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AUTOMORPHISMS OF CERTAIN P-GROUPS (P ODD).

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This paper shows that amongst the p-groups of order p^5, where p denotes an odd prime, there is only one group whose automorphism group is again a p-group. This automorphism group has order p^6 and it is shown that this is the smallest order a p-group may have when it occurs as an automorphism group. The paper also shows that all groups of order p^5 have an automorphism of order 2 apart from the group above and three other related groups.

In [**6**] MacHale considers p-groups which can occur as the automorphism group of a finite group. In particular he conjectures that for p odd, the smallest p-group which can arise in this way is of order p^{10} and is the automorphism group of a certain 3-generator class 2 p-group of order p^6. He also conjectures in connection with this that every group of order p^5, p odd, has an automorphism of order 2. The purpose of this paper is to show that both conjectures are false. Amongst the groups of order p^5, $p > 3$, there is one group

$$G_0 = \langle \alpha_1, \alpha \mid \alpha^p = [\alpha_1, \alpha]^p = [\alpha_1, \alpha, \alpha]^p = [\alpha_1, \alpha, \alpha, \alpha]^p = [\alpha_1, \alpha, \alpha, \alpha, \alpha] = 1,$$
$$\alpha_1^p = [\alpha_1, \alpha, \alpha, \alpha] = [\alpha_1, \alpha, \alpha_1]^{-1} \rangle$$

defined when $(p-1, 3) = 1$, and three groups defined when $(p-1, 3) = 3$, which have no automorphisms of order 2. The latter three groups however possess automorphisms of order 3. We show that G_0 is the unique group of order p^5 whose automorphism group is again a p-group, that $|\operatorname{Aut} G_0| = p^6$, and that p^6 is the smallest order which can occur when a p-group, p odd, is an automorphism group.

The notation used is standard and that of Gorenstein [**3**]. In particular the commutator $[x, y, z, \ldots] = [[[x, y], z], \ldots]$, where $[x, y] = x^{-1}y^{-1}xy$. We do however denote the lower central series of a group G by $\gamma_1(G) = G$, $\gamma_2(G) = [G, G]$, $\gamma_i(G) = [\gamma_{i-1}(G), G]$ for $i > 2$, and the cyclic group of order n by C_n.

Throughout, p always denotes an odd prime.

We first consider which groups of order p^5 have an automorphism of order 2. James [**5**] gives the groups of order p^n, $n \leqslant 6$, using the Hall-Senior method of classification

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The group theory program CAYLEY was helpful in considering these groups of order p^5.

by isoclinic families. We use James' list and adopt the same notation for the presentations. In particular, in any such presentation, trivial relations of the form $[\alpha,\beta]=1$ between generators are omitted for economy of space. We also use throughout the following standard procedure to produce an appropriate automorphism. If a group has presentation $G=\langle X|R\rangle$ and $\theta\colon X\to G$, we will denote the image x_i^θ of any generator $x_i\in X$ by $\overline{x}_i$. Then if $G=\langle\overline{X}\rangle$, where $\overline{X}=\{\overline{x}_i\mid x_i\in X\}$ and the generators $\overline{x}_i$ also satisfy the relations R, θ extends to an automorphism of G.

We begin by extending a result of Heineken and Liebeck in Theorem 2, but first state a lemma proved using standard properties of commutators.

LEMMA 1. *Let G be a class 3 group. For any a, $b\in G$*

(i) $[a^{-1},b,b]=[a,b,b]^{-1}$ *(ii)* $[a,b^{-1},b^{-1}]=[a,b,b]$
(iii) $[a^{-1},b^{-1},b^{-1}]=[a,b,b]^{-1}$ *(iv)* $[a^{-1},b^{-1},a^{-1}]=[a,b,a]^{-1}$

THEOREM 2. *If G is a p-group of order p^n, $n\leqslant 5$, and nilpotency class at most 3, then G has an automorphism of order 2.*

PROOF: The result is certainly true if G is abelian (Lemma 5 in [**2**]) or of class 2 (Lemma 7 in [**6**]) or of order p^4 (Lemma 9 in [**6**]). We may thus assume G is a class 3 group of order p^5. From James' list there are four families, ϕ_3, ϕ_6, ϕ_7 and ϕ_8 of such groups. For each of these families we show it is possible to define an automorphism θ which inverts at least one of the generators and fixes the remainder, so θ has order 2.

The family ϕ_3 consists of groups G with $G/Z(G)\approx\phi_2(1^3)$, the nonabelian group of order p^3 and exponent p, and $G'\approx C_p\times C_p$. Thus $G/G'\approx C_p^2\times C_p$ or $G/G'\approx C_p\times C_p\times C_p$ and $G=\langle\alpha,\alpha_1\rangle$ is a 2-generator group or $G=\langle\alpha,\alpha_1,\gamma\rangle$ is a 3-generator group respectively. However for convenience, as in [**5**], define additional generators $\alpha_{i+1}=[\alpha_i,\alpha]$, $i=1,2$. Then in either the 2 or 3 generator case all the groups G satisfy the relations: $\alpha_{i+1}^p=1$, $i=1$, 2; together with 2 (or 3) additional relations of the form: $\alpha^{p^t}=\alpha_3^l$, $\alpha_1^{p^t}=\alpha_3^m$, ($\gamma^p=\alpha_3^n$), where $t=1$ or 2, $0\leqslant l$, m $n<p$. Note $\gamma_2(G)=\langle\alpha_2,\alpha_3\rangle$ and $\gamma_3(G)=\langle\alpha_3\rangle$. Now define θ so that $\overline{\alpha}=\alpha^{-1}$, $\overline{\alpha}_1=\alpha_1^{-1}$, ($\overline{\gamma}=\gamma^{-1}$). Then $\overline{\alpha}_2$ is a conjugate of α_2 and, by Lemma 1, $\overline{\alpha}_3=[\alpha_1^{-1},\alpha^{-1},\alpha^{-1}]=\alpha_3^{-1}$. Thus the barred generators also satisfy the above relations, so θ is the required automorphism.

The family ϕ_6 consists of groups G with $G/Z(G)=\phi_2(1^3)$ and $G'=C_p\times C_p\times C_p$. Thus $G/G'\approx C_p\times C_p$ and $G=\langle\alpha_1,\alpha_2\rangle$ is a 2-generator group. Define additional generators $\beta=[\alpha_1,\alpha_2]$, $\beta_i=[\beta,\alpha_i]$, $i=1,2$. Then all the groups G satisfy the relations: $\beta^p=\beta_i^p=1$, $i=1$, 2; together with 2 further relations of the form: $\alpha_i^p=\beta_1^{m_i}\beta_2^{n_i}$, $i=1$, 2. Note $\gamma_2(G)=\langle\beta,\beta_1,\beta_2\rangle$ and $Z(G)=\gamma_3(G)=\langle\beta_1,\beta_2\rangle$.

Now define θ so $\overline{\alpha}_i = \alpha_i^{-1}$, $i = 1, 2$. Then $\overline{\beta}$ is a conjugate of β and, by Lemma 1, $\overline{\beta}_1 = [\alpha_1^{-1}, \alpha_2^{-1}, \alpha_i^{-1}] = [\alpha_1, \alpha_2, \alpha_i]^{-1} = \beta_i^{-1}$, $i = 1, 2$. Again the barred generators satisfy the relations, so θ is an automorphism.

The family ϕ_7 consists of groups G with $G/Z(G) \approx \phi_2(1^3) \times C_p$ and $G' \approx C_p \times C_p$. Thus $\gamma_3(G) = Z(G) \approx C_p$ and since $G/Z(G)$ is of exponent p, $G/G' \approx C_p \times C_p \times C_p$, so $G = \langle \alpha, \alpha_1, \beta \rangle$ is a 3-generator group. Define additional generators $\alpha_{i+1} = [\alpha_i, \alpha]$, $i = 1, 2$. Then all the groups G satisfy the relations: $\alpha_{i+1}^p = 1$, $i = 1, 2$, $[\alpha_1, \beta] = \alpha_3$; together with 3 further relations of the form: $\alpha_1^p = \alpha_3^l$, $\alpha^p = \alpha_3^m$, $\beta^p = \alpha_3^n$, where $0 \leqslant l, m, n < p$ and m and n are not both nonzero. Note $\gamma_2(G) = \langle \alpha_2, \alpha_3 \rangle$ and $\gamma_3(G) = \langle \alpha_3 \rangle$. When $n = 0$ define θ so $\overline{\alpha}_1 = \alpha^{-1}$, $\overline{\alpha} = \alpha^{-1}$, $\overline{\beta} = \beta$. Then $\overline{\alpha}_2$ is a conjugate of α_2, $\overline{\alpha}_3 = [\alpha_1^{-1}, \alpha^{-1}, \alpha^{-1}] = \alpha_3^{-1}$ by Lemma 1, and $[\overline{\alpha}_1, \overline{\beta}] = [\alpha_1^{-1}, \beta] = [\alpha_1, \beta]^{-1}$ since $[\alpha_1, \beta] \in Z(G)$. When $n \neq 0$ and $m = 0$ define θ so that $\overline{\alpha}_1 = \alpha_1$, $\overline{\alpha} = \alpha^{-1}$, $\overline{\beta} = \beta$. Then $\overline{\alpha}_2$ is a conjugate of α_2^{-1}, $\overline{\alpha}_3 = [\alpha_1, \alpha^{-1}, \alpha^{-1}] = \alpha_3$ by Lemma 1 and $[\overline{\alpha}_1, \overline{\beta}] = [\alpha_1, \beta]$. In either case, the barred generators again satisfy the relations and θ is an automorphism.

Finally the family ϕ_8 consists of just one group $G = \langle \alpha_1, \alpha_2, \beta \mid [\alpha_1, \alpha_2] = \beta = \alpha_1^p, \beta^{p^2} = \alpha_2^{p^2} = 1 \rangle$, which is a split extension of the cyclic group $\langle \alpha_1 \rangle$ by the cyclic group $\langle \alpha_2 \rangle$. Inverting α_1 and fixing α_2 thus gives the desired automorphism θ. ∎

We now deal with the groups of order p^5 and nilpotency class 4, again with the aid of James' list. But first we define the following groups G_r which are denoted by $\phi_{10}(2111)b_r$ in that list:

Definition. For $p > 3$, define G_r by

$$G_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k = \alpha_1^p, \alpha^p = \alpha_{i+1}^p = 1, i = 1, 2, 3 \rangle,$$

where $r + 1 = 1, \ldots, (p-1, 3)$ and $k = g^r$, g being the smallest positive integer which is a primitive root (mod p).

Theorem 3. *Excluding the group(s) G_r defined above, all remaining class 4 groups of order p^5 have an automorphism of order 2.*

Proof: In James' list of groups of order p^5 there are two class 4 families ϕ_9 and ϕ_{10}.

When $p = 3$ these families have a slightly different form, so we consider the groups of order 3^5 first. Each of these is of the form $G = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$, where $\alpha_{i+1} = [\alpha_i, \alpha]$, $i = 1, 2, 3$. For each we content ourselves with giving below James' designation of the group, the defining relations, and the action of an automorphism of

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order 2 on the two basic generators α_1 and α.

$\phi_9(2111)a$: $\alpha^3 = \alpha_4, \alpha_1^3\alpha_3 = \alpha_2^3\alpha_4 = \alpha_3^3 = \alpha_4^3 = 1;\ \overline{\alpha}_1 = \alpha_1^{-2}, \overline{\alpha} = \alpha_1^{-2}\alpha\alpha_1.$

$\phi_9(2111)b_1$: $\alpha_1^3\alpha_3 = \alpha_4, \alpha^3 = \alpha_2^3\alpha_4 = \alpha_3^3 = \alpha_4^3 = 1;\ \overline{\alpha}_1 = \alpha_1^{-1}, \overline{\alpha} = \alpha^{-1}.$

$\phi_9(1^5)$: $\alpha^3 = \alpha_1^3\alpha_3 = \alpha_2^3\alpha_4 = \alpha_3^3 = \alpha_4^3 = 1;\ \overline{\alpha}_1 = \alpha_1^{-1}, \overline{\alpha} = \alpha.$

$\phi_{10}(2111)a_r$: $[\alpha_1, \alpha_2]^{r+1} = \alpha_4^{r+1} = \alpha^3, \alpha_1^3\alpha_3 = \alpha_2^3\alpha_4 = \alpha_3^3 = \alpha_4^3 = 1;\ r = 0, 1;$
$\overline{\alpha}_1 = \alpha_1, \overline{\alpha} = \alpha_1\alpha^2\alpha_1^{-2}.$

$\phi_{10}(1^5)$: $[\alpha_1, \alpha_2] = \alpha_4, \alpha^3 = \alpha_1^3\alpha_3 = \alpha_2^3\alpha_4 = \alpha_3^3 = \alpha_4^3 = 1;$
$\overline{\alpha}_1 = \alpha_1, \overline{\alpha} = \alpha_1\alpha^{-1}\alpha_1.$

For $p > 3$, the two families ϕ_9 and ϕ_{10} consist of groups G with $G/Z(G) \approx \phi_3(1^4)$, the non-abelian class 3 group of order p^4 and exponent p, and $G' \approx C_p \times C_p \times C_p$. Thus $G/G' \approx C_p \times C_p$ and $G = \langle \alpha, \alpha_1 \rangle$ is a 2-generator group. Define the additional generators $\alpha_{i+1} = [\alpha_i, \alpha]$ for $i = 1, 2, 3$. Then the groups G all satisfy the relations: $\alpha_{i+1}^p = 1$ for $i = 1, 2, 3$; together with 2 relations of the form: $\alpha^p = \alpha_4^m$, $\alpha_1^p = \alpha_4^n$, $0 \leqslant m, n < p$; where m and n are not both non-zero. The groups G in ϕ_{10} also satisfy the additional relation: $[\alpha_1, \alpha_2] = \alpha_4$. Note $\gamma_2(G) = \langle \alpha_2, \alpha_3, \alpha_4 \rangle$, $\gamma_3(G) = \langle \alpha_3, \alpha_4 \rangle$ and $Z(G) = \gamma_4(G) = \langle \alpha_4 \rangle$.

When $n = 0$, define θ so that $\overline{\alpha} = \alpha^{-1}$ and $\overline{\alpha}_1 = \alpha_1$. Then using standard commutator properties $[\alpha_1, \alpha^{-1}, \alpha^{-1}, \alpha^{-1}] = [\alpha_1, \alpha, \alpha, \alpha]^{-1}$ so $\overline{\alpha}_4 = \alpha_4^{-1}$ and $[\alpha_1, \alpha^{-1}, \alpha_1] = [\alpha_1, \alpha, \alpha_1]^{-1}$ so $[\overline{\alpha}_1, \overline{\alpha}_2] = [\alpha_1, \alpha_2]^{-1}$. Also $\overline{\alpha}_2$ and $\overline{\alpha}_3$ are conjugates of α_2^{-1} and α_3 respectively.

