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TOPOLOGICAL SPACES WITH A UNIQUE COMPATIBLE QUASI-UNIFORMITY

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ABSTRACT. We show that a topological space X admits a unique quasi-uniformity if and only if every interior-preserving open collection of X is finite.

1. In [5], p. 45 Problem B, it is asked a) whether any topological space admitting only one quasi-proximity is hereditarily compact and b) whether any space in which every interior-preserving open cover is finite admits only one quasi-uniformity. The first part of Problem B is answered negatively in [7]; in this note we answer the second part positively. Topological spaces that admit a unique quasi-uniformity are considered in [1–5, 8–10]. Related questions are dealt with in [6, 7].

It is known that a topological space that admits a unique quasi-uniformity is hereditarily compact ([9]), [5] Theorem 2.36). The converse does not obtain ([8], [5] Example 2.38). In [3] it is noted that the class of hereditarily compact quasi-sober spaces is strictly contained in the class of the topological spaces that admit a unique quasi-uniformity. Note that the class of topological spaces with a finite topology is properly contained in the class of hereditarily compact quasi-sober spaces. Topological spaces that admit a unique quasi-proximity are characterized in [7]. It is shown that they are the topological spaces in which the topology is the unique base of open sets that is closed under finite unions and finite intersections. In particular every hereditarily compact topological space admits a unique quasi-proximity (compare [8], [5] Theorem 2.36). Example 1 of [7] is a non-compact topological space that admits a unique quasi-proximity.

In this note we will use the terminology of [5]. If $\mathcal{V}$ is a quasi-uniformity on a non-empty set X, then $\mathcal{V}^*$ denotes the uniformity that is generated by $\{U \cap U^{-1} | U \in \mathcal{V}\}$ on X. Recall that a topological space X is called *transitive* if the fine quasi-uniformity for X is contained in the fine-transitive quasi-uniformity for X. We will use the well-known result that a topological space is hereditarily compact if and only if every strictly increasing sequence of open sets is finite.

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2. THEOREM: *A topological space X admits a unique quasi-uniformity if and only if every interior-preserving open collection in X is finite.*

PROOF: It is well known that in a topological space that admits a unique quasi-uniformity every interior-preserving open collection is finite ([5], Theorem 2.36) and as is observed in [5] p. 45, in order to establish the converse it suffices to show that a topological space in which every interior-preserving open collection is finite is a transitive space (see [5], Corollary 2.8 and [9], Corollary 3.5).

Let $(X, \mathcal{J})$ be a topological space in which every interior-preserving open collection is finite.

Let $\mathcal{P}(\mathcal{J})$ denote the Pervin quasi-uniformity for X and let $Fine(\mathcal{J})$ denote the fine quasi-uniformity for X. Let $U \in Fine(\mathcal{J})$. Then for each $n \in \mathbb{N}$ there is a $U_n \in Fine(\mathcal{J})$ such that $U_{n+1}^2 \subset U_n$ and $U_1 \subset U$. We show that U belongs to the fine-transitive quasi-uniformity for X.

Let $\mathcal{V}$ denote the quasi-uniformity on X generated by $\{U_n | n \in \mathbb{N}\}$. We denote the topology $\mathcal{J}(\mathcal{V})$ by $\mathcal{J}'$ and the Pervin quasi-uniformity for $\mathcal{J}'$ by $\mathcal{P}(\mathcal{J}')$.

Since $\mathcal{J}' \subset \mathcal{J}$ and $\mathcal{J}$ is hereditarily compact, $\mathcal{J}'$ is hereditarily compact. Therefore $\mathcal{P}(\mathcal{J}')$ is the unique $\mathcal{J}'$-compatible quasi-uniformity that is totally bounded. In particular, $\mathcal{P}(\mathcal{J}') \subset \mathcal{V}$ (e.g. [5], 1.37). Note that $\mathcal{P}(\mathcal{J}') \subset \mathcal{P}(\mathcal{J})$, because we have that $\mathcal{J}' \subset \mathcal{J}$ If $\mathcal{V}$ is totally bounded, then $U \in \mathcal{V} = \mathcal{P}(\mathcal{J}') \subset \mathcal{P}(\mathcal{J})$ and we are finished. Hence it suffices to show that $\mathcal{V}$ is totally bounded. We will prove that every ultrafilter on X is a $\mathcal{V}^*$-Cauchy filter. By ([5], Proposition 3.14) it will follow that $\mathcal{V}$ is totally bounded.

Let $\mathcal{G}$ be an ultrafilter on X and let $V \in \mathcal{V}$. Let $x \in X$. Set $H(x) = \mathrm{int}_{\mathcal{J}'} V(x)$ and $H_x = [H(x) \times H(x)] \cup [(X \backslash H(x)) \times X]$. Then $H_x \in \mathcal{P}(\mathcal{J}') \subset \mathcal{V}$. We conclude that there is an $n \in \mathbb{N}$ such that $U_n \in \mathcal{V}$ and $U_n \subset H_x$. Hence $U_n(H(x)) = H(x)$. For each $n \in \mathbb{N}$ set $\mathcal{A}_n = \{H(x) | U_n(H(x)) = H(x)\}$. If $\mathcal{A} \subset \mathcal{A}_n$ and $x \in \cap \mathcal{A}$, then $U_n(x) \subset \cap\, \mathcal{A}$. Since $U_n(x)$ is a $\mathcal{J}$-neighborhood of x, $\mathcal{A}_n$ is an interior-preserving open collection of $(X, \mathcal{J})$; hence $\mathcal{A}_n$ is finite. We conclude that $\{H(x) | x \in X\}$ is countable. Let $H = \cup\{\{x\} \times H(x) | x \in X\}$.

Since $(X, \mathcal{J}')$ is hereditarily compact, the ultrafilter $\mathcal{G}$ contains a minimal $\mathcal{J}'$-closed set F. Obviously, every $\mathcal{J}'$-open set that meets F is a member of $\mathcal{G}$. Let $\mathcal{M} = \{H(x) | x \in F\}$. We assume in the following that the elements of $\mathcal{M}$ are indexed by the set of positive integers $\mathbb{N}$ or one of its initial segments. Call this index set N. Let $\mathcal{B} = \{[\bigcap_{n=1}^{k} H_n] \cup (X \backslash F) | H_n \in \mathcal{M}$ for $n \in \{1, \ldots, k\}$ and $k \in N\}$. If $\cap \mathcal{B} \cap F = \phi$, then $\mathcal{B}$ is an interior-preserving open collection in $(X, \mathcal{J})$. Hence $\mathcal{B}$ is finite and $\phi = \cap \mathcal{B} \cap F = \bigcap_{n=1}^{m} H_n \cap F \in \mathcal{G}$ for some $m \in N$ — a contradiction. We conclude that $\cap \mathcal{B} \cap F \neq \phi$. Let $y \in \cap \mathcal{B} \cap F = \cap \mathcal{M} \cap F$. Then $F \subset H^{-1}(y)$ so that $H^{-1}(y) \in \mathcal{G}$. Since $y \in F$, we have that $H(y) \in \mathcal{G}$. It follows that $(V \cap V^{-1})(y) \in \mathcal{G}$, because $H = \cup\{\{x\} \times H(x) | x \in X\} \subset V$. Therefore, $\mathcal{G}$ is a $\mathcal{V}^*$-Cauchy filter on X.

REMARK: Recall that a topological space with an infinite topology has a strictly increasing infinite sequence of open sets or a strictly decreasing infinite sequence of open sets (e.g. [6], Lemma 3). In [2] it was essentially proved that the following two conditions are equivalent for a topological space X.

1) Every interior-preserving open collection of X is finite.

2) There is no infinite strictly decreasing sequence of open sets with open intersection in X and there is no infinite strictly increasing sequence of open sets in X.

Note that a topological space that admits two different quasi-uniformities, admits two different transitive quasi-uniformities. For in a topological space that has an infinite interior-preserving open collection, the fine-transitive quasi-uniformity is different from the Pervin quasi-uniformity ([5], Corollary 2.8).

In the proof of the Theorem we used a variant of the following proposition, which may be of independent interest.

PROPOSITION: *Let $\mathcal{U}$ be a local quasi-uniformity with a countable base on a non-empty set X. If $\mathcal{U}$ contains a transitive quasi-uniformity $\mathcal{V}$ compatible with $\mathcal{J}(\mathcal{U})$, then $\mathcal{J}(\mathcal{U})$ is non-Archimedeanly quasi-pseudo-metrizable.*

PROOF: Let $\{U_n | n \in \mathbb{N}\}$ be a countable base of $\mathcal{U}$ and let $U_n^\infty = \cup\{U_n^k | k \in \mathbb{N}\}$ for each $n \in \mathbb{N}$. It suffices to show that $\{U_n^\infty(x) | x \in X, n \in \mathbb{N}\}$ is a base for $\mathcal{J}(\mathcal{U})$. