COMPLEXES OVER A COMPLETE ALGEBRA OF QUOTIENTS

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1. Introduction. Let R be a commutative ring with unit and A be a unitary commutative R-algebra. Let A_S be a generalized algebra of quotients of A with respect to a multiplicatively closed subset S of A. If $\mathfrak{C}(A)$ and $\mathfrak{C}(A_S)$ denote the categories of complexes and their homomorphisms over A and A_S respectively, then one easily sees that there exists a covariant functor T: $\mathfrak{C}(A) \to \mathfrak{C}(A_S)$ such that T is onto and $T(X, d)$ is universal over A_S whenever (X, d) is universal over A. Actually the category $\mathfrak{C}(A_S)$ is equivalent to a subcategory $\mathfrak{K}_S(A)$ of $\mathfrak{C}(A)$ where $\mathfrak{K}_S(A)$ contains all those complexes (X, d) over A such that for each s in S, the module homomorphism ϕ_s: $x \to sx$ of X_n into itself is one–one and onto for each $n \geqslant 1$. In this paper, it is shown that if A is an R-algebra such that every dense ideal in A contains a finitely generated projective dense ideal, then there exists a covariant functor F: $\mathfrak{C}(A) \to \mathfrak{C}(Q(A))$, $Q(A)$ being a complete algebra of quotients of A, such that F is onto and carries universal complexes over A to the universal complexes over $Q(A)$. In order to deal with the problem, a particular covariant functor from the category of A-modules to the category of $Q(A)$-modules is introduced and studied in some detail.

2. Preliminaries. Let X and Y be two graded algebras over a commutative ring R with unit; and let f: $X \to Y$ be a graded R-algebra homomorphism. We recall **(4)** that an R-linear mapping d: $X \to Y$ is called an *R-derivation of degree* 1 if (i) d is a homogeneous linear mapping of degree 1; and (ii) for any x, x' in X with x homogeneous of degree n,

$$d(xx') = dxf(x') + (-1)^n f(x)dx'.$$

In particular, if $Y = X$ and f: $X \to Y$ is the identity mapping, then a derivation of degree 1 of X into itself is called a *derivation of degree* 1 *of* X. For any unitary commutative R-algebra A, a pair (X, d) where X is an anticommutative graded R-algebra **(4)** such that $X_0 = A$ and where d: $X \to X$ is an R-derivation of degree 1 of X such that $d \circ d = 0$, is called a *complex over* A. For any two complexes (X, d), (Y, δ) over A, a graded R-algebra homomorphism **(4)** f: $X \to Y$ is called a *complex homomorphism over* A if (i) f maps A identically; and (ii) $f \circ d = \delta \circ f$. We write f: $(X, d) \to (Y, \delta)$. Moreover, a complex (U, d) over A is called *universal* **(5)** if given any other complex (V, ∂) over A there exists a

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unique complex homomorphism f: $(U, d) \to (V, \partial)$ over A. Finally, a homogeneous ideal $J \subseteq X$ is called a *complex ideal* if $dJ \subseteq J$.

Next, we shall recall some basic notions regarding a complete algebra of quotients. An ideal D in an R-algebra A is called *dense* if for all a in A, $aD = 0$ implies $a = 0$. In the following we list some properties of dense ideals.

PROPOSITION 2.1. (i) *A is dense.*

(ii) *If D is a dense ideal in A and $D \subseteq D'$, D' being an ideal in A, then D' is dense.*

(iii) *If D and D' are dense, then so are DD' and $D \cap D'$.*

(iv) *If D is a dense ideal and for each d in D, D_d is a dense ideal, then*

$$\sum dD_d \ (d \in D)$$

is a dense ideal.

(v) *If $R \neq 0$, then 0 is not dense.*

We give a proof of (iii) and (iv), the other properties being obvious. Let $aDD' = 0$. Then for any $d \in D$, $adD' = 0$, and so $ad = 0$, since D' is dense. Thus $aD = 0$; hence $a = 0$, since D is dense. Therefore DD' is dense. But $DD' \subseteq D \cap D'$; and, so, $D \cap D'$ is dense by (ii). Now to prove (v), take $\sum_{d \in D} adD_d = 0$. Then $adD_d = 0$ for each $d \in D$; and, so, $aD = 0$ since D_d is dense. Thus $a = 0$, since D is dense. Hence $\sum dD_d \ (d \in D)$ is dense.

It follows by known methods, as for example in **(6)**, that if $\mathfrak{D}$ is the set of all dense ideals of A, then

$$\varinjlim_{D \in \mathfrak{D}} \operatorname{Hom}_A(D, A)$$

exists and is a unitary commutative R-algebra containing an isomorphic copy of A. If we denote this injective limit by $Q(A)$, then

$$Q(A) = \bigcup_{D \in \mathfrak{D}} \operatorname{Hom}_A(D, A)/\Theta$$

where Θ is the following equivalence relation: "$f_1 \,\Theta\, f_2$ if and only if f_1 and f_2 agree on the intersection of their domains." This statement is equivalent to saying that $f_1 \,\Theta\, f_2$ if and only if f_1 and f_2 agree on some dense ideal. $Q(A)$ is called a *complete algebra of quotients* of A. The natural embedding of A into $Q(A)$ is given by the mapping $a \to \Theta\ (a/1)$ which is called the *natural mapping*. For the sake of convenience, we shall identify A with its natural image in $Q(A)$. Finally we note that, for any q in $Q(A)$, the set $q^{-1}A = \{a \in A \mid qa \in A\}$ is a dense ideal in A.

We end this section by stating the following well-known lemma, which we shall need later.

LEMMA 2.1. *Let M and N be A-modules. If either N or M is a finitely generated projective A-module, then the natural homomorphism*

$$\operatorname{Hom}_A(M, A) \otimes_A N \to \operatorname{Hom}_A(M, N)$$

is an isomorphism.

3. The functor M^*. The set $\mathfrak{D}$ of dense ideals of A is directed under inclusion since the intersection of any two dense ideals is again a dense ideal. Also, for an A-module M, $(\mathrm{Hom}_A(D, M))$ $(D \in \mathfrak{D})$ is a family of A-modules indexed by the directed set $\mathfrak{D}$. For each $D \subseteq E$, for D and E in $\mathfrak{D}$, the mappings

$$\lambda_{DE}\colon \mathrm{Hom}_A(E, M) \to \mathrm{Hom}_A(D, M)$$

given by $\lambda_{DE}(f) = f \mid D$, where $f \mid D$ denotes the restriction of f to D, have the following properties:

(i) $\lambda_{DE} \in \mathrm{Hom}_A(\mathrm{Hom}_A(E, M), \mathrm{Hom}_A(D, M))$, for each $D \subseteq E$;

(ii) λ_{DD} is the identity on $\mathrm{Hom}_A(D, M)$; and

(iii) $D \subseteq E \subseteq F$ implies $\lambda_{DE} \circ \lambda_{EF} = \lambda_{DF}$ since $(f \mid E) \mid D = f \mid D$ for all $f \in \mathrm{Hom}_A(F, M)$.

Hence, $(\mathrm{Hom}_A(D, M), \lambda_{DE})$ $(D \subseteq E)$ is an injective system of A-modules. Set

$$M^* = \varinjlim_{D \in \mathfrak{D}} \mathrm{Hom}_A(D, M).$$

Then

$$M^* = \bigcup_{D \in \mathfrak{D}} \mathrm{Hom}_A(D, M)/\equiv$$

where $\equiv$ is the following equivalence relation:

$f_1 \equiv f_2$ if and only if f_1 and f_2 coincide on some dense ideal.

Remarks. (1) If f belongs to $\mathrm{Hom}_A(D, M)$ for some dense ideal D, then the equivalence class of f will be *denoted by* $[f]$.

(2) Each $x \in M$ determines the homomorphism $a \to ax$ of A into M; if $\pi_M(x) \in M^*$ denotes the equivalence class of this homomorphism, then the mapping $\pi_M\colon x \to \pi_M(x)$ is a homomorphism of M into M^*, called the *natural homomorphism*. $\pi_M(x) = 0$ if and only if the homomorphism $a \to ax$ is zero on some dense ideal D, i.e., the order ideal $\{a | a \in A, ax = 0\}$ of x is dense.

(3) If M is the A-module A, then $A^* = Q(A)$.

PROPOSITION 3.1. *M^* is a $Q(A)$-module for each A-module M. Moreover, if $\phi\colon M \to N$ is an A-module homomorphism, then ϕ induces a uniqe $Q(A)$-module homomorphism $\phi^*\colon M^* \to N^*$ such that $\pi_N \circ \phi = \phi^* \circ \pi_M$. Finally, if*

$$0 \to M \xrightarrow{\phi} N \xrightarrow{\psi} P$$

is an exact sequence of A-modules, then

$$0 \to M^* \xrightarrow{\phi^*} N^* \xrightarrow{\psi^*} P^*$$

is an exact sequence of $Q(A)$-modules.

Proof. To begin with, we have to define an additive mapping

$$h\colon Q(A) \otimes_A M^* \to M^*$$

such that $h(1 \otimes x) = x$ for all x in M^*. For this we first define

$$h^0{}_{DE}\colon \operatorname{Hom}_A(D, A) \times \operatorname{Hom}_A(E, M) \to M^*$$

for any two dense ideals D and E of A as follows. If $\phi\colon D \to A$, then $\phi^{-1}(E)$ is again a dense ideal, for it contains ED since $\phi(ED) = E\phi(D) \subseteq E$, and ED is dense. Thus the homomorphism $f \circ (\phi|\phi^{-1}(E))$ is defined on a dense ideal for $f\colon E \to M$, and we can put

$$h^0{}_{DE}(\phi, f) = [f \circ (\phi|\phi^{-1}(E))].$$

Clearly, $h^0{}_{DE}$ is A-bilinear and hence determines an A-homomorphism $h_{DE}\colon \operatorname{Hom}_A(D, A) \otimes_A \operatorname{Hom}_A(E, M) \to M^*$. Moreover, if $D' \subseteq D$ and $E' \subseteq E$ are two other dense ideals and $\mu_{D'D}$ and $\lambda_{E'E}$ are the respective restriction homomorphisms, then $h_{DE} = h_{D'E'} \circ (\mu_{DD'} \otimes \lambda_{E'E})$. Therefore, h_{DE} induces a homomorphism into M^* on the injective limit of the injective system,

$$(\operatorname{Hom}_A(D, A) \otimes_A \operatorname{Hom}_A(E, M), \mu_{D'D} \otimes \lambda_{E'E}).$$

Since the latter is isomorphic to $Q(A) \otimes_A M^*$, we have thereby obtained a homomorphism $h\colon Q(A) \otimes_A M^* \to M^*$. Now take $x \in M^*$. Let $f \in \operatorname{Hom}_A(D,M)$ be chosen such that $[f] = x$. If i_D denotes the natural injection $D \to A$, one has

$$h^0(i_D, f) = [f \circ i_D] = [f] = x,$$

and this shows that $h(1 \otimes x) = x$ for all $x \in M^*$.

Now, let $(\operatorname{Hom}_A(D, M), \lambda_{DE})$ and $(\operatorname{Hom}_A(D, N), \mu_{DE})$ be in injective systems whose limits are M^* and N^* respectively. For each D in $\mathfrak{D}$, ϕ induces an A-module homomorphism $\phi_D\colon \operatorname{Hom}_A(D, M) \to \operatorname{Hom}_A(D, N)$ given by $\phi_D(f) = \phi \circ f$ for all f in $\operatorname{Hom}_A(D, M)$. Moreover, for each $E \subseteq D$, the diagram

$$\begin{array}{ccc}
\operatorname{Hom}_A(D, M) & \xrightarrow{\lambda_{ED}} & \operatorname{Hom}_A(E, M) \\
\downarrow{\scriptstyle \phi_D} & & \downarrow{\scriptstyle \phi_E} \\
\operatorname{Hom}_A(D, N) & \xrightarrow[\mu_{ED}]{} & \operatorname{Hom}_A(E, N)
\end{array}$$

commutes since

$$(\phi_E \circ \lambda_{ED})(f) = (\phi \circ f)|E = (\mu_{ED} \circ \phi_D)\ (f)$$

for any $f \in \operatorname{Hom}_A(D, M)$. Thus (ϕ_D) $(D \in \mathfrak{D})$ is an injective system of A-module homomorphisms. Set

$$\phi^* = \varinjlim_{D \in \mathfrak{D}} \phi_D.$$

Clearly, ϕ^* is an A-homomorphism. In order to show that ϕ^* is a $Q(A)$-homomorphism, take any $q \in Q(A)$ and $m \in M^*$. Then for some suitable dense ideal D of A, one has $f \in \operatorname{Hom}_A(D, M)$ and $\alpha \in \operatorname{Hom}_A(D, A)$ such that $m = [f]$, $q = [\alpha]$, and hence $qm = [f \circ \alpha]$. Then

$$\phi^*(qm) = [\phi \circ (f \circ \alpha)], \qquad q\phi^*(m) = q[\phi \circ f] = [(\phi \circ f) \circ \alpha],$$

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which shows that $\phi^*(qm) = q\phi^*(m)$. Also, for any $x \in M$,

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M^* & \xrightarrow[\phi^*]{} & N^* \end{array}$$

$\pi_N(\phi(x)) = [\phi(x)] = [h_{\phi(x)}]$ where $h_{\phi(x)}\colon A \to N$ is given by

$$h_{\phi(x)}\, a = a\phi(x) = \phi(ax).$$

Since $\phi^*(\pi_M(x)) = \phi^*[k_x] = [\phi \circ k_x]$ where $k_x\colon A \to M$ maps each a onto ax, it follows that $\pi_N(\phi(x)) = \phi^*\pi_M(x)$.

We next show that the exactness of the sequence

$$0 \to M \xrightarrow{\phi} N \xrightarrow{\psi} P$$

implies the exactness of the sequence

$$0 \to M^* \xrightarrow{\phi^*} N^* \xrightarrow{\psi^*} P^*;$$

that is, we have to show that (i) $\ker(\phi) = 0$ implies $\ker(\phi^*) = 0$ and (ii) $\ker(\psi) = \phi(M)$ implies $\ker(\psi^*) = \phi^*(M^*)$.

(i) Let m be any element in $\ker(\phi^*)$. Then $\phi^*(m) = 0$ implies $[\phi \circ f] = 0$ for every $f \in \operatorname{Hom}_A(D, M)$ such that $[f] = m$. Therefore $(\phi \circ f)\,(D') = 0$ for some dense ideal D' in A; and so $f(D') = 0$ since $\ker \phi = 0$. Thus $[f] = m = 0$ since D' is dense. Hence $\ker \phi^* = 0$.

