RIGHT HEREDITARY AFFINE PI RINGS ARE LEFT HEREDITARY

by ELLEN KIRKMAN and JAMES KUZMANOVICH

(Received 30 May, 1986)

Small [**11**] gave the first example of a right hereditary PI ring which is not left hereditary. Robson and Small [**9**] proved that a prime PI right hereditary ring is a classical order over a Dedekind domain, and hence is Noetherian (and therefore left hereditary). The authors have shown [**4**] that a right hereditary semiprime PI ring which is finitely generated over its center is left hereditary. In this paper we consider right hereditary PI rings Γ which are affine (i.e. finitely generated as an algebra over a central subfield k). Such rings need not be Noetherian; for example the ring $\Gamma = \begin{bmatrix} k[x] & k[x,y] \\ 0 & k[y] \end{bmatrix}$ is an affine hereditary PI ring which is not right or left Noetherian.

The right and left global dimensions of an affine PI ring need not be identical. In [**5**] an example is given of an affine PI ring Γ with $\operatorname{rgldim}\Gamma = 2$ and $\operatorname{lgldim}\Gamma = 3$ (Γ is not semiprime). This example is then used to produce an affine prime PI ring with differing global dimensions.

The authors are grateful to Professor Lance W. Small for several useful conversations about this material, and to the referee for some helpful suggestions.

If Γ is an affine PI right hereditary ring, then Γ has no infinite sets of orthogonal idempotents [**8**, Theorem 2.5, p. 108] and hence Γ is a piecewise domain having the following triangular structure [**3**]:

$$\Gamma = \begin{bmatrix} P_1 & M_{12} & \dots & M_{1n} \\ 0 & P_2 & \dots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_{n-1,n} \\ 0 & 0 & \dots & P_n \end{bmatrix}$$

with P_i prime rings and M_{ij} P_i-P_j-bimodules. Furthermore, by [**12**], Γ is left semihereditary. These results will be used throughout.

Palmer and Roos calculated the global dimension of a triangular matrix ring [**7**]. The following characterization of Goodearl [**2**] follows from their results.

LEMMA 1. *The ring* $\Gamma = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ *is left hereditary if and only if*

(a) *the rings R and S are left hereditary,*
(b) *the module M_S is flat,*
(c) *for every left ideal I of S, the R-module M/MI is R-projective.*

Page **[6]** proved the following corresponding result characterizing when a triangular matrix ring is semihereditary; it follows from results of Goodearl **[2]**.

LEMMA 2. *The ring* $\Gamma = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ *is left semihereditary if and only if*

(a) *the rings R and S are left semihereditary,*
(b) *the module M_S is flat,*
(c) *for every finitely generated left ideal I of S, every finitely generated R-module of M/MI is R-projective.*

The following lemma is used in several later arguments.

LEMMA 3. *Let* $\Gamma = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ *be an affine PI ring with* gldim $S = 0$ *and R right and left hereditary. Then Γ is left hereditary if and only if Γ is right hereditary.*

Proof. By the weak Nullstellensatz, S is finite dimensional as a vector space over k, $S = \sum y_j k$. Hence $M = \sum Rx_i S = \sum Rx_i y_j k = \sum Rx_i y_j$; so M is a finitely generated left R-module.

If Γ is left hereditary then M is R-flat because M is R-projective. Since every S-module is S-projective, Γ is right hereditary by Lemma 1.

If Γ is right hereditary then, since Γ is left semihereditary, the fact that M is finitely generated as an R-module implies that M is R-projective by Lemma 2. Since M is S-projective, to show that Γ is left hereditary it suffices to show that M/MK is left R-projective for all left ideals K of S. Since K is generated by an idempotent $K = Se$, $MK = Me$ and $M/Me \simeq M(1-e)$ is a direct summand of M, and hence is left R-projective.

The main theorem will be proved first in the case $\Gamma = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ with R, S prime rings. Then we will handle the general case involving larger triangular matrix rings.

PROPOSITION 4. *Let* $\Gamma = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ *be an affine PI ring with R and S affine, hereditary Noetherian prime PI rings and M a finitely generated R-S-bimodule. Then Γ is right hereditary if and only if Γ is left hereditary.*

Proof. Assume that Γ is right hereditary. By Lemma 1, to show that Γ is left hereditary it suffices to show that M/ML is a projective R-module for any left ideal L of S. By Lemma 3, we are done if S is simple Artinian. By Robson–Small **[9]**, S is a classical order over its center which is a Dedekind domain; so an argument using the Noether Normalization Lemma shows that there is no loss of generality in assuming that S is a finitely generated free extension of $k[y]$, where y is a central transcendental element.