With the exception of the groups G_r defined above, $n \neq 0$ and $m = 0$ occurs only for groups G in ϕ_9. For these groups in ϕ_9 define θ so that $\overline{\alpha} = \alpha$ and $\overline{\alpha}_1 = \alpha_1^{-1}$. Then as above $[\alpha_1^{-1}, \alpha, \alpha, \alpha] = [\alpha_1, \alpha, \alpha, \alpha]^{-1}$ so $\overline{\alpha}_4 = \alpha_4^{-1}$, and $\overline{\alpha}_2$ and $\overline{\alpha}_3$ are conjugates of α_2^{-1} and α_3^{-1} respectively. So in both cases the barred generators satisfy the relations above, and θ is the required automorphism of order 2. ∎

We now treat the exceptional group(s) G_r.

THEOREM 4. . *The groups G_r defined above have no automorphisms of order 2. In particular, for $p > 3$, if $(p-1, 3) = 3$ then $|\operatorname{Aut} G_r| = 3p^6$, $r = 0, 1, 2$, and if $(p-1, 3) = 1$ then $|\operatorname{Aut} G_0| = p^6$.*

PROOF: $G = G_r$ has $\gamma_2(G) = \langle \alpha_2, \alpha_3, \alpha_4 \rangle$, $\gamma_3(G) = \langle \alpha_3, \alpha_4 \rangle$ and $Z(G) = \gamma_4(G) = \langle \alpha_4 \rangle$. Since $G/G' \approx C_p \times C_p$, G has $p+1$ maximal subgroups $M_0 = \langle \alpha_1, G' \rangle$, $M_i = \langle \alpha_1^i\alpha, G' \rangle$, $1 \leqslant i \leqslant p$. Now $M_0' = \langle [\alpha_1, \alpha_2] \rangle = \langle \alpha_4 \rangle$, whereas for $i > 0$, $M_i' = \langle \alpha_3, \alpha_4 \rangle$ since $[\alpha_2, \alpha_1^i\alpha] = [\alpha_2, \alpha][\alpha_2, \alpha_1]^i = \alpha_3\alpha_4^{-i}$ and $[\alpha_3, \alpha_1^i\alpha] = [\alpha_3, \alpha] = \alpha_4$. Thus M_0 is characteristic.

G has exponent p^2 since $\alpha_1 \in G$ has order p^2 but G has no elements of larger order since $G/Z(G) \approx \phi_3(1^4)$ has exponent p. Thus $M_0, \ldots, M_{p-1}$ all have exponent p^2 since $\alpha_1 \in M_0$, and for $1 \leqslant i \leqslant p-1$, $\alpha_1^i\alpha \in M_i$ and $(\alpha_1^i\alpha)^p = (\alpha_1^p)^i = \alpha_4^{ki}$, using that G is regular. However, M_p clearly has exponent p, again using that G is regular. Thus M_p is characteristic.

Therefore since M_0, M_p and G' are characteristic, any automorphism θ of G must be defined so that $\overline{\alpha}_1$ and $\overline{\alpha}$ have the form: $\overline{\alpha}_1 = \alpha_1^i x$, $\overline{\alpha} = \alpha^j y$, where $1 \leqslant i$, $j < p$ and x, $y \in G'$. In fact we use the relations $\alpha_1^p = \alpha_4^k = [\alpha_1, \alpha_2]^k$ to show that if θ is an automorphism then $i^3 \equiv 1 \pmod p$ and $j \equiv i^{-1} \pmod p$. (*)

First

(1) $$\overline{\alpha}_1^p = (\alpha_1^i x)^p = \alpha_4^{ik}.$$

Next $\overline{\alpha}_2 = [\overline{\alpha}_1, \overline{\alpha}] = [\alpha_1^i x, \alpha^j y]$ and by considering the image of this commutator in $G/\gamma_3(G)$, we see that $\overline{\alpha}_2 = \alpha_2^{ij} w$, for some $w \in \gamma_3(G)$. So $[\overline{\alpha}_1, \overline{\alpha}_2] = [\alpha_1^i, \alpha_2^{ij} w] = [\alpha_1, \alpha_2]^{i^2 j} = \alpha_4^{i^2 j}$. Thus

(2) $$[\overline{\alpha}_1, \overline{\alpha}_2]^k = \alpha_4^{i^2 jk}.$$

Equating (1) and (2) gives

(3) $$ij \equiv 1 \pmod p.$$

Also $\overline{\alpha}_3 = [\overline{\alpha}_2, \overline{\alpha}] = [\alpha_2 w, \alpha^j y]$ and by considering the image of this commutator in $G/\gamma_4(G)$, we see that $\overline{\alpha}_3 = \alpha_3^j z$, for some $z \in \gamma_4(G)$. Thus $\overline{\alpha}_4 = [\overline{\alpha}_3, \overline{\alpha}] = [\alpha_3^j z, \alpha^j y] = [\alpha_3, \alpha]^{j2} = \alpha_4^{j2}$, so $\overline{\alpha}_4^k = \alpha_4^{j^2 k}$ so

(4) $$\overline{\alpha}_4^k = \alpha_4^{j^2 k}.$$

Equating (1) and (4) gives

(5) $$j^2 \equiv i \pmod p.$$

Thus from (3) and (5) $i^3 \equiv 1 \pmod p$ and $j \equiv i^{-1} \pmod p$, which is (*). Conversely, if i and j satisfy (*), then the barred generators satisfy the defining relations of G, so any map θ of form $\overline{\alpha}_1 = \alpha^i x$, $\overline{\alpha} = \alpha^j y$, for any x, $y \in G'$, extends to an automorphism of G.

Now $|G'| = p^3$. Thus if $(p-1, 3) = 3$, there are 3 solutions for i, so $|\operatorname{Aut} G_r| = 3p^6$, $r = 0, 1, 2$. But if $(p-1, 3) = 1$, then $i = j = 1$ so $|\operatorname{Aut} G_0| = p^6$. ∎

COROLLARY 5. *The group G_0, defined for $p > 3$ and $(p-1,3) = 1$, is the unique group of order p^5 whose automorphism group is again a p-group.*

Note that in contrast to Ying's result (Theorem 2 in [**8**]) when $(p-1,3) = 3$, the 2-generator groups G_r, $0 \leqslant r \leqslant 2$, have no automorphisms of order 2 yet $\operatorname{Aut} G_r$ is not a p-group either.

Finally we show in Theorem 9 that p^6 is the smallest order that a p-group may have when it occurs as an automorphism group. We first state the following results, in which G denotes a finite group.

THEOREM 6. *If $\operatorname{Aut} G$ is a p-group then G is also a non-abelian p-group P or $G \approx C_2 \times P$.*

PROOF: See Theorem 2 in [**7**]. ∎

THEOREM 7. *Every nilpotent group G with $|G| > 2$ has an outer automorphism.*

PROOF: See Lemma 11 in [**2**]. ∎

THEOREM 8. *If G is a non-cyclic p-group of order greater than p^2 such that $|G/Z(G)| \leqslant p^4$, then $|G|$ divides $|\operatorname{Aut} G|$.*

PROOF: This is the main result in Davitt [**1**]. ∎

THEOREM 9. *There is no group G such that $|\operatorname{Aut} G| = p^n$, $n \leqslant 5$.*

PROOF: Suppose on the contrary $|\operatorname{Aut} G| = p^n$, for $n \leqslant 5$. By Theorem 6, we may suppose G is a non-abelian p-group. By Theorem 7 Inn G is a proper subgroup of $\operatorname{Aut} G$ so Inn G divides p^4. Thus by Theorem 8 $|G|$ divides $|\operatorname{Aut} G|$, so $|G| = p^n$, $n \leqslant 5$. But by Theorems 2 and 3, all such groups apart from the groups G_r have an automorphism of order 2, so G cannot be any of them. Finally, by Theorem 4, G cannot be one of the groups G_r either, so there is no such group G. ∎

COROLLARY 10. *Let G_0 be the group defined above for $p > 3$ and $(p-1,3) = 1$. Then $\operatorname{Aut} G_0$ has the smallest order a p-group may have when it occurs as an automorphism group.*

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