(ii) Take $n \in \ker(\psi^*)$ arbitrary. Then $n = [g]$ for some $g \in \operatorname{Hom}_A(E, N)$, E a dense ideal in A. $\psi^*(n) = 0$ implies $(\psi \circ g)(D') = 0$ for some dense ideal D' in A. Thus, $g(D') \subseteq \phi(M)$. Define $f\colon D' \to M$ by $f(d) = \phi^{-1}(g(d)) = m_d$ for each $d \in D'$. Since ϕ is one–one, f is well defined. Clearly, $f \in \operatorname{Hom}_A(D', M)$ and $\phi \circ f = g$ on D'. So

$$n = [g] = [\phi \circ f] = \phi^*[f] \in \phi^*(M^*).$$

Therefore, $\ker(\psi^*) \subseteq \phi^*(M^*)$.

Conversely, choose $\phi^*(m) \in \phi^*(M)$ arbitrarily. Then

$$\psi^*(\phi^*(m)) = [\psi \circ \phi \circ f]$$

where $f \in \operatorname{Hom}_A(D, M)$ such that $[f] = m$. Since $f(D) \subseteq M$, $(\psi \circ \phi \circ f)\;(D) \subseteq \psi(\phi(M)) = 0$. Thus $[\psi \circ \phi \circ f] = 0$; and so $\phi^*(M^*) \subseteq \ker(\psi^*)$. Hence $\phi^*(M^*) = \ker(\psi^*)$ and the proposition is proved.

Remark. The association of M^* with every A-module M, and the association of ϕ^* with every A-module homomorphism $\phi\colon M \to N$ is a covariant functor from the category of A-modules into the category of $Q(A)$-modules.

LEMMA 3.1. *For any A-module M, $\pi_{M^*} = \pi^*{}_M$.*

Proof. Let $x \in M^*$ arbitrary. Then $x = [f]$ with $f \in \operatorname{Hom}_A(D, M)$ for some dense ideal D in A. Now $\pi_{M^*}(x) = [\phi_x]$ where $\phi_x\colon A \to M^*$ is given by

$$\phi_x(a) = ax = [af]$$

for each $a \in A$. Thus, for each a, d in D, $af(d) = df(a) = \phi_{f(a)}(d)$ implies

$$[af] = [\phi_{f(a)}] = \pi_M(f(a)) = (\pi_M \circ f)(a)$$

for each $a \in D$. Thus $\phi_x(a) = (\pi_M \circ f)(a)$ for each $a \in D$; and so

$$[\phi_x] = [\pi_M \circ f] = \pi^*{}_M[f] = \pi^*{}_M(x).$$

Therefore $\pi_{M^*}(x) = \pi^*{}_M(x)$; hence $\pi_{M^*} = \pi^*{}_M$.

DEFINITION. M is *torsion free* if $x \in M$, $Dx = 0$ for some dense ideal D implies $x = 0$.

For any A-module M, the set T of those $x \in M$ for which there exists a dense ideal D in A with $Dx = 0$ is an A-submodule of M and M/T is a torsion-free A-module. Let $\nu\colon M \to M/T$ be the natural A-homomorphism. Then

LEMMA 3.2. *$\nu^*\colon M^* \to (M/T)^*$ is one–one.*

Proof. Let $x \in \ker(\nu)$. Then $x = [f]$ for some $f \in \operatorname{Hom}_A(D, M)$, D being a dense ideal in A. Now $\nu^*(x) = 0$ implies $[\nu \circ f] = 0$ and so $(\nu \circ f)\ (D') = 0$ for some dense ideal D' in A. Thus $f(D') \subseteq \ker(\nu) = T$. In view of the definition of T, for each $d \in D'$, there exists a dense ideal D_d such that $f(d)D_d = f(dD_d) = 0$. Thus f vanishes on the ideal $\sum_{d \in D'} dD_d$ which, by Proposition 2.1, is dense. So $[f] = x = 0$.

LEMMA 3.3. *For any A-module M, M^* is torsion free.*

Proof. Let m be an element in M^* such that $Em = 0$ for some dense ideal E in A. To show that $m = 0$, we recall that $m \in M^*$ implies $m = [f]$ with $f \in \operatorname{Hom}_A(D, M)$, D being a dense ideal. Since $Em = 0$ implies $xm = x[f] = [x \circ f] = 0$ for each $x \in E$, it follows that for each $x \in E$, there exists a dense ideal D_x in A such that $xf(D_x) = f(xD_x) = 0$. Thus $f(\sum_{x \in E} xD_x) = 0$. But, by Proposition 2.1, (iv) $\sum_{x \in E} xD_x$ is dense; and, so $[f] = m = 0$. Hence M^* is torsion free.

PROPOSITION 3.2. *Let M be a torsion-free A-module and D be a dense ideal. Then $f \in \operatorname{Hom}_A(D, M^*)$ has a unique extension $A \to M^*$.*

Proof (B. Banaschewski). First we show that the ideal

$$E = f^{-1}(\pi_M(M)) = \{x \in D | f(x) \in \pi_M(M)\}$$

is dense. For any $a \in D$, $f(a)^{-1}M = \{x | \, xf(a) \in \pi_M(M)\}$ contains D_a, the domain of $\phi \in f(a)$; and hence $f(a)^{-1}M$ is dense. Now let $x \in f(a)^{-1}M$. Then $f(xa) = xf(a) \in \pi_M(M)$. Thus $xa \in E$; hence $af(a)^{-1}M \subseteq E$ and so $\sum_{a \in D} af(a)^{-1}M \subseteq E$. Therefore, in view of Proposition 2.1, (iv), E is dense.

Now, $x \in E$ implies $f(x) = \pi_M(m_x)$ for some m_x in M; and so $f(x) = h_x$ with $h_x : D_x \to M$ given by $h_x(y) = ym_x$ for each $y \in D_x$. For any $z \in A$, $zf(x) = f(zx)$ implies $ym_{zx} = zym_x = xym_z$ for all y in some dense ideal; then for $z \in E$, $xym_z = ym_{zx}$; or

$$y(xm_z - m_{zx}) = 0 \text{ for all } y \text{ in some dense ideal.}$$

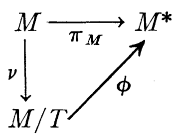
Since M is torsion free, it follows that $xm_z = m_{zx}$ for all x, z in E; that is, $h_x(z) = zm_x = xm_z$ for all x, z in E. Since the natural homomorphism π_M: $M \to M^*$ is one–one in this case, the mapping g: $E \to M$ given by $g(x) = m_x$ is well defined and A-linear. Also, $h_x(z) = xg(z)$ implies $f(x) = x[g]$ for all x, z in E. Hence for $x \in E$, $y \in D$, $x(f(y) - y[g]) = 0$. Since E is dense and M is torsion free, $f(y) = y[g]$ for all $y \in D$. Then the mapping h: $A \to M$ given by $x \to x[g]$ is an A-module homomorphism and extends f. Since M is torsion free, h is clearly unique.

COROLLARY. *If M is torsion free, then π_{M^*}: $M^* \to M^{**}$ is an isomorphism.*

Proof. Since M^* is torsion free, ker $\pi_{M^*} = 0$. To show π_{M^*} is onto, take any $[f]$ in M^{**}. By the above lemma we can take f to be defined on A. Then $f(1) \in M^*$ and clearly $[f] = \pi_{M^*}f(1)$.

