mathcal{G}_2),\ \sigma \mapsto F \circ \sigma$.

Remark: So far we have avoided Lie groupoid morphisms as Lie groupoids and general morphisms exhibit a more complicated interplay as they form a 2-category (thus the moniker higher geometry). We will not delve into the details of this construction.

6.3 (Re-)construction of a Lie Groupoid from Its Bisections

We shall now consider whether a Lie groupoid is determined by its group of bisections or can even be reconstructed from this group. For the reconstruction, we consider again the Lie group action of the bisections on the manifold of arrows. This action turns out to be a submersion.

6.19 Proposition *Let $\mathcal{G}$ be a finite-dimensional Lie groupoid. Then the following mappings are submersions:*

(a) $\gamma_m\colon \operatorname{Bis}(\mathcal{G}) \to \mathbf{s}^{-1}(m), \quad \sigma \mapsto \sigma(m), \quad$ for all $m \in M$,

(b) $\mathrm{ev}\colon \operatorname{Bis}(\mathcal{G}) \times M \to G, \quad \sigma \mapsto \sigma(m),$

(c) $\gamma\colon \operatorname{Bis}(\mathcal{G}) \times G \to G, \quad (\sigma,g) \mapsto \sigma(\mathbf{t}(g)) \cdot g.$

Proof (a) In Exercise 2.3.4 we saw that the tangent of $\mathrm{ev}_m\colon C^\infty(M,G) \to G$, $f \mapsto f(m)$ is given by $T_f\,\mathrm{ev}_m\colon C^\infty_f(M,TG) \to T_{f(m)}G$, $h \mapsto h(m)$. By assumption, $\mathbf{s}^{-1}(m) \subseteq G$ is a finite-dimensional manifold, whence 1.56 shows that it suffices to prove that for each $\sigma \in \mathrm{Bis}(\mathcal{G})$ the tangent map $T_\sigma\,\mathrm{Bis}(\mathcal{G}) \to T_{\sigma(m)}\,\mathbf{s}^{-1}(m)$ is surjective.[1] By construction $(\mathbf{s}_*)^{-1}(\mathrm{id}_M) \cap (\mathbf{t}_*)^{-1}(\mathrm{Diff}(M)) = \mathrm{Bis}(\mathcal{G}) \subseteq C^\infty(M,G)$ and since $\mathbf{s}_*$ is a submersion we have, with Exercise 1.7.3 and arguments as in Exercise 6.2.6(b), that

$$T_\sigma\,\mathrm{Bis}(\mathcal{G}) = \ker T_\sigma(\mathbf{s}_*) = \ker(T\,\mathbf{s})_*|_{T_\sigma C^\infty(M,G)} \cong \Gamma(\sigma^* T^{\mathbf{s}}G) \subseteq C^\infty_\sigma(M,G).$$

This shows that $T_\sigma \gamma_m = T_\sigma\,\mathrm{ev}_m|_{T_\sigma\,\mathrm{Bis}_\mathcal{G}}$ is surjective as every element of $T^{\mathbf{s}}_{\sigma(m)}G$ can be written as $X(\sigma(m))$ for some $X \in \Gamma(T^{\mathbf{s}}G) \cong \Gamma(\sigma^* T^{\mathbf{s}}G)$ (see Exercise 6.3.2). We deduce that γ_m is a submersion.

(b) The proof turns out to be quite involved and involves a reduction step to the case already dealt with in (a). Note that this is not obvious as $T\,\mathbf{s}^{-1}(m)$ is, in general, properly contained in TG. For similar reasons we cannot deduce the submersion property of ev from the submersion property of $\mathrm{ev}\colon C^\infty(M,G) \times M \to G$. For the proof and the necessary details we refer to Schmeding and Wockel (2016, Proposition 2.8 and Corollary 2.10).

(c) First note that we can write $\gamma(\sigma,g) = \mathbf{m}(\mathrm{ev}(\sigma,\mathbf{t}(g)),g)$ and both $\mathbf{t}$ and $\mathbf{m}$ are submersions (see Exercise 6.2.4). Hence γ is a submersion as $\mathrm{ev}\colon\ \mathrm{Bis}(\mathcal{G})\times M \to G$ is a submersion by (b). □

We now have a submersion $\mathrm{ev}\colon\ \mathrm{Bis}(\mathcal{G}) \times M \to G$ which is a Lie group action of the bisections on the arrow manifold of the groupoid from which we constructed the bisections. Note that the unit embeds M as a submanifold of G and $\mathbf{t}_*(\mathrm{Bis}(\mathcal{G})) \subseteq \mathrm{Diff}(M)$ Exercise 6.2.1 (b). This observation allows us to prolong the action of the bisections to an action on M:

$$A\colon\ \mathrm{Bis}(\mathcal{G}) \times M \to M, \quad (\sigma,m) \mapsto \mathbf{t}(\sigma(m)).$$

We will now show that the action groupoid constructed from this action determines (under certain conditions) the Lie groupoid $\mathcal{G}$.

6.20 Definition Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid and $A\colon\ \mathrm{Bis}(\mathcal{G})\times M \to M$ the canonical Lie group action of the bisections on the units. Then we call the action groupoid $\mathrm{Bis}(\mathcal{G}) \ltimes M := (\mathrm{Bis}(\mathcal{G}) \times M \rightrightarrows M)$ the *bisection action groupoid*. Furthermore, the map $\mathrm{ev}\colon\ \mathrm{Bis}(\mathcal{G})\times M \to G$ induces a Lie groupoid morphism **ev** over the identity.[2]

[1] While the above formula immediately shows that ev_m is a submersion, we cannot directly conclude this for γ_m without identifying the subspace of $C^\infty_\sigma(M, G)$ associated to $T_\sigma\,\mathrm{Bis}(\mathcal{G})$.

[2] A Lie groupoid morphism over the identity is a smooth map $f\colon G \to G'$ (for groupoids

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The question is now of course whether the bisection action groupoid completely determines the Lie groupoid from which the bisections were constructed. In general, this will not be the case as there will not be enough bisections to obtain all elements in the arrow manifold of a Lie groupoid.

6.21 Example Let M be a compact manifold with two connected components $M = M_1 \sqcup M_2$ such that M_1 and M_2 are *not* diffeomorphic. We have seen in the previous chapter that for the pair groupoid $\mathfrak{p}(M) = (M \times M \rightrightarrows M)$, the bisection group can be identified as $\mathrm{Bis}(\mathfrak{p}(M)) \cong \mathrm{Diff}(M)$. Now consider an element $(m_1, m_2) \in M$ such that $m_1 \in M_1$ and $m_2 \in M_2$. If there were a bisection σ such that $\sigma(m_2) = (m_1, m_2)$, this would imply that there must be a diffeomorphism $\phi\colon M \to M$ such that $\phi(m_2) = m_1$. As this entails $\phi(M_2) = M_1$ (since diffeomorphisms permute the connected components of a manifold), this is clearly impossible. We conclude that for every pair such that the elements come from different components, there cannot be a bisection through this element of the Lie groupoid.

So if there should be any hope that the bisection group identifies the Lie groupoid from which it was derived, we need to require that there are enough bisections in the following sense.

6.22 Definition A Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ is said to have *enough bisections* if for every $g \in G$ there exists a bisection $\sigma_g \in \mathrm{Bis}(\mathcal{G})$ with $\sigma_g(\mathbf{s}(g)) = g$.

Fortunately, sufficient conditions for a Lie groupoid to possess enough bisections are known. Indeed it turns out that the deficiency pointed out in Example 6.21 is caused by a lack of connectedness. This can be remedied by requiring that the groupoid $\mathcal{G}$ is *source connected*, that is, for every $m \in M$ the source fibre $\mathbf{s}^{-1}(m)$ is connected.

6.23 Remark If $\mathcal{G}$ is the pair groupoid of a manifold, $\mathrm{Bis}(\mathcal{G}) \cong \mathrm{Diff}(M)$, the groupoid is source connected if and only if M is connected. Our next result will entail that $\mathrm{Diff}(M)$ acts transitively on M. We remark that connectedness of M was required in the statement of the Takens–Filipkiewicz result in §6.1. Note that transitivity of the group action was an essential ingredient in the proof of the result.

6.24 Lemma *If $\mathcal{G}$ is source connected, then $\mathcal{G}$ has enough bisections.*

$G \rightrightarrows M$ and $G' \rightrightarrows M$ which relates the structural maps of the groupoids, that is, $\mathbf{s}' \circ f = \mathbf{s}$, $\mathbf{t} \circ f = \mathbf{t}$, $\mathbf{m}' \circ (f, f) = f \circ \mathbf{m}$ and $\mathbf{i}' \circ f = f \circ \mathbf{i}$. We verify the conditions for ev in Exercise 6.3.2.

