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ON FIXED POINTS AND MULTIPARAMETER ERGODIC THEOREMS IN BANACH LATTICES

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We present here multiparameter results about positive operators acting on a weakly sequentially complete Banach lattice. Sections 1, 2 and 3 generalize results obtained by M. A. Akcoglu and the second author in the case of a contraction. Even in that case, the classical L_1 theory extends to Banach lattices only under an additional monotonicity assumption (C), introduced in [**3**], without which the TL (or stochastic) ergodic theorem fails. The example proving this in [**4**] also shows that, without (C), the decomposition of the space into the "positive" part P, the largest support of a T-invariant element, and the "null" part N on which the TL limit is zero (see, e.g., [**22**], p. 141), also fails. If T is not a contraction but only mean-bounded, then the space decomposes into the "remaining" part Y, the largest support of a T^*-invariant element, and the "disappearing part" Z (see, e.g., [**22**], p. 172). Here we obtain, for Banach lattices and in the multiparameter case, a unified proof of both decompositions, and of the TL ergodic theorem. The main novelty in the argument is that the (difficult) proof of the TL convergence and the decomposition $P + N$ under (C), establishes also the (easier) decomposition $Y + Z$. The idea is to apply this proof to semi-norms, which become norms contracted by the operators in the proof of the decomposition $P + N$, and expressions $H(|f|)$, $H \in E^*+$, in the proof of the decomposition $Y + Z$.

The pointwise operator ergodic theorem (Cesaro convergence of iterates) cannot hold in Banach lattices including L_1 since it fails in L_1. However, one has *demiconvergence*: lim inf is equal to the TL (or stochastic) limit. In the last section this result is extended to the multiparameter case. This is accomplished by reducing this case to the one-parameter one by a general argument similar to the one given in [**28**] and [**16**]. However, these papers study convergence for operators that act both on L_1 and L_∞, in which case, as well known, pointwise convergence holds, but a higher degree of integrability is needed for higher dimension: in dimension d, f has to be in $L \log^{d-1} L$. Demiconvergence results are simpler in that the integrability requirement does not change with the dimension.

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1. Definition and notation. Let E be a sigma-complete Banach lattice. We assume the knowledge of the elements of the Banach lattice theory, as presented, e.g., in the beginning pages of the book of Lindenstrauss-Tzafriri [**23**]. Consider the following assumptions (A), (B) and (OCN) on E:

(A). There exists an element u in E_+, called a *weak unit*, such that if f is in E_+ and if $u \wedge f = 0$, then $f = 0$. The existence of a weak unit is assured if E is separable ([**23**], p. 9).

(B). Every norm-bounded increasing sequence in E has a strong limit. An equivalent condition is that E does not contain an isomorphic copy of c_0 ([**23**], p. 34).

(OCN), or *order continuity* (of the norm). For every downward directed net $(f_i, i \in I)$ with $\wedge_{i \in I} f_i = 0$, one has $\lim_i \|f_i\| = 0$. An equivalent condition is that every order interval

$$[f, g] = \{h: f \leqq h \leqq g\}$$

is weakly compact ([**23**], p. 28).

We usually assume (A) and (B). It is known and easy to see that (B) implies (OCN). If E is a Banach lattice satisfying (A) and (OCN), then there exists a *strictly positive* element U in the positive cone of the dual, that is, a U such that if $f \in E_+$ and $U(f) = 0$, then $f = 0$ ([**23**], p. 25). Therefore an order continuous Banach lattice with weak unit admits a representation as a Köthe function space over a probability space $(\Omega, \mathscr{F}, \mu)$. In fact, we often prefer to allow μ to be sigma finite, because in the important case of operators induced by point-transformations preserving an infinite invariant measure, the reduction to an equivalent probability measure hinders the intuitive understanding of the action of the operators. The representation as a Köthe space over a probability space means that E is order isometric to an order ideal X of $L^1(\Omega, \mathscr{F}, \mu)$ such that

(i) X is dense in $L^1(\Omega, \mathscr{F}, \mu)$ and $L^\infty(\Omega, \mathscr{F}, u)$ is dense in X^*, and

(ii) The dual of the isometry between E and X maps E^* onto the Banach lattice X^* of all μ measurable functions g for which

$$\|g\|_{X^*} = \sup\left\{\int fgd\mu: \|f\|_X \leqq 1\right\} < \infty.$$

One has $g(f) = \int fgd\mu$ for $f \in X$ and $g \in X^*$.

We may and do assume that E is a Köthe space of functions over a σ-finite measure space. Strong convergence in E is denoted by slim; weak convergence by wlim. Order convergence for monotone nets will be denoted by $\uparrow$ or $\downarrow$. Since order convergence corresponds to essential convergence, which usually is simply almost everywhere (a.e.) convergence, also the measure-theoretic terminology of a.e. convergence will be used.

For each $\delta \in E_+$, there exists a linear operator P_δ, called *band projection*, defined by

$$P_\delta f = \text{slim}_n\, f\Lambda(n\delta);$$

the limit exists in the strong topology by (OCN), since $f - f\Lambda(n\delta) \downarrow$. In the functional representation, $P_\delta f$ is the restriction of f to the support of δ.

Assuming that the Banach lattice E satisfies (A) and (B), we now recall the properties of truncated limits, a notion introduced by Akcoglu and Sucheston [**3**] and [**4**]. We refer to these articles for proofs, where they are given only for sequences, but the arguments for nets are the same. Let I be a directed sets filtering to the right, let $\delta \in E_+$, and let $(f_i, i \in I)$ be a net of elements of E_+. The *truncated limit* of f_i is δ, in symbols $TLf_i = \delta$, if for a weak unit u,

$$\text{slim}_i(f_i\Lambda ku) = \delta_k$$

exists for each k, and $\delta_k \uparrow \delta$. This definition does not depend on the choice of the weak unit u. For a net (f_i) in E such that TLf_i^+ and TLf_i^- exist, one sets

$$TLf_i = TLf_i^+ - TLf_i^-.$$

It is shown [**4**] that $TLf_i = 0$ if and only if $TL|f_i| = 0$, which holds if and only if f_i converges to zero in measure on sets of finite measure ("stochastic" convergence). One defines analogously the *weak truncated limit* of f_i, in symbols $WTLf_i$, requiring only that $f_i\Lambda(ku)$ converge weakly to δ_k. If in these definitions the role of a weak unit u is played by an arbitrary fixed positive element g, one writes

$$TL_g f_i = \delta \quad \text{or} \quad WLT_g f_i = \delta.$$

Thus for positive f_i, $TL_g f_i = \delta$ means that $\lim f_i\Lambda kg = \delta_k$ and $\delta_k \uparrow \delta$.

1.1. *Compactness for WTL. Every norm-bounded net has a subnet for which the weak truncated limit exists.*

It is this property which renders weak truncated limits useful.

1.2. *Let U be a strictly positive element in E^* and let (f_i) be a net in E_+ such that* $\lim U(f_i) = 0$. (a) *Then* $TLf_i = 0$. (b) *If furthermore* $\sup f_i \in E$, *then* $\text{slim}\, f_i = 0$.

1.3. *Additivity and Fatou for operators. If f_i, g_i, h_i are nets of elements in E_+ with $(W)TLf_i = f$, $(W)TLg_i = g$, $(W)TLh_i = h$, and $f_i + g_i = h_i$, then*

(a) $f + g = h$;

(b) $f = g$ *implies* $(W)TL|f_i - g_i| = 0$;

(c) If $T{:}E \rightarrow E$ *is a positive linear operator and* $Tf_i = g_i$, *then* $Tf \leqq g$.

1.4. (a) *If* $f_i \in E_+$, $\sup\|f_i\| < \infty$ *and* $WTLf_i = f$, *then* $WTL_f f_i = f$.
(b) *Let* $\delta \in E_+$; *then* $TL_\delta f_i = f$ *if and only if* $TLP_\delta f_i = f$.

Order continuous seminorms. We wish to treat simultaneously certain asymptotic positivity conditions on functionals and certain strict monotonicity conditions on the norm. For this purpose, we introduce order continuous (OC) seminorms. A *seminorm* is a map $N:E_+ \to \mathbf{R}$ such that (1.5) holds; N is called *order continuous* if also (1.6) holds:

1.5. (i) $N(f + g) \leqq N(f) + N(g)$ *for* $f, g \in E_+$.
(ii) $N(\alpha f) = \alpha Nf$ *for* $\alpha \geqq 0$, $f \in E_+$.
(iii) $0 \leqq f \leqq g$ *implies* $0 = N(0) \leqq N(f) \leqq N(g)$.

1.6. *If* $f_n \downarrow 0$ *then* $\lim N(f_n) = 0$.

Note that an OC seminorm is necessarily continuous at every element of f of E_+, that is:

1.7. *If* N *is an* OC *seminorm,* (f_i) *is a net in* E_+ *and* slim $f_i = f$, *then* $\lim N(f_i) = N(f)$.

Proof. Clearly, $f \leqq f_i + |f - f_i|$ and $f_i \leqq f + |f_i - f|$. The subadditivity of N on E_+ now implies that

$$|N(f_i) - N(f)| \leqq N(|f_i - f|),$$

which reduces the proof to the positive net $|f_i - f|$, converging strongly to zero because the norm of an element is equal to the norm of its module. Therefore we may and do assume $f = 0$. Suppose that there is a number $\gamma > 0$ and a net f_i in E_+ with slim $f_i = 0$ and $N(f_i) > \gamma$. Let α_n be a sequence of numbers such that $\sum \alpha_n < \infty$, and for each positive integer n, choose an index $i(n)$ so that $\|f_{i(n)}\| < \alpha_n$. Let $g_n = \sup_{k \geqq n} f_{i(k)}$; then

$$\|g_n\| \leqq \sum_{k \geqq n} \alpha_n \to 0 \quad \text{and} \quad g_n \downarrow 0.$$

Now, by (1.6), $N(g_n) \to 0$, while, by (1.5),

$$N(g_n) \geqq N(f_{i(n)}) \geqq \gamma.$$

This is a contradiction.