We show first that M is R-projective by considering M as a left module over $R \otimes_k k[y] = R[y]$, a prime Noetherian PI ring. Let T be the $R[y]$-torsion submodule of M and let I be the annihilator of T in $R[y]$. If $I \neq 0$ then I contains a nonzero central element $f(y)$; we may suppose that $f(0) \neq 0$, and we have $f(0)T \subseteq Ty$. Since M is a finitely generated $R[y]$-module, M/My is a finitely generated R-module, and M/My is R-projective by Lemma 2. We may embed T/Ty in M/My, which is R-torsionfree. Since $f(0)$ is a nonzero central element of R, $f(0)$ is regular in R and hence $T = Ty$, and so $T = 0$ since M is S-projective. Hence M embeds in a free $R[y]$-module; since R is hereditary, M is R-projective.

It remains to show that M/ML is R-projective for a nonzero left ideal L of S. Suppose first that L is an essential left ideal of S. Then L contains a nonzero $g(y) \in k[y]$. Since M is a finitely generated $R[y]$-module, $M/Mg(y)$ is finitely generated as an R-module and hence M/ML is finitely generated as an R-module. Since Γ is left semihereditary, M/ML is R-projective by Lemma 2. For any left ideal L of S, $L \oplus K$ is an essential left ideal in S for some left ideal K of S. We know that $M/M(L \oplus K)$ is R-projective; so that $M(L \oplus K)$ is a direct summand of M. Because M is S-projective we have $ML \cap MK = M(L \cap K) = 0$, and therefore $ML \oplus MK \simeq M(L \oplus K)$. Hence ML is a direct summand of $M(L \oplus K)$, and thus also of M. Therefore M/ML is isomorphic to a direct summand of M and so is R-projective.

Note that when Γ is a ring as in Proposition 4, M must be torsionfree as an $R[y]$-module. However, the following example shows that M is $R[y]$-torsionfree is not sufficient to imply that Γ is hereditary.

Example 5. Let $R = k[x]$, $S = k[y]$, and $M = \langle x, y \rangle$. If Γ were left hereditary, M/My would be R-torsionfree, which it is not; similarly Γ is not right hereditary. It is not difficult to show that if M is any non-principal ideal of $k[x, y]$, Γ is not left hereditary. We know of no ring $\Gamma = \begin{bmatrix} k[x] & M \\ 0 & k[y] \end{bmatrix}$ which is left hereditary where M is not $k[x, y]$-free.

The following proposition is used in inducting on the size of the triangular matrix rings of the general case.

Proposition 6. *Let* $\Gamma = \begin{bmatrix} P_1 & M_{12} & M_{13} \\ 0 & \Gamma_2 & M_{23} \\ 0 & 0 & P_3 \end{bmatrix}$ *be an affine* PI *ring with* P_1 *and* P_3 *prime Noetherian rings and with* $\Lambda = \begin{bmatrix} P_1 & M_{12} \\ 0 & \Gamma_2 \end{bmatrix}$ *and* $\Lambda' = \begin{bmatrix} \Gamma_2 & M_{23} \\ 0 & P_3 \end{bmatrix}$ *right and left hereditary rings. Then* Γ *is right hereditary if and only if it is left hereditary.*

Proof. We will need to think of Γ as $\Gamma = \begin{bmatrix} \Lambda & M \\ 0 & P_3 \end{bmatrix}$ with $M = \begin{bmatrix} M_{13} \\ M_{23} \end{bmatrix}$ and as $\Gamma = \begin{bmatrix} P_1 & M' \\ 0 & \Lambda' \end{bmatrix}$ with $M' = [M_{12}\, M_{13}]$. We will assume that Γ is right hereditary and show that Γ is left hereditary.

To show that M is left Λ-projective we will show first that $M_{13}/M_{12}M_{23}$ is a projective left P_1-module. Consider the ring $\Lambda^* = \begin{bmatrix} P_1 & M_{13}/M_{12}M_{23} \\ 0 & P_3 \end{bmatrix}$ which is affine. We will show that Λ^* is right hereditary, and hence by Proposition 4 left hereditary, to prove that $M^* = M_{13}/M_{12}M_{23}$ is left P_1-projective. For any right ideal I of P_1 (including $I = 0$), we must show that M^*/IM^* is right P_3-projective. Since I is a right ideal of P_1, $J = \begin{bmatrix} I & M_{12} \\ 0 & 0 \end{bmatrix}$ is a right ideal of Λ. Since Γ is right hereditary, $M/JM = \begin{bmatrix} M_{13}/(IM_{13} + M_{12}M_{23}) \\ M_{23} \end{bmatrix}$ is right P_3-projective; furthermore $M_{13}/(IM_{13} + M_{12}M_{23})$ is a direct summand of M/MJ as a right P_3-module and hence is P_3-projective. But

$$M^*/IM^* = (M_{13}/M_{12}M_{23})/(I(M_{13}/M_{12}M_{23})) \simeq M_{13}/(IM_{13} + M_{12}M_{23})$$

and hence is right P_3-projective. To show that M^* is flat as a P_1-module we will show that it is torsionfree as a P_1-module. If p is a regular element of P_1 with $pm \in M_{12}M_{23}$ for some $m \in M_{13}$ then $pm = \sum a_ib_i$ for $a_i \in M_{12}$, $b_i \in M_{23}$. Let $K = \begin{bmatrix} 0 & \sum \Gamma_2 b_i \\ 0 & 0 \end{bmatrix}$ which is a finitely generated left ideal of the left semihereditary ring Λ'; hence finitely generated submodules of $M'/M'K$ are projective as left P_1-modules. Now

$$M'/M'K = [M_{12}, M_{13}/(\textstyle\sum M_{12}b_i)],$$

and so

$$[0, (P_1m + \textstyle\sum M_{12}b_i)/(\sum M_{12}b_i)]$$

is left P_1-projective and hence P_1-torsionfree; thus $pm \in \sum M_{12}b_i$ implies that $m \in \sum M_{12}b_i \subset M_{12}M_{23}$, and hence M^* is a flat left P_1-module.

Having shown $M_{13}/M_{12}M_{23}$ is a projective left P_1-module, we proceed to show that $M = \begin{bmatrix} M_{13} \\ M_{23} \end{bmatrix}$ is left Λ-projective. We have that $M_{13} \simeq M_{12}M_{23} \oplus C$ for a left P_1-module C. Hence $M = \begin{bmatrix} M_{12}M_{23} \\ M_{23} \end{bmatrix} \oplus \begin{bmatrix} C \\ 0 \end{bmatrix}$, and it remains to show that $\begin{bmatrix} M_{12}M_{23} \\ M_{23} \end{bmatrix}$ is a projective left Λ-module. Since Λ' is left hereditary, Γ_2 is a left hereditary ring and M_{23} is projective as a left Γ_2-module. Hence M_{23} is isomorphic to a direct sum, $M_{23} \simeq \oplus I_\alpha$, with each I_α a finitely generated ideal of Γ_2 [**1**]. Then for each α, $\begin{bmatrix} 0 & M_{12}I_\alpha \\ 0 & I_\alpha \end{bmatrix}$ is a left ideal of Λ and hence is left Λ-projective, since Λ is left hereditary. Consider the Λ-map

$$\sigma : \bigoplus_\alpha \begin{bmatrix} 0 & M_{12}I_\alpha \\ 0 & I_\alpha \end{bmatrix} \xrightarrow{\sigma} \begin{bmatrix} M_{12}M_{23} \\ M_{23} \end{bmatrix}$$

induced by the isomorphism $M_{23} \simeq \oplus I_\alpha$. Since σ is clearly surjective, to show that $\begin{bmatrix} M_{12}M_{23} \\ M_{23} \end{bmatrix}$ is Λ-projective, it suffices to show that σ is one-to-one; in fact, it is enough to show σ restricted to a finite sum is injective:

$$A = \bigoplus_{i=1}^{n} \begin{bmatrix} 0 & M_{12}I_{\alpha_i} \\ 0 & I_{\alpha_i} \end{bmatrix} \xrightarrow{\sigma|A} \sigma(A) \to 0.$$

We have that

$$\ker(\sigma \mid A) \subset \bigoplus_{i=1}^{n} \begin{bmatrix} 0 & M_{12} I_{\alpha_i} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & M_{12} \\ 0 & 0 \end{bmatrix} A,$$

and $\sigma(A)$ is a finitely generated Λ-submodule of M; since Γ is left semihereditary, $\sigma(A)$ is a projective left Λ-module. Hence $A \simeq \ker(\sigma \mid A) \oplus D$, where $D \simeq \sigma(A)$. Thus $A = \begin{bmatrix} 0 & M_{12} \\ 0 & 0 \end{bmatrix} A + D$ which implies that $A = D$, since $\begin{bmatrix} 0 & M_{12} \\ 0 & 0 \end{bmatrix}$ is a nilpotent ideal of Λ, and hence $\ker(\sigma \mid A) = 0$. This completes the proof that M is Λ-projective.

The proof that M/ML is left Λ-projective for any left ideal L of P_3 is the same as the proof in the previous proposition. We thus have shown that Γ is left hereditary.

We can now prove our main result.

THEOREM 7. *Let Γ be an affine PI ring. Then Γ is right hereditary if and only if Γ is left hereditary.*

Proof. Assume that Γ is right hereditary. As noted earlier Γ is a PWD and hence

$$\Gamma \simeq \begin{bmatrix} P_1 & M_{12} & \cdots & M_{1n} \\ 0 & P_2 & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n-1,n} \\ 0 & 0 & \cdots & P_n \end{bmatrix}.$$

Each P_i is right hereditary [**10**], and hence also left hereditary [**9**]. The proof is by induction on the number of prime rings on the diagonal. The proof then follows by Proposition 6 since we can think of Γ as

$$\Gamma \simeq \begin{bmatrix} P_1 & M_{12} & \cdots & M_{1n} \\ 0 & P_2 & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n-1,n} \\ 0 & 0 & \cdots & P_n \end{bmatrix} = \begin{bmatrix} P_1 & M_{12}^* & M_{1n}^* \\ 0 & \Gamma_2 & M_{23}^* \\ 0 & 0 & P_3 \end{bmatrix};$$

the corresponding rings Λ, Λ' of Proposition 6 are right hereditary by [**10**], are clearly affine, and hence are left hereditary by the induction hypothesis.

REFERENCES

1. F. Albrecht, On projective modules over semi-hereditary rings, *Proc. Amer. Math. Soc.* **12** (1961), 638–639. *MR* **23** #A3766.

2. K. R. Goodearl, *Ring theory*: *Nonsingular rings and modules*, (Marcel Dekker, 1976).

3. R. Gordon and L. W. Small, Piecewise domains, *J. Algebra* **23** (1972), 553–564. *MR* **46** #9087.

4. E. Kirkman and J. Kuzmanovich, Hereditary finitely generated algebras satisfying a polynomial identity, *Proc. Amer. Math. Soc.* **83** (1981), 461–466. *MR* **82**k: 16004.

5. E. Kirkman and J. Kuzmanovich, Matrix subrings having finite global dimension. *J. Algebra.* **109** (1987), 74–92.

6. A. Page, Sur les anneaux héréditaires ou semi-héréditaires, *Comm. Algebra* **6** (1978), 1169–1186. *MR* **57** #12594.

7. I. Palmér and J.-E. Roos, Formules explicites pour la dimension homologique des anneaux de matrices généralisées, *C. R. Acad. Sci. Paris Sér. A-B* **273** (1971), A1026–A1029. *MR* **45** #1977.

8. C. Procesi, *Rings with polynomial identities* (Marcel Dekker, 1973).

9. J. C. Robson and L. W. Small, Hereditary prime PI rings are classical hereditary orders, *J. London Math. Soc.* (2) **8** (1974), 499–503. *MR* **50** #2236.

10. F. L. Sandomierski, A note on the global dimension of subrings, *Proc. Amer. Math. Soc.* **23** (1969), 478–480. *MR* **39** #6930.

11. L. W. Small, An example in Noetherian rings, *Proc. Nat. Acad. Sci. U.S.A.* **54** (1965), 1035–1036. *MR* **32** #5691.

12. L. W. Small, Semihereditary rings, *Bull. Amer. Math. Soc.* **73** (1967), 656–658. *MR* **35** #2926.

WAKE FOREST UNIVERSITY
P.O. BOX 7311
REYNOLDA STATION
WINSTON-SALEM
N.C. 27109 U.S.A.

https://doi.org/10.1017/S0017089500007096 Published online by Cambridge University Press