PROPOSITION 3.3. *$(M/T)^*$ is isomorphic to M^{**}.*

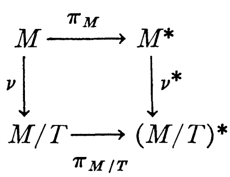
Proof. We recall that ker $(\pi_M) = T =$ ker ν. Therefore, there exists an A-module monomorphism ϕ: $M/T \to M^*$ such that the diagram



commutes; that is $\phi \circ \nu = \pi_M$. But ν induces the monomorphism

$$\nu^*: M^* \to (M/T)^*$$

such that the following diagram commutes:



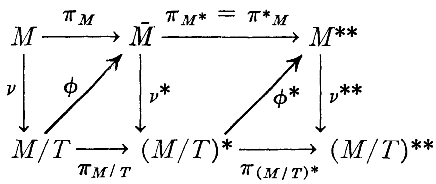
i.e., such that $\nu^* \circ \pi_M = \pi_{M/T} \circ \nu$. Since $\pi_M = \phi \circ \nu$, it follows that

$$\nu^* \circ \phi \circ \nu = \pi_{M/T} \circ \nu;$$

hence $\nu^* \circ \phi = \pi_{M/T}$ since ν is an epimorphism. Thus

$$(\nu^* \circ \phi)^* = \nu^{**} \circ \phi^* = \pi^*_{M/T} = \pi_{(M/T)^*};$$

where ν^{**}, ϕ^* are the monomorphisms induced by ν^* and ϕ respectively. Thus we have the following diagram:



Since $\pi_{M/T}$ is an isomorphism (by Proposition 3.2, Corollary), it follows that ν^{**} is an epimorphism. Hence, $\nu^{**}: M^{**} \to (M/T)^{**}$ is an isomorphism. Hence ϕ^* is an isomorphism, which proves that $(M/T)^*$ is isomorphic to M^{**}.

LEMMA 3.4. *For any A-module M, M^* is a rational extension of $\pi_M(M)$.*

Proof. We first show that for any $y \in M$, the ideal

$$y^{-1}M = \{a \in A \mid ya \in \pi_M(M)\}$$

is dense. For this recall that $y = [f]$ with $f \in \mathrm{Hom}_A(D, M)$ for some dense D. Therefore,

$$yd = [f]d = [f \circ d] = \phi_{f(d)} = \pi_M(f(d)) \in \pi_M(M) \qquad \text{for each } d \in D,$$

and hence $D \subseteq y^{-1}M$. Since D is dense, it follows that $y^{-1}M$ is dense.

Now we show that M^* is a rational extension of $\pi_M(M)$. For this we have to show that to any $x, y \in M^*$, $x \neq 0$, there is an $a \in A$ such that $xa \neq 0$ and $ya \in \pi_M(M)$. By its very definition for every $a \in y^{-1}M \subseteq A$, $ya \in \pi_M(M)$. Moreover, since M^* is torsion free, $xa \neq 0$ for at least one $a \in y^{-1}M$. This proves the assertion.

COROLLARY 1. *If M is torsion free, then M^* is a rational extension of M.*

COROLLARY 2. $Q(A) = (Q(A))^*$.

Proof. Since $Q(A)$ is torsion free, $(Q(A))^*$ is a rational extension of $Q(A)$. Now our assertion follows from the fact that $(Q(A))^*$ is rationally complete.

PROPOSITION 3.4. *Let A be an R-algebra such that every dense ideal in A contains a finitely generated projective dense ideal. Then*

(i) *the natural mapping $Q(A) \otimes_A M \to M^*$ given by*

$$q \otimes m \to q\pi_M(m) \text{ for all } m \in M,\ q \in Q(A)$$

is a $Q(A)$-module isomorphism.

(ii) *$Q(A)$ is a flat A-module.*

(iii) *For any $Q(A)$-module N, the natural homomorphism $\pi_N: N \to N^*$ is an isomorphism.*

Proof. (i) Let $\mathfrak{D}'$ denote the set of all finitely generated projective dense ideals in A. Then since for each dense ideal D in $\mathfrak{D}$ there exists a D' in $\mathfrak{D}'$ with $D' \subseteq D$, it follows that $\mathfrak{D}'$ is a co-initial subset of $\mathfrak{D}$. Thus,

$$(\mathrm{Hom}_A(D, A), \mu_{DE}) \ (D, E \in \mathfrak{D}')$$

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is an injective system of A-modules and

$$Q'(A) = \varinjlim_{D \in \mathfrak{D}'} \operatorname{Hom}_A(D, A)$$

is isomorphic to

$$Q(A) = \varinjlim_{D \in \mathfrak{D}} \operatorname{Hom}_A(D, A).$$

Similarly, $(\operatorname{Hom}_A(D, M), \lambda_{DE})$ $(D, E \in \mathfrak{D}')$ is an injective system and

$$M^* = \varinjlim_{D \in \mathfrak{D}'} \operatorname{Hom}_A(D', M)$$

is isomorphic to

$$M^* = \varinjlim_{D \in \mathfrak{D}} \operatorname{Hom}_A(D, M)$$

as $Q(A)$-modules. Now

$$\begin{aligned} Q(A) \otimes_A M &= \left(\varinjlim_{D \in \mathfrak{D}} \operatorname{Hom}_A(D, A) \right) \otimes_A M, \\ &\cong \left(\varinjlim_{D \in \mathfrak{D}'} \operatorname{Hom}_A(D, A) \right) \otimes_A M, \\ &\cong \varinjlim_{D \in \mathfrak{D}'} (\operatorname{Hom}_A(D, A) \otimes_A M), \\ &\simeq \varinjlim_{D \in \mathfrak{D}'} \operatorname{Hom}_A(D, M), \qquad \text{(by Lemma 2.1)} \\ &\simeq \varinjlim_{D \in \mathfrak{D}} \operatorname{Hom}_A(D, M) = M^*. \end{aligned}$$

Thus it is enough to show that this isomorphism is given by $q \otimes x \to q\pi_M(x)$. But $q \otimes x = \Theta(f) \otimes x \to [f \otimes x]$ where $f \in \operatorname{Hom}_A(D, A)$ such that $\Theta(f) = q$. We recall that the isomorphism $\operatorname{Hom}_A(D, A) \otimes_A M \to \operatorname{Hom}_A(D, M)$ is given by $f \otimes x \to \phi_x \circ f$ where $\phi_x \colon A \to M$ is given by $a \to ax$ for all $a \in A$. Therefore, $[f \otimes x] = [\phi_x \circ f] = q\pi_M(x)$. This proves (i).

(ii) Now to prove that $Q(A)$ is a flat A-module, we must show that for any exact sequence

$$M \xrightarrow{\phi} N \xrightarrow{\psi} P$$

of A-modules, the sequence

$$Q(A) \otimes_A M \xrightarrow{I \otimes \phi} Q(A) \otimes_A N \xrightarrow{I \otimes \psi} Q(A) \otimes_A P$$

is exact; here I denotes the identity mapping on $Q(A)$ (1). Since $Q(A) \otimes_A M$ is isomorphic to M^*, we only have to show that

$$M^* \xrightarrow{\phi^*} N^* \xrightarrow{\psi^*} P^*$$

is exact. The exactness of the sequence

$$M \xrightarrow{\phi} N \xrightarrow{\psi} P$$

implies $\ker(\psi) = \phi(M)$; thus we have the following commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow{\psi} & P \\ {\scriptstyle \nu}\downarrow & \nearrow{\scriptstyle \chi} & \\ N/\phi(M) & & \end{array}$$

where $\nu: N \to N/\phi(M)$ is the natural A-homomorphism and $\chi: N/\phi(M) \to P$ is the unique A-monomorphism induced by ν. But $\chi \circ \nu = \psi$ implies $\chi^* \circ \nu^* = \psi^*$. Since χ^* is a monomorphism (Proposition 3.1), it follows that $\ker(\psi^*) = \ker(\nu^*)$. Thus, we have to show that $\ker(\nu^*) = \phi^*(M^*)$.

For this consider the exact sequence

$$0 \to \phi(M) \xrightarrow{\tau} N \xrightarrow{\nu} N/\phi(M)$$

where τ is the natural injection. By Proposition 3.1, the sequence

$$0 \to (\phi(M))^* \xrightarrow{\tau^*} N^* \xrightarrow{\nu^*} (N/\phi(M))^*$$

is also exact. So $\ker(\nu^*) = \tau^*((\phi(M))^*)$. Thus, it only remains to show that $\phi^*(M^*) = \tau^*((\phi(M))^*)$. This, however, follows from the fact that

$$\phi^* = \varinjlim_{D \in \mathfrak{D}} \phi_D$$

and that $\phi_D(\mathrm{Hom}_A(D, M)) = \tau\, \mathrm{Hom}_A(D, \phi(M))$ for each $D \in \mathfrak{D}'$.

(iii) It remains to show that for any $Q(A)$-module N, N^* is isomorphic to N. This follows immediately from the following identities:

$$N^* \simeq Q(A) \otimes_A N \text{ and } N = Q(A) \otimes_{Q(A)} N = Q(A) \otimes_A Q(A) \otimes_{Q(A)} N$$

(since $Q(A) = (Q(A))^*$).

To prove that this isomorphism is equal to π_N, take an arbitrary $qx \in N$. Then

$$qx \to q \otimes x \to 1 \otimes q \otimes x \to 1 \otimes qx \to \pi_N(qx)$$

gives the effect of the above isomorphism.

COROLLARY. *If every dense ideal in A contains a finitely generated projective dense ideal, then the natual homomorphism $\pi_{M^*}: M^* \to M^{**}$ is an isomorphism.*

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We have seen in Proposition 3.1 that the functor $M \to M^*$ takes monomorphisms into monomorphisms. If A is an R-algebra satisfying the conditions of Proposition 3.4, then the following holds:

PROPOSITION 3.5. *If ϕ: $M \to N$ is an epimorphism, then ϕ^*: $M^* \to N^*$ is an epimorphism.*

Proof. Since the domain of f contains a finitely generated projective dense ideal D', there exists for any $y = [f]$ in N^* a $g \in \mathrm{Hom}_A(D', M)$ such that $\phi \circ g = f$ on D'. Thus $[f] = [\phi \circ g] = \phi^*[g]$. Since $[g] \in M^*$, ϕ^* is an epimorphism.

PROPOSITION 3.6. *Let A be an R-algebra such that every dense ideal in A contains a finitely generated dense ideal. Let M be a free A-module. Then M^* is isomorphic to $Q(A) \otimes_A M$.*

Proof. Let $(x_\tau)_{\tau \in I}$ be an A-basis for M. Then

$$M = \sum_{\tau \in I} Ax_\tau \quad \text{(direct)}.$$

Let D be a finitely generated dense ideal. Then

$$\mathrm{Hom}_A\Big(D, \sum_{\tau \in I} Ax_\tau\Big)$$

is isomorphic to

$$\sum_{\tau \in I} \mathrm{Hom}_A(D, Ax_\tau).$$

To see this, let f_τ: $D \to Ax_\tau$ be an element of $\mathrm{Hom}_A(D, Ax_\tau)$. Then the family (f_τ) $(\tau \in I)$, with $f_\tau = 0$ for all but finitely many τ, belongs to

$$\sum_{\tau \in I} \mathrm{Hom}_A(D, Ax_\tau);$$

since if $f_\tau = 0$ unless $\tau \neq \tau_1, \ldots, \tau_n$, then

$$\sum_{i=1}^{n} f_{\tau_i}(d) \text{ belongs to } \sum_{i=1}^{n} Ax_{\tau_i} \subseteq \sum_{\tau \in I} Ax_\tau \text{ for each } d \in D.$$

Thus

$$\phi : (f_\tau)_{\tau \in I} \to \sum_{\tau \in I} f_\tau$$

is an A-homomorphism of

$$\sum_{\tau \in I} \mathrm{Hom}_A(D, Ax_\tau) \text{ into } \mathrm{Hom}_A\Big(D, \sum_{\tau \in I} Ax_\tau\Big).$$

If

$$\sum_{i=1}^{n} f_{\tau_i}(d) = 0 \quad \text{for each } d \in D,$$

then the directness of the sum $\sum_{\tau\in I} Ax_\tau$ implies that $f_{\tau_i}(d) = 0$ for each i and each $d \in D$; thus ϕ is one–one. To show that ϕ is onto, take any

$$f \in \operatorname{Hom}_A(D, \textstyle\sum_{\tau\in I} Ax_\tau).$$

If, for each $\tau \in I$, π_τ: $\sum_{\tau\in I} Ax_\tau \to Ax_\tau$ denotes the natural projection, then $f = \sum_{\tau\in I} \pi_\tau \circ f$, where $\pi_\tau \circ f = f_\tau$ belongs to $\operatorname{Hom}_A(D, Ax_\tau)$ for each τ. That $f_\tau = 0$ for all but finitely many τ follows from the fact that D is finitely generated. Thus $(f_\tau)_{\tau\in I}$ belongs to $\sum_{\tau\in I} \operatorname{Hom}_A(D, Ax_\tau)$ and $\phi((f_\tau)_{\tau\in I}) = f$. Hence, ϕ is an A-isomorphism.

Now the proposition follows immediately from the identities:

$$Q(A) \otimes_A M = Q(A) \otimes_A \sum_{\tau\in I} Ax_\tau = \sum_{\tau\in I} (Q(A) \otimes_A A)x_\tau$$

and

$$M^* = \varinjlim \operatorname{Hom}_A\Big(D, \sum_{\tau\in I} Ax_\tau\Big) = \varinjlim \sum_{\tau\in I} \operatorname{Hom}_A(D, Ax_\tau)$$

$$= \sum_{\tau\in I} \varinjlim \operatorname{Hom}_A(D, Ax_\tau),$$

where the injective limit is taken over the set of all finitely generated dense ideals.

PROPOSITION 3.7. *If M is a finitely generated projective A-module, then the natural homomorphism $Q(A) \otimes_A M \to M^*$, given by $q \otimes x \to q\pi_M(x)$, is an isomorphism.*

Proof.

$$Q(A) \otimes_A M = \Big(\varinjlim \operatorname{Hom}_A(D, A)\Big) \otimes_A M,$$

$$\cong \varinjlim (\operatorname{Hom}_A(D, A) \otimes_A M),$$

$$\cong \varinjlim \operatorname{Hom}_A(D, M) \qquad \text{(Lemma 2.1)},$$

$$= M^*.$$

This isomorphism is given by

$$q \otimes x \to \Theta(f) \otimes x \to [f \otimes x]$$

where $\Theta(f) = q$, the domain of f being equal to D. Since the isomorphism $\operatorname{Hom}_A(D, A) \otimes_A M \to \operatorname{Hom}_A(D, M)$ is given by $f \otimes x \to \phi_x \circ f$ where ϕ_x: $A \to M$ is given by $a \to ax$, it follows that

$$[f \otimes x] \to [\phi_x \circ f] = q[\phi_x] = q\pi_M(x).$$

PROPOSITION 3.8. *Let M be an A-module such that the order ideal of x is dense for each x in M. Then $M^* = 0$.*

Proof. Here $M = T$ and so $M/T = 0$. Thus $0^* = 0$ implies $M^{**} = 0$ (Proposition 3.3). Since $\pi_{M^*}\colon M^* \to M^{**}$ is one–one, it follows that $M^* = 0$.

4. Complexes over a complete algebra of quotients.

THEOREM 4.1. *Let M be an A-module; and let $d\colon A \to M$ be an R-derivation. Then d induces a unique derivation $d^*\colon Q(A) \to M^{**}$ such that*

$$d^*|A = \pi_{M^*} \circ \pi_M \circ d.$$

Proof. For each $q \in Q(A)$, let $\phi_q\colon q^{-1}A \to M^*$ be given by

$$\phi_q(x) = \pi_M d(qx) - q(\pi_M(dx)).$$

One can easily check that ϕ_q is an A-homomorphism and so belongs to $\mathrm{Hom}_A(q^{-1}A, M^*)$. Since $q^{-1}A$ is a dense ideal in A, ϕ_q determines a class in M^{**}. Let this class be denoted by d^*q. Now consider the mapping $d^*\colon Q(A) \to M^{**}$ given by $d^*\colon q \to d^*q$. We claim that d^* is the required derivation. In order to prove this assertion take any q_1, q_2 in $Q(A)$ and r, r' in R. Then a straightforward calculation shows that for any $x \in q_1{}^{-1}A \cap q_2{}^{-1}A$:

$$\text{(i)} \qquad \phi_{q_1 r + q_2 r'}(x) = (r\phi_{q_1} + r'\phi_{q_2})(x).$$

Thus

$$\phi_{q_1 r + q_2 r'} - r\phi_{q_1} - r'\phi_{q_2} = 0$$

on the dense ideal $q_1{}^{-1}A \cap q_2{}^{-1}A$ and hence

$$d^*(q_1 r + q_2 r') = (d^* q_1)r + (d^* q_2)r.$$

Therefore d^* is R-linear. Also, for any $x \in (q_1 q_2)^{-1}A \cap q_2{}^{-1}A$.

$$\begin{aligned}
\text{(ii)} \quad \phi_{q_1 q_2}(x) &= \pi_M\, d((q_1 q_2)x) - (q_1 q_2)\pi_M(dx),\\
&= \pi_M\, d(q_1(q_2 x)) - (q_1 q_2)\pi_M(dx),\\
&= \pi_M\, d(q_1(q_2 x)) - q_1\pi_M\, d(q_2 x) + q_1\pi_M\, d(q_2 x) - (q_1 q_2)\pi_M(dx),\\
&= \phi_{q_1}(q_2\, x) + q_1\, \phi_{q_2}(x).
\end{aligned}$$

Hence,

$$x\phi_{q_1 q_2}(y) = \phi_{q_1 q_2}(xy) = \phi_{q_1}(q_2\, xy) + q_1\, \phi_{q_2}(xy) = x(q_2\, \phi_{q_1}(y) + q_1\, \phi_{q_2}(y))$$

for any x and y in $(q_1 q_2)^{-1}A \cap q_2{}^{-1}A$.

Since M^* is torsion free, we obtain

$$\phi_{q_1 q_2}(y) = (q_2\, \phi_{q_1} + q_1\, \phi_{q_2})(y) \qquad \text{for all } y \text{ in } (q_1 q_2)^{-1}A \cap q_2{}^{-1}A.$$

Thus $d^*(q_1 q_2) = q_2\, d^*q_1 + q_1\, d^*q_2$. Therefore, (i) and (ii) together show that d is an R-derivation.

Finally if $a \in A$, then

$$\phi_a(x) = \pi_M\, d(ax) - a\pi_M(dx) = \pi_M(xda) = x\pi_M(da) \text{ for all } x \in A.$$

Hence $d^*a = [\phi_a] = \pi_{M^*}(\pi_M(da))$; that is, $d^*|A = \pi_{M^*} \circ \pi_M \circ d$. Thus we have shown that d^* is a derivation from $Q(A)$ into M^{**} with the required properties.

To show the uniqueness of d^*, let $\bar{d}$ be another R-derivation from $Q(A)$ into M^{**} such that $d^*|A = \pi_{M^*} \circ \pi_M \circ d$. Then $d^* - \bar{d} = 0$ on A. Since $d^* - \bar{d}$ is a derivation on $Q(A)$, it follows that for any $q \in Q(A)$ and any $x \in q^{-1}A$,

$$(d^* - \bar{d})(qx) = ((d^* - \bar{d})q)x + q(d^* - \bar{d})x.$$

Since $d^* - \bar{d} = 0$ on A, we have $((d^* - \bar{d})q)x = 0$; thus $(d^* - \bar{d})q$ is annulled by the dense ideal $q^{-1}A$. Since M^{**} is torsion free, $(d^* - \bar{d})\ q = 0$ for each $q \in Q(A)$. Hence $d^* = \bar{d}$ on $Q(A)$, which proves the uniqueness of d^*.

COROLLARY 1. *If M is a torsion free A-module and $d : A \to M$ is any R-derivation, then d induces a unique derivation d^*: $Q(A) \to M^*$ such that $d|A = \pi_M \circ d$.*

Proof. This follows immediately from the Corollary of Proposition 3.2.

As a special case we obtain:

COROLLARY 2. *Any R-derivation of A into itself has a unique extension to an R-derivation of $Q(A)$ into itself.*

The Corollary to Proposition 3.4 also implies:

COROLLARY 3. *If every dense ideal of A contains a finitely generated projective dense ideal, then any R-derivation d: $A \to M$ induces a unique R-derivation d^*: $Q(A) \to M^*$ such that $d^*|A = \pi_M \circ d$.*

THEOREM 4.2. *Let A be an R-algebra and let (X, d) be a complex over A. Suppose*

(i) *every dense ideal in A contains a finitely generated projective dense ideal; or*

(ii) *every dense ideal in A contains a finitely generated dense ideal and X is a free A-module; or*

(iii) *X is a finitely generated projective A-module.*

Then there exists a unique derivation d^*: $X^* \to X^*$ such that (X^*, d^*) is a complex over $Q(A)$ and the natural homomorphism π_X: $X \to X^*$ is a graded algebra homomorphism such that $\pi_X \circ d = d^* \circ \pi_X$.

Proof. Since X is an A-module, Propositions 3.4, 3.6, and 3.7 imply that X^* is isomorphic to $Q(A) \otimes_A X$ under any of the conditions (i), (ii), or (iii). Therefore, X^* is an anticommutative graded R-algebra such that the module X^*_0 of homogeneous elements of degree 0 is equal to $Q(A)$. Also since

$\pi_X(xx') = 1 \otimes xx' = (1 \otimes x)(1 \otimes x') = \pi_X(x)\pi_X(x')$ so for each x, x' in X,

it follows that π_X is a graded algebra homomorphism.

We now wish to define a derivation d^*: $X^* \to X^*$ of degree 1 such that $d^* \circ d^* = 0$. First, the derivation d_0: $A \to X_1$ induces a unique derivation d^*_0: $Q(A) \to X^*_1$ under any of the conditions (i), (ii), (iii). For (i), this is

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Corollary 3 of Theorem 4.1; for (ii) and (iii), it results from Corollary 1 of that theorem since X is torsion free in these cases. Now consider the mapping $\delta: Q(A) \times X \to X^*$ given by

$$\delta(q, x) = q\pi_X(dx) + (-1)^n \pi_{X_2} x) d^*{}_0 q$$

for all $q \in Q(A)$ and all homogeneous x of degree n of X. Clearly, δ is A-bilinear. Therefore δ induces a unique mapping $d^*: Q(A) \otimes_A X = X^* \to X^*$ given by

$$d^*(q \otimes x) = q\pi_X(dx) + (-1)^n \pi_X(x)(d^*{}_0 q)$$

for all $q \in Q(A)$ and all homogeneous x of degree n in X. Clearly, d^* is a homogeneous R-linear mapping of degree 1 such that $d^*(1 \otimes x) = \pi_X(dx)$ for each $x \in X$. We claim that d^* is the required derivation. Checking of the product rule is straightforward and left to the reader. To show that $d^* \circ d^* = 0$, let $q \otimes x$ be any element of degree n in $Q(A) \otimes_A X$. Then

$$\begin{aligned} d^*d^*(q \otimes x) &= d^*(q\pi_X(dx) + (-1)^n \pi_X(x) d^*{}_0 q), \\ &= d^*(q \otimes dx) + (-1)^n d^*((1 \otimes x) d^*{}_0 q), \\ &= q\pi_X(ddx) + (-1)^{n+1} \pi_X(dx)(d^*{}_0 q) + (-1)^n (d^*(1 \otimes x)(d^*{}_0 q) \\ &\quad + (-1)^n (1 \otimes x) d^*d^*q), \\ &= (1 \otimes x) d^*d^*q. \end{aligned}$$

Therefore, $d^*d^*(q \otimes x) = 0$ if and only if $d^*d^*q = 0$. But for all x in $(d^*q)^{-1}X \cap q^{-1}A$,

$$d^*(d^*q)\, x) = (d^*(d^*q))x - (d^*q)(d^*x).$$

$$\begin{aligned} \text{Therefore } (d^*(d^*q))x &= d^*(\pi_X\, d(qx) - q\pi_X(dx)) + (d^*q)\pi_X(dx), \\ &= \pi_X(dd(qx)) - d^*(q \otimes dx) + (d^*q)\pi_X(dx), \\ &= \pi_X(d^*x) d^*q + d^*q\pi_X(dx) = 0, \end{aligned}$$

by the anticommutativity of X. Hence $d^*(d^*q)$ annihilates the dense ideal $(d^*q)^{-1}X \cap q^{-1}A$. Since X^* is torsion free, it follows that $d^*(d^*q) = 0$ for all $q \in Q(A)$. Hence $d^*d^*(q \otimes x) = 0$ for all $q \otimes x$ in X^*, which shows that $d^*d^* = 0$. Hence (X^*, d^*) is a complex over $Q(A)$. Also, since

$$\pi_X\, dx = 1 \otimes dx = d^*(1 \otimes x) = d^*\pi_X(x) \qquad \text{for each } x \in X,$$

it follows that $\pi_X: X \to X^*$ satisfies the required condition. Thus the theorem is proved.

THEOREM 4.3. *Let (X, d) and (Y, δ) be two complexes over A; and let*

$$f: (X, d) \to (Y, \delta)$$

be a complex homomorphism over A. Under the hypotheses of Theorem 4.2, f induces a unique complex homomorphism $f^*: (X^*, d^*) \to (Y^*, \delta^*)$ over $Q(A)$ such that $f^* \circ \pi_X = \pi_Y \circ f$.

Proof. Since the hypotheses of Theorem 4.2 are satisfied, we have $X^* = Q(A) \otimes_A X$ and $Y^* = Q(A) \otimes_A Y$. Let I denote the identity mapping on $Q(A)$. Then

$$f^* = I \otimes f\colon Q(A) \otimes X \to Q(A) \otimes_A Y$$

is a graded $Q(A)$-algebra homomorphism such that $f^* \circ \pi_X = \pi_Y \circ f$. Moreover,

$$\begin{aligned}(f^* \circ d^*)(q \otimes x) &= f^*(q\pi_X(dx) + (-1)^n \pi_X(x) d^* q) \\ &= q\pi_X(f(dx)) + (-1)^n \pi_X(f(x)) f^*(d^* q)\end{aligned}$$

for any $q \otimes x$ in $Q(A) \otimes_A X$. But, for any $x \in (dq)^{-1}X$,

$$(f^*(d^*q))x = f^*((d^*q)x) = f^*(\pi_X(d(qx)) - q\pi_X(dx)) = \pi_X(f(dqx)) - q\pi_X(f(dx)).$$

By definition of f^* this is equal to

$$\pi_Y(\delta(qx)) - q\pi_Y(\delta x) = (\delta^* q)x.$$

Thus, $(f^*(d^*q) - \delta^* q)x = 0$ for all $x \in (dq)^{-1}X$, which is a dense ideal. Since Y^* is torsion free, it follows that $f^*(d^*q) = \delta^* q$ for all $q \in Q(A)$. Therefore,

$$\begin{aligned}(f^* \circ d^*)(q \otimes x) &= q\pi_Y(\delta f(x)) + (-1)^n \pi_Y(f(x))\delta^* q = \delta^*(q \otimes f(x)) \\ &= (\delta^* \circ f^*)(q \otimes x).\end{aligned}$$

Hence $f^* \circ d^* = \delta \circ f^*$ on X^*. Hence $f^*\colon (X^*, d^*) \to (Y^*, \delta^*)$ is a complex homomorphism over $Q(A)$.

Remark. Suppose every dense ideal in A contains a finitely generated projective dense ideal. Let $F\colon \mathfrak{C}(A) \to \mathfrak{C}(Q(A))$ be the mapping that associates with each complex (X, d) in $\mathfrak{C}(A)$ the complex (X^*, d^*) in $\mathfrak{C}(Q(A))$ and with every complex homomorphism $f\colon (X, d) \to (Y, \delta)$ over A the complex homomorphism $f^*\colon (X^*, d^*) \to (Y^*, \delta^*)$ over $Q(A)$. Then F is a covariant functor. Also, since $X^* = Q(A) \otimes_A X$, it follows that (X^*, d^*) is generated by $d^*(Q(A))$ whenever X is generated by dA.

Next, let $\mathfrak{K}_\pi(A)$ denote the subcategory of $\mathfrak{C}(A)$ consisting of those complexes (X, d) over A which have the following property:

For each $n \geqslant 1$, the natural homomorphism $\pi_{X_n}: X_n \to X^*_n$ is an isomorphism.

THEOREM 4.4. *$\mathfrak{C}(Q(A))$ is equivalent to $\mathfrak{K}_\pi(A)$.*

Proof. First, we shall define a covariant functor $F'\colon \mathfrak{C}(Q(A)) \to \mathfrak{K}_\pi(A)$. Let (X, d) be any complex over $Q(A)$. Then X can be made into an A-algebra as follows. Define $ax = \Theta(a)x$ for each $a \in A$ and $x \in X$. Then X_n is an A-module with respect to this scalar multiplication for each $n \geqslant 1$. Thus $A + \sum_{n \geqslant 1} X_n$ (X_n being considered as A-module) is an anticommutative graded R-algebra such that the module of homogeneous elements of degree 0 is equal to A. Moreover,

$$d^{\sim}: A + \sum_{n\geqslant 1} X_n \to A + \sum_{n\geqslant 1} X_n$$

given by $d^{\sim}_0 = d_0 \circ \pi_A = d_0|A$ on A and $d^{\sim}_n = d_n$ on X_n $(n \geqslant 1)$ is an R-derivation of degree 1 of $A + \sum_{n\geqslant 1} X_n$ such that $d^{\sim}d^{\sim} = 0$. Therefore, $(A + \sum_{n\geqslant 1} X_n, d^{\sim})$ is a complex over A. By proposition 3.4, the natural homomorphism π_{X_n} is an isomorphism for each $n \geqslant 1$. Therefore, $(A + \sum_{n\geqslant 1} X_n, d^{\sim})$ belongs to $\mathfrak{K}_\pi(A)$. Moreover, for each complex (Y, δ) over $Q(A)$ and every complex homomorphism f: $(X, d) \to (Y, \delta)$ over $Q(A)$, the mapping

$$f^{\sim}: A + \sum_{n\geqslant 1} X_n \to A + \sum_{n\geqslant 1} Y_n$$

which is equal to the identity on A and to f on $\sum_{n\geqslant 1} X_n$ is clearly a graded A-algebra homomorphism. One can easily check that $f^{\sim} \circ d^{\sim} = \delta^{\sim} \circ f^{\sim}$. Therefore, $f^{\sim}$ is a complex homomorphism over A.

Now, consider the mapping F': $\mathfrak{C}(Q(A)) \to \mathfrak{K}_\pi(A)$ given by

$$F'(X, d) = (A + \sum_{n\geqslant 1} X_n, d^{\sim})$$

and $F'(f) = f^{\sim}$. Then, obviously, F' is a covariant functor. Moreover,

$$F \circ F'(X, d) = F(A + \sum_{n\geqslant 1} X_n, d^{\sim}) = (Q(A) \otimes (A + \sum_{n\geqslant 1} X_n), d^{\sim *}) = (X, d^{\sim *})$$

since $Q(A) \otimes_A X_n \simeq X^*_n$ for each n. We claim that $d^{\sim *} = d$. This, however, follows from the observation that both d and $d^{\sim *}$ extend the derivation $d_0|A$ of A. Therefore, $F \circ F'(X, d) = (X, d)$ and hence $F \circ F'$ is the identity on $\mathfrak{C}(Q(A))$.

Conversely, take a complex (Y, δ) in $\mathfrak{K}_\pi(A)$. Then (Y, δ) is a complex over A such that π_{Y_n}: $Y_n \to Y^*_n$ is an isomorphism for each $n \geqslant 1$. Now

$$F(Y, \delta) = (Q(A) \otimes_A Y, \delta^*) = (Q(A) + \sum_{n\geqslant 1} Y_n, \delta^*);$$

hence $F' \circ F(Y, \delta) = (A + \sum_{n\geqslant 1} Y_n, \delta^{*\sim}) = (Y, \delta^{*\sim})$. In order to show that $\delta^{*\sim} = \delta$, we recall that $\delta^{*\sim}_0 = \delta^*_0|A = \delta_0$ and $\delta^{*\sim}_n = \delta^*_n$. But

$$\delta^*(1 \otimes y) = \pi_{Y_n}(\delta y) = \delta y \text{ for each } y \in Y_n \ (n \geqslant 1)$$

since π_{Y_n} is an isomorphism. Therefore $\delta^*_n = \delta_n$ $(n \geqslant 1)$. Hence $\delta^{*\sim} = \delta$ and $(Y, \delta^{*\sim}) = (Y, \delta)$, which proves that $F' \circ F$ is the identity on $\mathfrak{K}_\pi(A)$. Hence the two categories $\mathfrak{C}(Q(A))$ and $\mathfrak{K}_\pi(A)$ are equivalent.

THEOREM 4.5. F: $\mathfrak{C}(A) \to \mathfrak{C}(Q(A))$ *takes the universal complexes over* A *to the universal complexes over* $Q(A)$.

Proof. Let (U, d) be a universal complex over A. Then $(U^*, d^*) = F(U, d)$ is a complex over $Q(A)$. We claim that (U^*, d^*) is universal over $Q(A)$. Let (V, δ) be any complex over $Q(A)$. By Theorem 4.4, $(A + \sum_{n\geqslant 1} V_n, \delta^{\sim})$ is a

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complex over A. By the universality of (U, d), there exists a unique complex homomorphism f: $(U, d) \to A(+\sum_{n\geqslant 1} V_n, \delta^{\sim})$ over A. Then $F(f) = f^*$: $(U^*, d^*) \to (V, \delta)$ is a complex homomorphism over $Q(A)$. Since (U^*, d^*) is generated by $d^*(Q(A))$, f^* is unique. Hence (U^*, d^*) is a universal complex over $Q(A)$.

PROPOSITION 4.1. *Let A be an R-algebra such that every dense ideal in A contains a finitely generated dense ideal and let (U, d) be a universal complex over A. If U_1 is a free A-module, then (U^*, d^*) is a universal complex over $Q(A)$.*

Proof. Let (V, δ) be any other complex over $Q(A)$. We recall that (U, d) is a universal complex over A if and only if (U_1, d_0) is a universal derivation module of A and U is the exterior algebra of U_1. Since V_1 can be considered as an A-module, the universality of (U_1, d_0) implies that there exists a unique A-homomorphism f: $U_1 \to V_1$ such that $f \circ d_0 = \delta_0$ on A. We know that f induces a unique $Q(A)$-homomorphism f^*: $U^*_1 \to V^*_1 \simeq V_1$. Thus since V is anticommutative, f^* extends uniquely to a $Q(A)$-algebra homomorphism g: $E(U^*_1) \to V$ where $E(U^*_1)$ denotes the exterior algebra of U^*_1 over $Q(A)$. Since $U^*_1 \simeq Q(A) \otimes_A U_1$, it follows that $E(U^*_1) \simeq Q(A) \otimes_A E(U_1) \simeq U^*$. Thus g maps U^* into V. One can easily check that $g \circ d^* = \delta \circ g$. The uniqueness of g, however, follows from the fact that U^* is generated by $d^*Q(A)$. Hence (U^*, d^*) is a universal complex over $Q(A)$.

The following proposition is proved by similar arguments; it is left to the reader.

PROPOSITION 4.2. *If (U, d) is a universal complex over A such that U_1 is a finitely generated and projective A-module, then (U^*, d^*) is a universal complex over $Q(A)$.*

Finally, we observe that if a universal complex (U, d) over A is such that the order ideal of every element of U_1 is dense, then (U^*, d^*) is trivial; and hence a universal complex over $Q(A)$ is trivial.

REFERENCES

1. N. Bourbaki, *Algèbre commutative*, Chap. I (Paris, 1961).
2. ——— *Algèbre*, Chap. II (Paris, 1955).
3. H. Cartan and S. Eilenberg, *Homological algebra* (Princeton, 1956).
4. C. Chevalley, *Fundamental concepts of algebra* (New York), 1956.
5. E. Kähler, *Algebra und Differentialrechnung*, Bericht über die Mathematiker-Tagung in Berlin vom 14. bis 18. Januar 1953 (Berlin, 1954).
6. J. Lambek, *On the structure of semi-prime rings and their rings of quotients*, Can. J. Math., *13* (1961), 392–417.

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