It will be sometimes useful to make an assumption going in the direction opposite to (1.7).

1.8. Let N be an OC seminorm. The Banach lattice norm is said to be *continuous with respect to* N on E_+ if for each sequence f_n in E_+ with $\sup_n f_n \in E$ and $\lim_n N(f_n) = 0$, one has slim $f_n = 0$.

We also consider the following strict monotonicity assumptions (C_1) and (C), made about the lattice norm in [**4**]. Clearly, (C) implies (C_1).

(C_1) For any $f, g \in E_+$, if $N(f) > 0$, then $N(f + g) > N(g)$.

(C) For every $f \in E_+$ and for every $\alpha > 0$, there is a number $\beta = \beta(f, \alpha)$ such that if $g \in E_+$, $N(g) \leqq 1$, $0 \leqq h \leqq f$ and $N(h) \geqq \alpha$, then $N(g + h) \geqq N(g) + \beta$.

Examples. Clearly, an order continuous norm is an OC seminorm, and every element H of E_+^* defines an OC seminorm on E_+ by $N(f) = H(f)$. If $H = U$, a strictly positive element of E_+^*, then, by (1.2), this seminorm satisfies (1.8). If E is an Orlicz space, then (C) is equivalent with (C_1), and both hold if and only if the Orlicz function satisfies the classical condition Δ_2, as shown in **[5]**.

2. Existence of invariant elements. Let at first T be a linear operator on a Banach space E. For each positive integer n, let

$$A_n = A_n(T) = (1/n) \sum_{0 \leqq i < n} T^i$$

be the Cesaro average of iterates of T. We say that T is *power-bounded* if $\sup_n \|T^n\| < \infty$, *mean-bounded* if $\sup_n \|A_n\| < \infty$. Given a seminorm N on E, we say that T *contracts* N if $N(Tf) \leqq N(f)$ for each $f \in E_+$. The ergodic theorem of Kakutani-Yosida (cf. **[22]**, p. 72) states that if T is mean-bounded, $f \in E$, slim $T^n f/n = 0$, and there is a sequence of positive integers $n_{(i)} \nearrow \infty$ such that $A_{n(i)} f$ converges weakly, then $A_n f$ converges strongly, to a limit invariant under T, here denoted $A_\infty f$. It follows that if T is a positive mean-bounded operator on an order continuous Banach lattice, $\delta \in E_+$, $T\delta \leqq \delta$, and $0 \leqq f \leqq \delta$, then $A_n f$ converges strongly. Indeed,

$$0 \leqq T^n f/n \leqq \delta/n \downarrow 0$$

implies

$$\text{slim } T^n f/n = 0.$$

Also, the sequence $A_n f$ belongs to the interval $[0, \delta]$, and hence has a weak cluster point.

We now consider several operators (equivalently, several parameters) at the same time. Let $d \geqq 1$ be a fixed integer and let $I = \mathbf{N}^d$ be the index set, filtering to the right for the usual coordinatewise partial order:

$$s = (s_1, \ldots, s_d) \leqq t = (t_1, \ldots, t_d) \quad \text{if } s_i \leqq t_i \text{ for } i = 1, \ldots, d.$$

Given s, t in I, $s < t$ means $s_i < t_i$ for all i. Instead of s_i we sometimes write $s(i)$. We set

$$[s, t[\; = \{u \in I : s \leqq u < t\} \quad \text{and}$$

$$\Pi(t) = \prod_{1 \leqq i \leqq d} t_i = \text{card}([0, t[),$$

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where $0 = (0, \ldots, 0)$. Given $t \in I$ and operators $T_1, \ldots, T_d$, we denote by T^t the product of powers determined by t of "coordinatewise" operators, that is, we set

$$T^t = T_1^{t(1)} \ldots T_d^{t(d)}.$$

We set

$$S_t = \sum_{s \in [0,t[} T^s,$$

$$A_t = \prod (t)^{-1} S_t.$$

We say that the operators T_i *commute* if $T_i T_j = T_j T_i$ for $1 \leqq i, j \leqq d$.

2.1 LEMMA. *Let E be an order-continuous Banach lattice, $T_1, \ldots, T_d$ positive mean-bounded operators on E, and $\delta \in E_+$ such that $T_i \delta \leqq \delta$ for $i = 1, \ldots, d$. Then for each $f \in P_\delta E$, the nets $(A_n(T_i)f; n \in \mathbf{N})$, $1 \leqq i \leqq d$, respectively $(A_t; t \in I)$, converge strongly to $A_\infty(i)f$, $1 \leqq i \leqq d$; respectively $A_\infty f = A_\infty(1) \ldots A_\infty(d)f$. If the operators T_i commute, then the operators $A_\infty(i)$ commute on $P_\delta E$.*

Proof. Let $M = \sup_{n,i} \|A_n(T_i)\|$, and let $f \in P_\delta E_+$. Fix $\epsilon > 0$, and choose k so large that if $g = f \wedge k\delta$, then $\|f - g\| < \epsilon$. Then

$$\|A_n(T_i)f - A_n(T_i)g\| < \epsilon M \text{ for every } n \text{ and } i = 1, \ldots, d.$$

As noted, the Kakutani-Yosida theorem implies that $A_n(T_i)g$ converges strongly to some T_i invariant element of $P_\delta E_+$, and since ϵ is arbitrary, the same is true about the sequence $A_n(T_i)f$. Choose n_0 so that for each $i = 1, \ldots, d$, and each $n \geqq n_0$.

$$\|A_n(T_i)A_\infty(i+1) \ldots A_\infty(d)f - A_\infty(i) \ldots A_\infty(d)f\| < \epsilon.$$

Then for each $t \geqq (n_0, \ldots, n_0)$, one has that

$$\|A_t f - A_\infty(1) \ldots A_\infty(d)f\| \leqq \sum_{1 \leqq i \leqq d} \|A_{t(1)}(T_1)\|$$
$$\ldots \|A_{t(i-1)}(T_{i-1})\|$$
$$\times \|A_{t(i)}(T_i)A_\infty(i+1) \ldots A_\infty(d)f - A_\infty(i) \ldots A_\infty(d)f\|$$
$$\leqq \sum_{1 \leqq i \leqq d} M^{i-1}\epsilon.$$

Thus $A_t f$ converges strongly to $A_\infty(1) \ldots A_\infty(d)f$. If the operators T_i commute, then by the same argument the averages $A_t f$ converge strongly to $A_\infty(\sigma(1)) \ldots A_\infty(\sigma(d))$ for each permutation σ of $(1, \ldots, n)$, and hence the operators $A_\infty(i)$ commute.

Capital letters below denote linear operators, and inequalities between operators are defined by their actions on E_+, i.e., $Q \geqq R$ means that $Qf \geqq Rf$ for each $f \in E_+$. The symbols slim applied to operators denotes convergence in the strong operator topology.

2.2 LEMMA. *Let $T_1, \dots, Td$ be positive mean-bounded operators on E that commute. Fix $u \in I$, $\delta \in E_+$, and an order continuous seminorm N with* (1.8). *Then one can write*

(i) $A_{u+t}f \geqq A_t f + G_u(t)$ *with* $\text{slim}_t\, G_u(t) = 0$.

(ii) $\text{slim}_t[A_{u+t} - T^u A_t] = 0$.

(iii) *If* $(T_i^n/n;\ n \geqq 1)$ *converges strongly to zero for every* i, *then*

$$\text{slim}_t[A_{u+t} - \text{A}_t] = 0.$$

(iv) *If each* T_i *contracts* N, *then for each* $f \in E_+$,

$$\lim_t N(|A_{u+t}f - A_t f|) = 0, \quad \textit{and} \quad \text{slim}_t[A_{u+t}f\Lambda\delta - A_t f\Lambda\delta] = 0.$$

Proof. It suffices to prove this lemma for $u = (1, 0, \dots, 0)$, since the case of $u = (0, \dots, 0, 1, 0, \dots, 0)$ follows by a permutation of the T_i, and the general case is proved by induction. Set $u = (1, 0, \dots, 0)$, then for each $t \in I$,

$$A_{t+u} = A_t + \frac{1}{\pi(t+u)} \sum_{v \in [0,t[, v(1)=t(1)} T^v - G_t,$$

with

$$G_t = \frac{t(1)}{\pi(t)}\left[\frac{1}{t(1)} - \frac{1}{t(1)+1}\right] \sum_{v \in [0,t[} T^v = [t(1)+1]^{-1} A_t,$$

which proves (i), since A_t is bounded. To prove (ii), observe that

$$A_{t+u} - T_1 A_t = H_t$$

with

$$H_t = \frac{t(1)}{[t(1)+1]\pi(t)} \sum_{v \in [0,t[, v(1)=0} T^v - T_1 G_t$$

$$= [t(1)+1]^{-1} A_{0,t(2),\dots,t(d)} - T_1 G_t,$$

which converges strongly to zero. Furthermore, (iii) follows from

$$\|A_{t+u} - A_t\| \leqq \|G_t\| + [t(1)+1]^{-1}\|T_1^{t(1)}\|\,\|A_{0,t(2),\dots,t(d)}\|.$$

Finally, fix $f \in E_+$, and set

$$A_{t+u}f - A_t f = -G_t f + R_t f.$$

Since T_1 and $A_{0,t(2),\dots,t(d)}$ contract N,

$$\begin{aligned} N(|A_{t+u}f\Lambda\delta - A_t f\Lambda\delta|) &\leqq N(|A_{t+u}f - A_t f|) \\ &\leqq N(G_t f) + N(R_t f) \\ &\leqq N(G_t f) + [t(1)+1]^{-1} N(f). \end{aligned}$$

By (1.7), $\lim N(G_t f) = 0$, and hence

$$\lim_t N(|A_{t+u}f - A_t f|) = 0.$$

Since

$$|A_{t+u}f\Lambda\delta - A_t f\Lambda\delta| \leqq \delta,$$

the condition (1.8) on N concludes the proof of the lemma.

We now show that unless the truncated limit of multiple averages is zero, there is in E a positive element δ subinvariant under the semigroup generated by the T_i's. The argument, a particular case of which appears in [**3**], is an application of compactness for weak truncated limits.

2.3 PROPOSITION. *Let $T_1, \dots, T_d$ be positive, mean-bounded commuting operators on E. Let $f \in E_+$ and let $\eta \in E_+$ be such that the net $(P_\eta A_t f)$ is not TL null. Then there exists a sequence $t(n)$ of multiparameter indices for which*

$$WTL(P_\eta A_{t(n)}f) = \delta_0 \neq 0,$$

and there is in E a $\delta \geqq \delta_0$ such that $T_i\delta \leqq \delta$ for every $i = 1, \dots, d$. If each sequence T_i^n/n, $i = 1, \dots, d$, converges strongly to zero, then one may choose $\delta = WTL(A_{t(n)}f)$.

Proof. Let k be an integer such that the net $(ku)\Lambda(P_\eta A_t f)$ does not converge to zero. Let $t(n)$ be a sequence of elements of I such that $t(n) \geqq (n, \dots, n)$, and

$$\|(ku)\Lambda(P_\eta A_{t(n)}f)\| > \alpha$$

for some number $\alpha > 0$. Suppose that there is an integer $q \geqq k$ such that a subsequence of $(qu)\Lambda(P_\eta A_{t(n)}f)$ converges weakly to zero. Then by (1.2) this subsequence also converges strongly, which is a contradiction. It follows that by the diagonal procedure one can obtain a subsequence of $t(n)$, still denoted by $t(n)$, such that

$$\text{wlim}(qu)\ \Lambda(P_\eta A_{t(n)}f) = f_q \neq 0$$

for every $q \geqq k$. Hence

$$WTL(P_\eta A_{t(n)}f) = \delta_0 \neq 0.$$

Applying again the diagonal procedure, we choose a further subsequence, still denoted $t(n)$, such that

$$WTL(A_{t(n)+u}f) = \delta_u \neq 0$$

exists for every $u \in I$. Lemma 2.2 (i) shows that for every $n \in \mathbf{N}$ and every $u \leqq v \in I$, one has

$$A_{t(n)+v}f \geqq A_{t(n)+u}f - g_n,$$

with $g_n \geqq 0$ and slim $g_n = 0$. This implies that the net $(\delta_u;\ u \in I)$ is increasing. Since this net is norm-bounded by $\|f\|\sup_t\|A_t\|$, one has that

$$\delta = \lim \uparrow \delta_u \in E,$$

and $\delta_u \geqq \delta_0$ implies that $\delta \geqq \delta_0$. We now show that δ is subinvariant under each T_i. Lemma 2.2(ii) implies that for every $u \in I$,

$$WTLT_1 A_{t(n)+u} f = WTLA_{t(n)+u+(1,0\dots 0)} f = \delta_{u+(1,0,\dots,0)},$$

and by (1.3)(c),

$$T_1 \delta_u \leqq \delta_{u+(1,0,\dots,0)}.$$

Hence

$$T_1 \delta = \lim_u T_1 \delta_u \leqq \lim_u \delta_{u+(1,0,\dots,0)} = \delta;$$

a similar argument shows that $T_i \delta \leqq \delta$ for each i.

If T_i^n/n converges strongly to zero for every i, then

$$\text{slim}_n [A_{t(n)+v} - A_{t(n)+u}] = 0$$

for every $u \leqq v \in I$, hence $\delta = WTLA_{t(n)} f$.

By a standard measure-theoretic argument, we now show that there always exists a subinvariant element δ with the maximal support. Our settings are still appropriate, namely Köthe, function spaces over sigma-finite measure spaces, but in search for maximality we pass to an equivalent probability measure. Part (iii) of the following lemma says that outside of the support of δ, the Cesaro averages of iterates converge stochastically to zero. They may of course converge stochastically to zero also on the support of δ, if δ is only subinvariant and not invariant.

2.4 LEMMA. *Let $T_1, \dots, T_d$ be positive, mean-bounded commuting operators on E. Then there exists an $\delta \in E_+$, called maximal subinvariant element, such that*

(i) *$T_i \delta \leqq \delta$ for $1 \leqq i \leqq d$,*

(ii) *If $\gamma \in E_+$ and $T_i \gamma \leqq \gamma$ for $1 \leqq i \leqq d$, then $P_\delta \gamma = \gamma$.*

(iii) *For every $f \in E$, $TL(I - P_\delta)A_t f = 0$.*

Proof. Let ν be a probability measure equivalent with μ. Let $\mathcal{ST}$ denote the class of functions in E_+ subinvariant under all the T_i's. Set

$$\alpha = \sup\{\nu(f > 0), f \in \mathcal{ST}\}.$$

Let (f_n) be a sequence in $\mathcal{ST}$ such that $\|f_n\| = 1$, and $\nu(f_n > 0) \to \alpha$, set

$$\delta = \sum_{n \geqq 0} 2^{-n} f_n.$$

Then clearly (i) holds. Let $\gamma \in \mathcal{ST}$, then $\delta + \gamma \in \mathcal{ST}$ and hence $\{\delta + \gamma > 0\} = \{\delta > 0\}$, and $P_\delta \gamma = \gamma$, which proves (ii). To prove (iii), let $f \in E_+$ be such that the relation

$$TL(I - P_\delta)A_t f = 0$$

fails. Let P_η be the projection $I - P_\delta$, then by Proposition 2.3 there is a sequence $t(n)$ contained in I such that

$$WTL(P_\eta A_{t(n)} f) = \gamma_0 \neq 0,$$

hence $P_\eta \gamma_0 = \gamma_0 \neq 0$, and there is a function $\gamma \in \mathscr{S}\mathscr{T}$ with $\gamma \geqq \gamma_0$. Then, by (ii) above, $P_\delta \gamma = \gamma$, hence $P_\delta \gamma_0 = \gamma_0$, which contradicts $P_\eta \gamma_0 \neq 0$.

It is of interest to determine when the subinvariant elements obtained above by the method of WTL are actually invariant. This holds if the operators T_i are contractions of an order continuous seminorm N satisfying the conditions (1.8) and (C) above. If N is the norm $\| \ \|$, and the T_i's are contractions, then (1.8) will be automatically satisfied, and (C) will be a known monotonicity assumption on the norm [**4**], implying the ergodic theorem for TL. If N is a linear functional, then (C) will hold automatically, and one will have to postulate that the functional N is, in the sense of (1.8), "stronger" than the norm.

2.5 LEMMA. *Let E be a Banach lattice satisfying* (A) *and* (B). *Let N be an order continuous seminorm with properties* (1.8) *and* (C), *and let T_i, $i = 1, \dots, d$ be positive, mean-bounded, commuting operators on E, contracting the seminorm N, i.e., such that $N(T_i f) \leqq N(f)$ for each i and each $f \in E_+$. Let $f \in E_+$, and let $\eta \in E_+$ be such that the net $(P_\eta A_t f)$ is not TL null. Then there exists a sequence of indices $t(n)$ for which*

$$WTL(P_\eta A_{t(n)} f) = \delta_0 \neq 0,$$

and there is in E a $\delta > \delta_0$ such that $T_i \delta = \delta$ for all $i = 1, \dots, d$.

Proof. Let $t(n)$ be the sequence of indices constructed in Proposition 2.3, such that

$$WTL_n(A_{t(n)+u} f) = \delta_u \neq 0$$

exists for every $u \in I$. Then $\delta = \lim_u \delta_u$ satisfies $T_i \delta \leqq \delta$ for every $i = 1, \dots, d$.

Set $\delta' = \delta - T_1 \delta$ and suppose that $\delta' \neq 0$, and hence that $N(\delta') > 0$. Choose $v \in I$ such that

$$N(\delta - \delta_v) < N(\delta')/8.$$

Replacing $t(n)$ by a further subsequence, we may assume that

$$\text{wlim}_n (l\delta_u \wedge A_{t(n)+u} f) = \gamma_u(l)$$

exists for each $u \in I$ and $l > 0$. The property (1.4) of weak truncated limits shows that

$$\gamma_u(l) \nearrow \delta_u \quad \text{as } l \to +\infty.$$

Fix l_0 such that

$$N(\delta_v - \gamma_v(l_0)) \leqq N(\delta')/8.$$

Lemma 2.2 shows that for every $u \in I$, there exists a sequence $g_n \in E_+$ with

$$\lim_n \|g_n\| = 0,$$

such that

$$l\delta_{u+(1,0,\ldots,0)} \Lambda A_{t(n)+u+(1,0,\ldots,0)} f \geqq (l\delta_u \Lambda A_{t(n)+u} f) - g_n$$

for every $l \geqq 1$. Hence letting $n \to \infty$ yields that

$$\gamma_{u+(1,0,\ldots,0)}(l) \geqq \gamma_u(l),$$

and a similar argument shows that for every $l \geqq 1$ and every $u < u' \in I$, one has

$$\gamma_u(l) \leqq \gamma_{u'}(l).$$

Hence the net $(\gamma_u(l);\ l \in \mathbf{N},\ u \in I)$ increases to δ, and

$$N(\delta - \gamma_u(l)) < N(\delta')/4 \quad \text{for } u \geqq v \text{ and } l \geqq l_0.$$

Fix l, and set

$$r(n, u) = (A_{t(n)+u} f) \Lambda l\delta, \quad s(n, u) = A_{t(n)+u} f - r(n, u),$$

$$r'(n, u) = (T_1(A_{t(n)+u} f) \Lambda l\delta, \quad s'(n, u) = T_1 A_{t(n)+u} f - r'(n, u).$$

Clearly,

$$T_1 A_{t(n)+u} f = T_1 r(n, u) + T_1 s(n, u)$$
$$= r'(n, u) + s'(n, u).$$

Since $T_1 \delta \leqq \delta$, one has that $T_1 r(n, u) \leqq l\delta$, and hence that

$$0 \leqq T_1 r(n, u) \leqq r'(n, u).$$

Thus

$$T_1 s(n, u) = (r'(n, u) - T_1 r(n, u)) + s'(n, u)$$

is the sum of two positive elements of E. Furthermore, by Lemma 2.2 (ii) and (iv)

$$\operatorname{slim}_n [T_1 A_{t(n)+u} f \Lambda l\delta - A_{t(n)+u+(1,0,\ldots,0)} f \Lambda l\delta] = 0$$
$$= \operatorname{slim}_n [T_1 A_{t(n)+u} f \Lambda l\delta - A_{t(n)+u} f \Lambda l\delta],$$

and hence it follows that

$$\operatorname{wlim}_n r(n, u) = \gamma_u(l) = \operatorname{wlim}_n r'(n, u).$$

Clearly

$$\text{wlim}_n\ T_1 r(n, u) = T_1(\text{wlim}_n\ r(n, u)) = T_1\gamma_u(l),$$

and hence

$$\text{wlim}_n[r'(n, u) - T_1 r(n, u)] = \gamma_u(l) - T_1\gamma_u(l) \geqq 0.$$

Furthermore,

$$\gamma_u(l) - T_1\gamma_u(l) = (\delta - T_1\delta) + T_1(\delta - \gamma_u(l)) - (\delta - \gamma_u(l)).$$

Since T_1 contracts N, we obtain that

$$N(\delta') \leqq N(\gamma_u(l) - T_1\gamma_u(l)) + 2N(\delta - \gamma_u(l)).$$

Fix $u \geqq v$ and $l \geqq l_0$; then

$$N(\gamma_u(l) - T_1\gamma_u(l)) \geqq N(\delta')/2 > 0.$$

Suppose that

$$\liminf_n \|r'(n, u) - T_1 r(n, u)\| = 0,$$

and let n_k be a sequence of integers such that

$$\lim_k \|r'(n_k, u) - T_1 r(n_k, u)\| = 0.$$

Since

$$\text{wlim}_k(r'(n_k, u) - T_1(n_k, u)) = \gamma_u(l) - T_1\gamma_u(l),$$

one concludes that $\gamma_u(l) - T_1\gamma_u(l) = 0$, which gives a contradiction. Hence

$$\liminf_n \|r'(n, u) - T_1 r(n, u)\| > 0,$$

and the property (1.8) of N implies that

$$\liminf_n N(r'(n, u) - T_1 r(n, u)) > \alpha > 0.$$

Then by condition (C) on N, there exists $\beta > 0$ such that

$$\begin{aligned} N(T_1 s(n, u)) &= N[s'(n, u) + (r'(n, u) - T_1 r(n, u))] \\ &\geqq N(s'(n, u)) + \beta \end{aligned}$$

for large values of n. Lemma 2.2 (ii) and (iv) implies that

$$\lim_n N(|T_1 A_{t(n)+u} f - A_{t(n)+u} f|) = 0,$$

and hence

$$\lim_n N(|s(n, u) - s'(n, u)|) = 0.$$

This yields that

$$N(s(n, u)) \geqq N(T_1 s(n, u)) \geqq N(s(n, u)) + \beta/2$$

for large values of n. This gives a contradiction, and hence proves that

$T_1\delta = \delta$. A similar argument shows that δ is invariant under each operator T_i, $1 \leqq i \leqq d$.

LEMMA 2.5 yields the following decomposition of the space E.

2.6 PROPOSITION. *Let E be a Banach lattice with* (A) *and* (B), *let N be an order continuous seminorm satisfying the conditions* (1.8) *and* (C), *and let $T_1, \dots, T_d$ be positive, mean-bounded, commuting operators contracting the seminorm N. Then there exists $\delta \in E_+$ such that*

(i) $T_i\delta = \delta$ *for* $i = 1, \dots, d$.

(ii) *For each $\gamma \in E_+$ such that $T_i\gamma = \gamma$, for every $i = 1, \dots, d$, one has that $P_\delta\gamma = \gamma$,*

(iii) *For each $f \in E$, $TL(I - P_\delta)A_tf = 0$.*

Proof. Let

$$\mathcal{T} = \{f \in E_+ : T_if = f,\ 1 \leqq i \leqq d\},$$

and let ν be a finite measure equivalent with μ. Let δ satisfy (i) and (ii), and suppose that there exists $f \in E_+$ such that

$$TL(I - P_\delta)A_tf \neq 0.$$

Then Lemma 2.5 yields the existence of a function γ in $\mathcal{T}$ such that

$$\gamma \geqq TL(I - P_\delta)A_{t(n)}f$$

for some sequence of indices $t(n)$. The function $\gamma - \delta$ belongs to $\mathcal{T}$, and this contradicts the maximality of the support of δ stated in (ii).

3. On the existence of truncated limits of averages. We at first study the strong convergence of the net A_tf on the support of a function δ invariant under every operator T_i. We first consider the particular case $E = L^1$.

3.1 LEMMA. *Let $T_1, \dots, T_d$ be positive commuting contractions on L^1, and let $\delta \in L^1_+$ be such that $T_i\delta = \delta$, $1 \leqq i \leqq d$. Then for each function $f \in L^1_+(\delta > 0)$, the net $(A_tf, t \in I)$ converges in $L^1(\delta > 0)$ to $\bar{f}$ such that $\|\bar{f}\| = \|f\|$.*

Proof. The strong convergence of the net A_tf follows from Lemma 2.1. Thus it suffices to show that the averages A_t are isometries of $L^1_+(\delta > 0)$. Fix an $f \in L^1_+(\delta > 0)$ and let $\epsilon > 0$ be arbitrary. Let k be such that

$$\|f - f\Lambda k\delta\|_1 < \epsilon.$$

Then for each $t \in I$,

$$\begin{aligned}\|A_t(k\delta)\|_1 &= \|A_t(f\Lambda k\delta)\|_1 + \|A_t(k\delta - f\Lambda k\delta)\|_1 \\ &\leqq \|f\Lambda k\delta\|_1 + \|k\delta - f\Lambda k\delta\|_1 = \|k\delta\|_1.\end{aligned}$$

Since δ is invariant under the T_i's, the extreme terms agree,

$$\|A_t(f\Lambda k\delta)\|_1 = \|f\Lambda k\delta\|_1,$$

and hence

$$\|A_t f\|_1 \geqq \|f\Lambda k\delta\|_1 \geqq \|f\|_1 - \epsilon.$$

The following lemmas show that one can approximate $1_{\{\delta>0\}}A_t f$ by averages of a function in $L^1_+(\delta > 0)$. A set $A \subset \Omega$ is *absorbing* for T if for any function f with support included in A, the support of Tf is included in A.

3.2 LEMMA. *Let T be a positive contraction on L^1 and let $A \subset \Omega$ be absorbing for T. Then for any $f \in L^1$.*

$$\lim_k \limsup_n \|1_A A_n(T)f - A_n(T)(1_A A_k(T)f)\|_1 = 0.$$

Proof. Let T_A be the positive contraction of L^1 defined by $T_A f = 1_{A^c}Tf$. Then the operator

$$\sum_A = \sum_{k\geqq 0} 1_A T(T_A)^k$$

is a positive contraction of L^1 (see, e.g., **[25]**, p. 193). Lemma 2.2 shows that for every k,

$$\lim_n \|A_n(T)f - A_n(T)A_k(T)f\| = 0.$$

Thus, it is enough to prove that

$$\lim_k \sup_n \|1_A A_n(T)A_k(T)f - A_n(T)(1_A A_k(T)f)\| = 0,$$

or that

$$\lim_k \sup_n \|1_A A_n(T)(1_{A^c}A_k(T)f)\| = 0.$$

Fix $f \in L^1_+$; since A is absorbing for T, one has for each n and each $k \geqq l$

$$\|1_A A_n(T)(1_{A^c}A_k(T)f)\|$$
$$= \left\|\frac{1}{n}\sum_{1\leqq i<n} 1_A T^i(1_{A^c}A_k(T)f)\right\|$$
$$= \left\|\frac{1}{n}\sum_{1\leqq i<n}\sum_{0\leqq j<i} (1_A T)^{i-j}(1_{A^c}T^j A_k(T)f)\right\|$$
$$\leqq \frac{1}{n}\sum_{1\leqq i<n}\sum_{1\leqq j\leqq l}\left\|1_A T\left(1_{A^c}\frac{j}{k}T^{j-1}f\right)\right\|$$
$$+ \frac{1}{n}\sum_{1\leqq i<n}\sum_{l\leqq j<n+k-2} \|1_A T(1_{A^c}T^j f)\|$$

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$$\leqq \frac{n-1}{n}\frac{l(l+1)}{k}||f|| + \frac{n-1}{n}\sum_{j\geqq l}||1_A T(T_A)^j f||.$$

Fix $\epsilon > 0$; since

$$||f|| \geqq \left\|\sum_A f\right\| = \sum_{j\geqq 0}||1_A T(T_A)^j f||,$$

we may choose l such that

$$\sum_{j\geqq l}||1_A T(T_A)^j f|| < \epsilon,$$

and then choose k_0 such that $l(l+1)||f||/k_0 < \epsilon$.

Then for every $k \geqq k_0$ and every n,

$$||1_A A_n(T)(1_{A^c}A_k(T)f)|| < 2\epsilon.$$

The following lemma establishes an approximation for multiparameter averages of a semigroup, similar to that proved for one parameter in Lemma 3.2.

3.3 LEMMA. *Let $t_1, \ldots, T_d$ be positive, commuting contractions on L^1, and let $\delta \in L^1_+$ be invariant under each operator T_i, $1 \leqq i \in d$. Then*

$$\liminf_t \limsup_u ||1_{\{\delta>0\}}A_u f - A_u(1_{\{\delta>0\}}A_t f)||_1 = 0.$$

Proof. Fix $f \in L^1_+$, and for each $t \in I$, set

$$g_t = 1_{\{\delta>0\}}A_t f.$$

Lemma 2.2 shows that for each index t,

$$\lim_u ||A_u f - A_u A_t f|| = 0.$$

Since $A_u gt \leqq 1_{\{\delta>0\}}A_u A_t f$, one has

$$\limsup_u ||(A_u g_t - g_u)^+|| = 0 \quad \text{for each fixed } t.$$

Lemma 2.1 shows that for every t, the net $(A_u g_t;\ u \in I)$ converges in L^1, say to $\bar{g}_t$. Hence

$$\sup_t \limsup_u ||(\bar{g}_t - g_u)^+|| = 0,$$

and the proof of the lemma reduces to showing that

$$\liminf_t \limsup_u ||(g_u - \bar{g}_t)^+|| = 0.$$

Suppose the contrary; let $\epsilon_n > 0$ be a sequence such that $\sum_n \epsilon_n < \infty$, and choose $\alpha > 0$ such that

$$\liminf_t \limsup_u ||(g_u - \bar{g}_t)^+|| > \alpha.$$

Define a sequence of functions (h_n) by induction as follows. Choose t_0 such that

$$\limsup_u \| (g_u - \bar{g}_t)^+ \| > \alpha \quad \text{for every } t \geqq t_0,$$

and set $\bar{h}_0 = \bar{g}_{t_0}$. Suppose that $t_{n-1} \geqq t_0$ has been defined, and choose $t_n > t_{n-1}$ such that if $h_n = g_{t_n}$, $p_n = (h_n - \bar{h}_{n-1})^+$, $q_n = (\bar{h}_{n-1} - h_n)^+$, then $\|p_n\| > \alpha$ and $\|q_n\| < \epsilon_n$. Since

$$h_n = \bar{h}_{n-1} + p_n - q_n,$$

one has

$$\bar{h}_n = \bar{h}_{n-1} + \bar{p}_n - \bar{q}_n \quad \text{for every } n \geqq 1.$$

Lemma 3.1 shows that $\|\bar{p}_n\| > \alpha$, and $\|\bar{q}_n\| < \epsilon_n$; thus

$$\liminf \|\bar{h}_n - \bar{h}_{n-1}\| \geqq \alpha > 0.$$

We also prove that the sequence $(\bar{h}_n)$ converges strongly, which gives a contradiction. Indeed, the choice of (ϵ_n) ensures the strong convergence of the sequence $(\sum_{k \leqq n} \bar{q}_k)$. The sequence $(\bar{h}_n + \sum_{k \leqq n} \bar{q}_k)$ is increasing, norm-bounded, and hence converges in L^1. Hence the sequence $(\bar{h}_n)$ converges in L^1, which completes the proof of the lemma.

Let δ be a positive function invariant under each T_i. The following proposition relates the limits of the nets $1_{\{\delta>0\}}A_t$, and $1_{\{\delta>0\}}A_n(T_i)$, $1 \leqq i \leqq d$, in the particular case where $E = L^1$.

3.4 PROPOSITION. *Let $T_1, \dots, T_d$ be positive, commuting contractions on L^1, and let δ in L^1_+ be such that $\delta = T_i\delta$, $1 \leqq i \leqq d$. Then for each $f \in L^1$, the nets*

$$(1_{\{\delta>0\}}A_n(T_i)f;\ n \geqq 1)$$

for $i = 1, \dots, d$, respectively

$$(1_{\{\delta>0\}}A_t f;\ t \in I)$$

converge in L^1 to $A_\infty(T_i)f$, $i = 1, \dots, d$, respectively to $A_\infty(T_1) \dots A_\infty(T_d)f$.

Proof. The strong convergence of the nets is a direct consequence of Lemmas 3.1, 3.2 and 3.3. Indeed, fix $\epsilon > 0$, and by Lemma 3.3 choose t and u_0 such that

$$\|1_{\{\delta>0\}}A_u f - A_u(1_{\{\delta>0\}}A_t f\| < \epsilon$$

for each $u \geqq u_0$. Lemma 3.1 shows that the net

$$(A_u(1_{\{\delta>0\}}A_t f);\ u \in I)$$

converges strongly. This proves that the net $(1_{\{\delta>0\}}A_u f)$ is Cauchy, and hence converges strongly, say to $\bar{f}$. A similar argument shows that each sequence

$$(1_{\{\delta>0\}}A_n(T_i)f;\ n \geqq 1)$$

for $i = 1, \ldots, d$ converges strongly, say to $A_\infty(T_i)f$.

Thus, it remains to prove that

$$\bar{f} = A_\infty(T_1) \ldots A_\infty(T_d)f \quad \text{for } f \in L^1_+:$$

this is an immediate consequence of the equality

$$\lim_t \| 1_{\{\delta>0\}} A_t f - A_{(t(1),\ldots,t(d-1),0)}(1_{\{\delta>0\}} A_{t(d)}(T_d)f) \| = 0.$$

By Lemma 3.2, this equality follows from

$$\liminf_k \limsup_t \| 1_{\{\delta>0\}} A_t f - A_t(1_{\{\delta>0\}} A_k(T_d)f) \| = 0.$$

Clearly, for $f \in L_1^+$

$$A_t(1_{\{\delta>0\}} A_k(T_d)f) \leqq 1_{\{\delta>0\}} A_k(T_d) A_t f,$$

and Lemma 2.2 show

$$\lim_k \limsup_t \| (A_t(1_{\{\delta>0\}} A_k(T_d)f) - 1_{\{\delta>0\}} A_t f)^+ \| = 0.$$

Thus it suffices to show that

$$\liminf_k \limsup_t \| (1_{\{\delta>0\}} A_t f - A_{t(d)}(T_d)(1_{\{\delta>0\}} A_{(t(1),\ldots,t(d-1),k)} f))^+ \| = 0.$$

Suppose the contrary, and choose $\alpha > 0$ and k_0 such that for each $k \geqq k_0$,

$$\limsup_t \| (1_{\{\delta>0\}} A_t f - A_{t(d)}(T_d)(1_{\{\delta>0\}} A_{(t(1),\ldots,t(d-1),k)} f))^+ \| > \alpha.$$

Clearly

$$\lim_k \limsup_t \| (A_{t(d)}(T_d)(1_{\{\delta>0\}} A_{(t(1),\ldots,t(d-1),k)} f) - 1_{\{\delta>0\}} A_t f)^+ \| = 0.$$

Set $r_0 = k_0$, and suppose that r_{n-1} has been defined. Choose $r_n > r_{n-1}$, and $t_n \geqq (n, \ldots, n, r_n)$ such that if

$$s_n = (t_n(1), \ldots, t_n(d-1), r_n),$$

$$g(t) = 1_{\{\delta>0\}} A_t f,$$

$$p_n = (g(t_n) - A_{t_n}(d)(T_d) g(s_n))^+,$$

$$q_n = (A_{t_n}(d)(T_d) g(s_n) - g(t_n))^+,$$

then $\|p_n\| \geqq \alpha$, and $\|q_n\| \leqq \alpha/n$.

One has

$$g(t_n) = A_{t_n}(d)(T_d) g(s_n) + p_n - q_n;$$

applying A_t on both sides and letting t go to infinity yields that

$$\bar{g}(t_n) = \bar{g}(s_n) + \bar{p}_n - \bar{q}_n,$$

with $\|\bar{p}_n\| \geqq \alpha$, and $\|\bar{q}_n\| \leqq \alpha/n$ by Lemma 3.1. Hence

$$\limsup_n \|\bar{g}(t_n) - \bar{g}(s_n)\| \geqq \alpha.$$

Since the net $(g(t);\ t \in I)$ converges strongly, the sequences $\bar{g}(t_n)$ and $\bar{g}(s_n)$ have the same limit, which gives a contradiction, and completes the proof.

3.5 *Remark*. Under the assumptions of Proposition 3.4, similar approximation properties show that the sequences

$$(\delta \Lambda A_n(T_i)f;\ n \geqq 1)$$

for $1 \leqq i \leqq d$, respectively the net

$$(\delta \Lambda A_t f;\ t \in I),$$

converge strongly to $\bar{T}_i f$, $1 \leqq i \leqq d$, and, respectively, $\bar{T}_1 \dots \bar{T}_d f$.

Let $\delta \in E_+$ be an element invariant under each operator T_i. We give sufficient conditions for the strong convergence of the net $P_\delta A_t f$ for each $f \in E_+$; this extends Proposition 3.4.

3.6 Lemma. *Let $T_1, \dots, T_d$ be positive commuting operators on E. Let $\delta \in E_+$ be invariant under the T_i, i.e., $T_i\delta = \delta$ for $i = 1, \dots, d$, and suppose that there exists $H \in E_+^*$, subinvariant under the T_i^*, i.e., such that $T_i^* H \leqq H$ for $i = 1, \dots, d$. Then for each $f \in P_H E_+$, the sequences*

$$(\delta \Lambda P_H A_n(T_i)f;\ n \geqq 1)$$

for $i = 1, \dots, d$ and, respectively, the net

$$(\delta \Lambda P_H A_t f;\ t \in I),$$

converge strongly to $\bar{T}_i f$ for $i = 1, \dots, d$, and, respectively, to $\bar{T}_1 \dots \bar{T}_d f$.

Proof. For every function $f \in E_+$ with support included in the support of H, set

$$\tau_i f = H T_i\left(\frac{f}{H}\right), \quad i = 1, \dots, d,$$

with the convention $f/H = 0$ on $\{H = 0\}$. Then the operators τ_i commute, and are contractions of $L^1((H > 0), \mu)$. Indeed,

$$\int \tau_i |f| d\mu = \int H T_i\left(\frac{|f|}{H}\right) d\mu = \int T_i^* H \frac{|f|}{H} d\mu \leqq \int |f| d\mu.$$

For each $f \in E$, one has that $f \cdot H \in L^1((H > 0), \mu)$, and, if

$$a_t = \Pi\, (t)^{-1} \sum_{u \in [0,t[} \tau^u,$$

then $A_t f = (1/H) a_t (Hf)$ on $\{H > 0\}$ for every $t \in I$.

Since $H\delta$ is invariant under each operator τ_i, the Remark 3.5 shows that for each $f \in P_H E_+$, the sequences

$$(H\delta \Lambda A_n(\tau_i)(Hf);\ n \geqq 1)$$

for $i = 1, \ldots, d$, and, respectively, the net

$$(H\delta \Lambda a_t(Hf);\ t \in I)$$

converge in L^1 to $\bar{\tau}_i(Hf)$, $1 \leqq i \leqq d$, and, respectively to $\bar{\tau}_1 \ldots \bar{\tau}_d(Hf)$. Hence for $i = 1, \ldots, d$

$$\lim_n \int H\left|\delta \Lambda P_H A_n(T_i)f - \frac{1}{H}\bar{\tau}_i(Hf)\right| d\mu = 0,$$

$$\lim_t \int H\left|\delta \Lambda P_H A_n(T_i)f - \frac{1}{H}\bar{\tau}_1\left(H \cdot \frac{1}{H}\bar{\tau}_2(\ldots (\bar{\tau}_d(Hf)) \ldots)\right| d\mu = 0.$$

Set

$$\bar{T}_i f = \frac{1}{H}\bar{\tau}_i(Hf) \quad \text{for every } f \in P_H E,$$

and apply (1.2) to the sublattice $P_H E$ to conclude the strong convergence of

$$(\delta \Lambda P_H A_t f;\ t \in I)$$

to $\bar{T}_1 \ldots \bar{T}_d f$.

The following lemma compares the supports of elements of E_+ invariant under each operator T_i with that of elements of E^*_+ invariant under each adjoint operator T_i^*, under the assumption (C_1) on the seminorm N.

3.7 LEMMA. *Let N be an order continuous seminorm with the properties (1.8) and (C_1). Let $T_1, \ldots, T_d$ be positive, commuting operators on E, contracting N. Let $H \in E^*_+$ be such that $T_i^* H \leqq H$ for $i = 1, \ldots, d$, and $TLA_t g = 0$ if $H(|g|) = 0$. Then for every element $\delta \in E_+$ invariant under each T_i, $i = 1, \ldots, d$, one has that $P_H\delta = \delta$, i.e., $\{\delta > 0\} \subset \{H > 0\}$.*

Proof. Let $\delta = T_i\delta$ for $i = 1, \ldots, d$; set $\delta = f + g$ with

$$f = P_H\delta \quad \text{and} \quad g = (I - P_H)\delta.$$

Then for each $t \in I$,

$$\delta = A_t f + A_t g \quad \text{and} \quad TLA_t g = 0.$$

Since $A_t g$ belongs to the order interval $[0, \delta]$, we have that

$$\text{slim}_t\, A_t g = 0,$$

and hence $A_t f$ converges strongly to δ. Let (ϵ_n) be a sequence of positive reals, and let t_n be a sequence of indices such that

$$N(A_{t_n} f) \leqq N(f) + \epsilon_n.$$

Then

$$\begin{aligned} N(\delta) &= N(A_{t_n} f + A_{t_n} g) \\ &\leqq N(f) + \epsilon_n + N(A_{t_n} g). \end{aligned}$$

Letting $n \to \infty$ yields that $N(\delta) \leqq N(f)$. Since $0 \leqq f \leqq \delta$, the assumption (C_1) made on N shows that $N(\delta - f) = 0$, and hence that $(I - P_H)f = 0$ by (1.8).

Finally, the following lemma shows the existence of a positive element $H \in E^*$ invariant under the adjoint operators T_i^*, and satisfying the conditions in Lemma 3.7.

3.8 LEMMA. *Let $T_1, \dots, T_d$ be positive, mean-bounded operators on E which commute. Then there exists an $H \in E_+^*$ such that*

(i) *$T_i^* H = H$ for $i = 1, \dots, d$*
(ii) *$TLA_t f = 0$ for each $f \in E$ with $H(|f|) = 0$.*

Proof. Let U be a strictly positive element in E^*. Since $\|A_t\|$ is bounded, given any subset $\bar{I} \subset I$, there exists a subnet $J \subset \bar{I}$ such that the net $(A_t^* U;\ t \in J)$ is weak-star convergent to $e_0 \in E_+^*$. Then

$$\begin{aligned} T_1^* e_0 &= \text{weak* } \lim_{t \in J} T_1^* A_t^* U \\ &\geqq \text{weak* } \lim_{t \in J} \left[A_t^* U - \frac{1}{t(1)} A^*_{(0,t(2),\dots,t(d))} U \right] \\ &\geqq e_0, \end{aligned}$$

and a similar computation shows that $T_i^* e_0 \geqq e_0$ for every $i = 2, \dots, d$. Since the net $(T^{*t} e_0,\ t \in I)$ is increasing, the net $(A_t^* e_0,\ t \in I)$ is also increasing, norm-bounded, and hence converges strongly to $e' \in E_+^*$. Clearly $T_i^* e' = e'$ for every $i = 1, \dots, d$. Let

$$\mathscr{T}^* = \{h \in E_+^*,\ T_i^* h = h \text{ for } i = 1, \dots, d\},$$

and let ν be a finite measure equivalent with μ. Choose a sequence $h_n \in \mathscr{T}^*$ such that

$$\lim \nu(h_n > 0) = \sup\{\nu(h > 0);\ h \in \mathscr{T}^*\}.$$

The case

$$\sup\{\nu(h > 0);\ h \in \mathscr{T}^*\} = 0$$

is trivial. Thus we suppose that each h_n is non-null, and set

$$H = \sum_{n\geqq 0} 2^{-n}\|h_n\|^{-1}h_n \in E^*_+.$$

Then $H \in \mathscr{T}^*$, and H has the maximal support. Let $f \in E_+$ be such that $H(f) = 0$, and suppose that

$$\limsup_t U(A_t f) \neq 0.$$

Let $t_n \in I$ be a subsequence such that

$$\lim U(A_{t_n} f) = \alpha > 0.$$

Let (u_n) be a further subsequence such that wlim $A^*_{u_n} U = e$, and let

$$h = \lim A^*_t e.$$

Then $h \in \mathscr{T}^*$, and hence $h(f) = 0$. Now

$$h(f) \geqq e(f) = \lim_n A^*_{u_n} U(f)$$
$$= \lim_n U(A_{u_n} f) = \alpha > 0,$$

which gives a contradiction and concludes the proof.

The following theorem proves the stochastic convergence of the net $A_t f$, and relates the limit of the averages of the semigroup to the stochastic limits of the averages of each operator.

3.9 THEOREM. *Let E be a Banach lattice with* (A) *and* (B), *let $T_1, \dots, T_d$ be positive, commuting, mean-bounded operators on E. Let N be an order continuous seminorm on E with the properties* (1.8) *and* (C), *and such that each operator T_i contracts N. Then*

(i) *If E is represented as a function space over $(\Omega, \mathscr{F}, \mu)$, then Ω admits a decomposition $\Omega = Y + Z = P + D + Z$ such that*:

(a) *There exists $H \in E^*_+$ with $T^*_i H = H$ for $i = 1, \dots, d$, and $Y = \{H > 0\}$.*

(b) *There exists $\delta \in E_+$ with $T_i\delta = \delta$ for $i = 1, \dots, d$, and $P = \{\delta > 0\}$.*

(c) *For $f \in E_+$ with $\{f > 0\} \subset Z$, one has that $TLA_t f = 0$.*

(d) *For $f \in E_+$ with $\{f > 0\} \subset Y$, one has that*

$$TL\,(1_{D+Z} A_t f) = 0,$$

and, letting $s = (t_1, \dots, t_{d-1})$, one has that

$$\lim_t \delta \Lambda A_t f = \lim_s A_{(s,0)} \lim_{t_d}(\delta A_{t_d} f).$$

(ii) *For each $f \in E$, the truncated limits $TL_n A_n(T_i) f = A_\infty(T_i) f$, $1 \leqq i \leqq d$, and $TLA_t f = A_\infty(T_1) \dots A_\infty(T_d) f$ exist.*

Proof. (i) Lemma 3.8 shows the existence of $H \in E^*_+$, Proposition 2.6 shows the existence of $\delta \in E_+$, and Lemma 3.7 relates the supports P of δ, and $P + D = Y$ of H, which proves (a) and (b). Part (c) is a consequence of Lemma 3.9, and (d) follows from Proposition 2.6 and Lemma 3.6.