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Proof The image $\mathcal{U} := \mathrm{ev}(\mathrm{Bis}(\mathcal{G}) \times M)$ contains the image of the object inclusion $\mathbf{1}$, that is, $\mathbf{1}(m) \in \mathcal{U}$ for all $m \in M$. Define for $m \in M$ the set $\mathcal{U}_m = \mathcal{U} \cap \mathbf{s}^{-1}(m)$ and note that $\mathcal{U}_m = \mathrm{ev}(\mathrm{Bis}(\mathcal{G}) \times \{m\})$. Clearly the set $\mathcal{U}$ contains $\mathbf{1}(M)$ and forms a subgroupoid $\mathcal{U} \rightrightarrows M$ of $\mathcal{G}$ of $G \rightrightarrows M$ (see Exercise 6.3.3). As submersions are open maps, $\mathcal{U}$ is an open Lie subgroupoid of $\mathcal{G}$. Now as $\mathrm{ev}_m\colon \mathrm{Bis}(\mathcal{G}) \to \mathbf{s}^{-1}(m)$ is a submersion by Proposition 6.19(a) we infer that $\mathcal{U}_m$ is an open subset of $\mathbf{s}^{-1}(m)$. However, $\mathcal{U}_m$ is also closed: The complement $\mathbf{s}^{-1}(m) \setminus \mathcal{U}_m$ is the union $\bigcup_{g \in \mathbf{s}^{-1}(m)\setminus \mathcal{U}_m} \mathcal{U}_{\mathbf{t}(g)} \cdot g$. As $\mathcal{U}_{\mathbf{t}(m)} \Subset \mathbf{s}^{-1}(\mathbf{t}(g))$, we see that $\mathcal{U}_{\mathbf{t}(g)} \cdot g$ is open, whence $\mathcal{U}_m$ is also closed. We deduce that the clopen set $\mathcal{U}_m \subseteq \mathbf{s}^{-1}(m)$ equals $\mathbf{s}^{-1}(m)$ as $\mathcal{G}$ is source connected. □

If $\mathcal{G}$ has enough bisections, the evaluation map $\mathrm{ev}\colon \mathrm{Bis}(\mathcal{G}) \times M \to G$ becomes a surjective submersion. Hence the Lie groupoid structure of $\mathcal{G}$ is completely determined by the group of bisections. One can moreover show that the original groupoid is a quotient groupoid of the bisection action groupoid in this case (see Schmeding and Wockel, 2016, Theorem B). As we do not wish to introduce deeper concepts in groupoid theory, we do not go into details concerning this result. The main upshot, however, is that for a Lie groupoid with enough bisections, the groupoid is uniquely determined by the action of the bisection group.

6.25 Remark The results presented so far in this section are reconstruction results. This means that starting from a Lie groupoid, we can recover the Lie groupoid (under certain topological assumptions) as the quotient of a Lie groupoid we cook up from the action of the bisection group. One can of course ask whether there are construction results for Lie groupoids which do not require starting from a Lie groupoid. Instead, one would like to start from an action of a suitable infinite-dimensional group and construct a Lie groupoid such that, in the case we started with the canonical action of a bisection group, one would recover the Lie groupoid. At least partial answers to these questions exist. We refer to Schmeding and Wockel (2016), where transitive pairs, that is, a principal bundle version of Klein geometries (Sharpe, 1997, Chapter 3), are proposed as a starting point for a construction result.

Exercises

6.3.1 Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid. Show that the image of the canonical action $E := \mathrm{ev}(\mathrm{Bis}(\mathcal{G}) \times M) \subseteq G$ is closed under multiplication and inversion in G. Hence with the induced structure maps we obtain a subgroupoid $E \rightrightarrows M$ of $\mathcal{G}$.

6.3.2 Prove that $\mathrm{ev}\colon \mathrm{Bis}(\mathcal{G}) \times M \to G$ induces a Lie groupoid morphism $\mathrm{Bis}(\mathcal{G}) \ltimes M \to G$ over the identity.

6.3.3 Let $\pi\colon E \to M$ be a finite-dimensional vector bundle. Show that for every $e \in E$ there is $X^e \in \Gamma(E)$ with $X^e(\pi(e)) = e$.
Hint: Construct locally in trivialisations using bump functions. Note that the assumption of being finite dimensional can be replaced by requiring the existence of suitable bump functions.

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