(ii) Apply part (i) with $T_i = \ldots = T_{d-1} = I$, and let $\Omega = P(d) + D(d) + Z(d)$ be the corresponding decomposition. Then

$$P \subset P(d),$$

$$TL\ (1_{D(d)+Z(d)}A_n(T_d)f) = 0 \quad \text{for each } f \in E, \quad \text{and}$$

$$TL\ A_n(T_d)f = 0 \quad \text{for each } f \in E \text{ with } \{\,|f| \neq 0\} \subset Z(d).$$

Let $\gamma \in E_+$ be such that

$$T_d\gamma = \gamma, \quad \{\gamma > 0\} \subset P(d),$$

and fix $f \in E_+$ with

$$\{f > 0\} \subset Y(d) = P(d) + D(d).$$

Then for each $k > 0$, Lemma 3.6 shows that the sequence,

$$(k\gamma\Lambda A_n(T_d)f;\ n > 0)$$

converges strongly, say to $f(k)$. The sequence $f(k)$ is increasing, and converges strongly to

$$TL\ (1_{P(d)}A_n(T_d)f).$$

This completes the proof of the existence of

$$TL\ A_n(T)f = A_\infty f \quad \text{for each } f \in E_+,$$

and hence for each $f \in E$. If $\{\,|f| > 0\} \subset P(d)$, then

$$TL\ A_n(T_d)f = \text{slim}\ A_n(T_d)f.$$

Let $f \in E_+$ be such that $\{f > 0\} \subset Y$. For each fixed k, the net

$$(k\delta\Lambda A_t f;\ t \in I)$$

converges strongly, say to $\bar{f}_k$, and the sequence $\bar{f}_k$ increases to $\bar{f} = TL\ A_t f$. Fix $\epsilon > 0$; by the definition of TL, Proposition 3.4, Lemmas 3.6 and 3.7, we can choose $k > 0$, $t \in I$ such that

$$\|\bar{f} - k\delta\Lambda A_u f\| < \epsilon,$$

$$\|TL\ P_\delta A_n(T_d)f - k\delta\Lambda A_n(T_d)f\| < \epsilon,$$

$$\|k\delta\Lambda A_u f - A_{(u(1),\ldots,u(d-1),0)}(k\delta\Lambda A_n(T_d)f)\,\| < \epsilon,$$

and

$$\|A_\infty(T_1)\ldots A_\infty(T_{d-1})(TLP_\delta A_n(T_d)f) - A_{(u(1),\ldots,u(d-1),0)}TL(P_\delta A_n(T_d)f)\,\| < \epsilon$$

for every $u = (u(1), \ldots, u(d-1), n) \geqq t$. Set $M = \sup \|A_t\|$, then

$$\|\bar{f} - A_\infty(T_1) \dots A_\infty(T_{d-1})(TL\ P_\delta A_n(T_d)f)\| \leqq \epsilon(3 + M).$$

Hence

$$\bar{f} = A_\infty(T_1) \dots A_\infty(T_{d-1})(TL\ P_\delta A_n(T_d)f)$$
$$\leqq A_\infty(T_1) \dots A_\infty(T_{d-1})A_\infty(T_d)f \leqq \bar{f},$$

which identifies $\bar{f}$.

Clearly if $f \in E_+$, then

$$\bar{f} \geqq A_\infty(T_1) \dots A_\infty(T_d)f.$$

Thus for $f \in E_+$ with support $f \subset Z$, one has

$$0 = \bar{f} = A_\infty(T_1) \dots A_\infty(T_d)f.$$

This completes the proof of (ii).

We say that the Banach lattice satisfies the condition (C) if the norm $\|\cdot\|$ satisfies (C) (cf. [**3**], [**4**], [**5**]). Theorem 3.9 yields the existence of truncated limits for averages of a semigroup of contractions of a Banach lattice with the properties (A), (B) and (C); the one parameter case was proved in [**4**]. More precisely, we have the following:

3.10 COROLLARY. *Let E be a Banach lattice with* (A), (B) *and* (C), *let $T_1, \dots, T_d$ be positive commuting contractions of E. Then for each $f \in E$, the truncated limits*

$$TLA_n(T_i)f = A_\infty(T_i)f, \quad 1 \leqq i \leqq d, \quad \text{and}$$
$$TLA_tf = A_\infty(T_1) \dots A_\infty(T_d)f$$

exist.

Proposition 2.6, Lemma 3.6 and Theorem 3.9 also yield the existence of truncated limits for averages of a semigroup, provided there is a strictly positive subinvariant element for the adjoint semigroup, without any extra assumption on the norm of the Banach lattice.

3.11 COROLLARY. *Let E be a Banach lattice with* (A) *and* (B), *let $T_1, \dots, T_d$ be positive, commuting mean-bounded operators on E. Let $H \in E^*_+$ be strictly positive and such that $T_i^*H \leqq H$ for every $i = 1, \dots, d$. Then*

(i) *If E is represented as a function space over $(\Omega, \mathcal{F}, \mu)$, there exists a decomposition $\Omega = P + D$ such that*

(a) *There exists $\delta \in E_+$, with $T_i\delta = \delta$ for every $i = 1, \dots, d$, and $\{\delta > 0\} = P$.*

(b) *For each $f \in E$, $TL\ 1_DA_tf = 0$.*

(c) *For each $f \in E_+$, the net $(\delta \Lambda A_tf)$ converges strongly to*

$$\lim_s A_{(s,0)}(\lim_n \delta \Lambda A_n(T_d)f),$$

where $s = (t_1, \ldots, t_{d-1})$.

(ii) *For each* $f \in E$, *the truncated limits*

$$TL_n(A_n(T_i)f) = A_\infty(T_i f), \quad \textit{and}$$

$$TL_t A_t f = A_\infty(T_1) \ldots A_\infty(T_d) f$$

exist.

Proof. Set $N(f) = H(|f|)$ for each $f \in E$; then N is an order continuous seminorm with the properties (1.8) and (C). Apply Proposition 2.6 and Lemma 3.6 to prove (i), and Theorem 3.9 to prove (ii).

4. Demiconvergence and pointwise convergence. In this section we relate the truncated limit for semigroups of positive operators, obtained in Section 3, to the pointwise lower limit. We prove one-parameter results which we extend to several parameter semigroups by a simple general argument.

Recall that an order continuous Banach lattice with a weak unit admits a representation as a Köthe function space. Hence the Banach lattice E we consider is assumed to be a function space over a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$. Let I be directed set filtering to the right, and let $(f_t;\ t \in I)$ be a net of real-valued functions in E such that $TL\ f_t$ exists. Then

$$\liminf f_t \leqq TLf_t \leqq \limsup f_t.$$

We say that the net (f_t) is *lower* (*upper*) *demiconvergent* if

$$\liminf f_t = TLf_t \quad (\limsup f_t = TLf_t).$$

First, in the setting of the previous section, there is a one-parameter demiconvergence result.

4.1 THEOREM. *Let E be a Banach lattice satisfying* (A) *and* (B), *let N be an order continuous seminorm on E with the properties* (1.8) *and* (C), *and let T be a positive linear, mean-bounded operator on E, contracting N. Then, for each $f \in E_+$, the sequence $A_n(T)f$ is lower demiconvergent, i.e.,*

$$TLA_n(T)f = \liminf A_n(T)f.$$

Proof. Apply Theorem 3.9 with $d = 1$ and $T_1 = T$. For $f \in E_+$ with

$$\{f > 0\} \subset \{H > 0\} = Y = P + D,$$

set

$$\tau f = HT(f/H);$$

τ is a positive contraction of $L^1((H > 0), \mu)$. Then

$$\tau(H\delta) = H\delta,$$

and the Chacon-Ornstein theorem **[9]** implies that the ratio

$$\sum_{0 \leqq i \leqq n} \tau^i g/nH\delta$$

converges μ a.e. on $\{H\delta > 0\} = \{\delta > 0\}$ for each

$$g \in L^1_+(\{H > 0\}, \mu).$$

Hence for $f \in E_+$ with $\{f > 0\} \subset \{H > 0\}$, the sequence $A_n(T)f$ converges a.e. on $P = \{\delta > 0\}$, so that

$$\liminf A_n(T)f = TLA_n(T)f$$

on P. For $f \in E_+$ with $\{f > 0\} \subset \{H > 0\} = Y$, set

$$f_n = 1_{\{\delta > 0\}} A_n(T)f,$$

and for $f \in E_+$ with support contained in Z, set $f_n = A_n(T)f$. In both cases the inequalities

$$0 \leqq \liminf f_n \leqq TL\, f_n = 0$$

conclude the proof.

We now extend demiconvergence to multiparameter semigroups by an argument similar to that given in [**28**]; see also [**16**]. Let E be a Banach lattice, and let $F \subset E$.

A map $T:F \rightarrow E$ is *positively homogeneous* if $T(\alpha f) = \alpha T(f)$ for each $\alpha \in \mathbf{R}_+$ and $f \in F$; *increasing* if $f \leqq g$ implies $Tf \leqq Tg$; *subadditive* (*superadditive*) if $T(f + g) \leqq (\geqq) Tf + Tg$; *continuous at zero* if for every net (g_t) in F, $\|g_t\| \rightarrow 0$ implies $\|Tg_t\| \rightarrow 0$. We call T *monotonely continuous for order* (MCO) if for every net (f_t), $f_t \downarrow f$ (resp. $f_t \uparrow f$) implies that $Tf_t \downarrow Tf$ (resp. $Tf_t \uparrow Tf$).

4.2 LEMMA. *Let E be a Banach lattice with order continuous norm, let F be a* (*closed*) *sublattice of E, and let T be a positively homogeneous increasing map on F_+. Then T is continuous at zero.*

Proof. Let I be a directed set and let $(f_t;\ t \in I)$ be a net of elements of F_+ such that

$$\lim\|f_t\| = 0 \quad \text{and} \quad \lim\sup\|Tf_t\| > \epsilon > 0.$$

Choose a sequence $(t(n))$ of indices such that

$$\sum 2^n \|f_{t(n)}\| < \infty \quad \text{and} \quad \inf\|Tf_{t(n)}\| \geqq \epsilon;$$

set

$$g_n = \sum_{1 \leqq k \leqq n} 2^k f_{t(k)}.$$

The sequence (g_n) increases to some element $g \in E_+$. Since F is closed, one has that $g \in F_+$, and for every n,

$$Tg \geqq Tg_n \geqq T(2^n f_{t(n)}) = 2^n T(f_{t(n)}) \geqq 2^n \epsilon.$$

This gives a contradiction, and hence proves the lemma.

4.3 LEMMA. *Let E be a Banach lattice with an order continuous seminorm, let F be a closed sublattice of E, and let $T:F_+ \to E$ be an increasing, positively homogeneous, subadditive map. Then T is monotonely continuous for order.*

Proof. Let (f_t) be a net of elements of F_+ such that $f_t \uparrow f$ $(f_t \downarrow f)$. Then

$$\lim_t \|f_t - f\| = 0,$$

and Lemma 4.2 shows that

$$\lim_t \|T|f_t - f|\,\| = 0.$$

Now, for every index t,

$$Tf_t \leqq Tf \leqq Tf_t + T(f - f_t)$$

$$(\text{respectively } Tf \leqq Tf_t \leqq Tf + T(f_t - f));$$

thus $|Tf - Tf_t| \leqq T|f - f_t|$, which concludes the proof.

4.4 PROPOSITION. *Let E be a Banach lattice with a weak unit and an order continuous norm, and let F be a closed sublattice of E. Let I, J be directed sets with countable cofinal subsets. Let $T_t:F \to E$ be a net of positive linear operators indexed by I.*

(i) *Let $(f_j;\ j \in J)$ be a net of elements of F_+ such that* (a) *and* (b) *hold*:

(a) $f_\infty = \liminf_j f_j \in F_+$.

(b) *There exists an increasing, positively homogeneous, subadditive map $T_\infty:F_+ \to E$ such that*

$$T_\infty f \leqq \liminf_t T_t f \quad \textit{for every } f \in F_+.$$

Then $T_\infty f_\infty \leqq \liminf_{t,j} T_t f_j$.

(ii) *Let $(f_j;\ j \in J)$ be a net of elements of F_+ such that*

(c) $\sup_j f_j \in F_+$.

(d) $T_\infty f = \limsup_t T_t f \in E$ *for each* $f \in F_+$. *Set* $f_\infty = \limsup_j f_j$; *then*

$$T_\infty f_\infty \geqq \limsup_{t,j} T_t f_j.$$

(iii) *Let $(f_j;\ j \in J)$ be a net of elements of F such that*

(e) $\lim_j f_j = f_\infty \in F$ *and* $\sup_j |f_j| \in F$;

(f) $\lim_t T_t f = T_\infty f$ *for each* $f \in F$, *and T_∞ is monotonely continuous for order* (MCO).

Then $\lim_{t,j} T_t f_j = T_\infty f_\infty$.

Proof. We assume, to simplify the proof, that both directed sets I and J are filtering to the right.

(i) For each $j \in J$, set

$$m_j = \inf_{l \geqq j} f_l \in F_+.$$

By assumption, the net $(m_j;\ j \in J)$ increases to $f_\infty \in F_+$. Furthermore, for every $t \in I$,

$$\inf_{u \geqq t} \inf_{l \geqq j} T_u f_l \geqq \inf_{u \geqq t} T_u m_j.$$

Let $t \to \infty$; then

$$\liminf_t \inf_{l \geqq j} T_t f_l \geqq \liminf_t T_t m_j \geqq T_\infty m_j.$$

Since T_∞ is (MCO) by Lemma 4.3, we obtain, by letting $j \to \infty$,

$$\liminf_{t,j} T_t f_j \geqq T_\infty f_\infty.$$

(ii) For each $j \in J$, let $M_j = \sup_{l \geqq j} f_l \in F_+$. For each $t \in I$, one has that

$$\sup_{u \geqq t} \sup_{l \geqq l} T_u f_l \leqq \sup_{u \geqq t} T_u M_j.$$

Letting $t \to \infty$ yields that

$$\limsup_t \sup_{l \geqq j} T_t f_l \leqq \limsup_t T_t M_j = T_\infty M_j.$$

The map T_∞ is clearly subadditive, increasing and positively homogeneous; hence Lemma 4.3 shows that T_∞ is (MCO). Let $j \to \infty$; the net M_j decreases to $f_\infty \in F_+$, which concludes the proof.

(iii) Apply (i) and (ii), and analogous results for nets in F_-.

4.5 THEOREM. *Let $L(1) \supset L(2) \supset \dots \supset L(d)$ be Banach lattices with order continuous norm, each with weak unit, and let $I(i)$; $1 \leqq i \leqq d$ be directed sets with countable cofinal subsets. For each $i \in \{1, \dots, d\}$, let $(T(i, t);\ t \in I(i))$ be a net of positive linear operators from $L(i)$ to $L(1)$. Suppose that*

(i) $\limsup_t T(i, t)f \in L(1)$ *for each* $f \in L(i)_+$

and

(ii) $\sup_t T(i, t)f \in L(i - 1)$ *for each* $f \in L(i)_+$, $2 \leqq i \leqq d$.

Then for each $f \in L(d)_+$

$$\limsup_t T(1, t(1)) \dots T(d, t(d))f \leqq T(1, \infty) \dots T(d, \infty)f.$$

Proof. For $d = 2$, one has $\sup_t T(2, t)f \in L(1)_+$ for $f \in L(2)_+$; hence Proposition 4.4 (ii) applied with $E = F = L(1)$, and the nets

$$(f_j = T(2, j)f;\ j \in I(2)) \quad \text{and} \quad (T_t = T(1, t);\ t \in I(1))$$

gives the inequality.

We suppose that the thoerem holds for d, and prove it for $d + 1$. Let $f \in L(d + 1)_+$, $F = L(d)$, $E = L(1)$, $J = I(d + 1)$, $(f_j = T(d + 1, j)f;\ j \in J)$, $I = I(1) \times \dots \times I(d)$, and for $t = (t(1), \dots, t(d)) \in I$, set

$$U_t = T(1, t(1)) \dots T(d, t(d)).$$

Apply Proposition 4.4 (ii) with

$$f_\infty = \limsup_j T(d + 1, j)f \in F_+, \quad \text{and}$$

$$U_\infty = \limsup_t T(1, t(1)) \dots T(d, t(d)).$$

Then

$$U_\infty f_\infty \geqq \limsup_{t(1),\dots,t(d+1)} U_t T(d + 1, t(d + 1))f,$$

and the induction assumption applied to $L(1) \supset \dots \supset L(d)$, the operators $(U_t; t \in I(1) \times \dots \times I(d))$ and $f_\infty \in L(d)_+$ proves the inequality for the product of $d + 1$ operators.

4.6 THEOREM. *Let E be a Banach lattice with a weak unit and an order continuous norm, and let $I(i)$, $i = 1, \dots, d$ be directed sets with countable cofinal subsets. For each $i = 1, \dots, d$, let $(T(i, t); t \in I(i))$ be a net of positive linear operators on E such that there exists a positively homogeneous subadditive map $T(i, \infty):E_+ \to E$ such that*

$$T(i, \infty)f \leqq \liminf_t T(i, t)f \in E_+ \text{ for every } f \in E_+.$$

Then for each $f \in E_+$

$$\liminf_t T(1, t(1)) \dots T(d, t(d))f \geqq T(1, \infty) \dots T(d, \infty)f.$$

Proof. In the case $d = 2$, the inequality is a direct consequence of Proposition 4.4 (i). Suppose that the inequality holds for any product of d operators, and prove it for a product of $d + 1$ operators. Let $f \in E_+$, $F = E$, and let I and J, (f_j) and (U_t) be defined as in the proof of Theorem 4.5. Set

$$U_\infty = T(1, \infty) \dots T(m, \infty);$$

since each map $T(i, \infty)$, $1 \leqq i \leqq d$, is increasing, positively homogeneous and subadditive on E_+, the map U_∞ has the same properties. The induction hypothesis and Proposition 4.4 show that

$$f_\infty = \liminf_f T(d + 1, t)f \geqq T(d + 1, \infty)f,$$

hence

$$\begin{aligned} U_\infty T(d + 1, \infty)f &\leqq U_\infty f_\infty \\ &\leqq \liminf_{t(1),\dots,t(d),t(d+1)} T(1, t(1)) \\ &\qquad \dots T(d, t(d))T(d + 1, t(d + 1))f. \end{aligned}$$

Since the truncated limit of a net of positive linear operators on a Banach lattice E defines a new linear positive operator on E, Theorem 4.6 immediately yields the following:

4.7 PROPOSITION. *Let E be a Banach lattice with* (A) *and* (B). *Fix $d \geqq 1$ and for every $i = 1, \ldots, d$, let $I(i)$ be a directed set with a countable cofinal subset, and let $(T(i, t); t \in I(i))$ be a net of positive linear operators on E. Suppose that*:

(i) *For each $i = 1, \ldots, d$, and each $f \in E_+$, the net $T(i, t)f$ is lower demiconvergent, i.e.*,

$$T(i, \infty)f = TLT(i, t)f = \liminf T(i, t)f.$$

(ii) *For each $f \in E_+$, $TL(T(1, t(1)) \ldots T(d, t(d))f)$ exists and is equal to $T(1, \infty) \ldots T(d, \infty)f$.*

Then

$$\begin{aligned} &\liminf_{t(1),\ldots,t(d)} T(1, t(1)) \ldots T(d, t(d))f \\ &= T(1, \infty) \ldots T(d, \infty)f \\ &= TL(T(1, t(1)) \ldots T(d, t(d)f) \end{aligned}$$

for every $f \in E_+$.

Now the Theorems 3.9, 4.1 and 4.7 yield demiconvergence of multi-parameter semigroups, the main result of the present section.

4.8 THEOREM. *Let E be a Banach lattice with* (A) *and* (B), *let N be an order continuous seminorm with* (1.8) *an* (C), *let $T_1, \ldots, T_d$ be positive, mean-bounded, commuting operators which contract N. Then for each $f \in E_+$,*

$$TLA_tf = \liminf A_tf.$$

Specifying the seminorm N, we obtain the following corollaries, which strengthen Corollaries 3.10 and 3.11.

4.9 COROLLARY. *Let E be a Banach lattice with* (A), (B) *and* (C), *let $T_1, \ldots, T_d$ be positive, commuting contractions of E. Then for each $f \in E_+$,*

$$TLA_tf = \liminf_t A_tf.$$

4.10 COROLLARY. *Let E be a Banach lattice with* (A) *and* (B), *let $T_1, \ldots, T_d$ be positive, commuting mean-bounded operators on E, and let $H \in E_+^*$ be strictly positive, such that $T_i^*H \leqq H$ for $i = 1, \ldots, d$. Then for each $f \in E_+$,*

$$TLA_tf = \lim_t \inf A_tf.$$

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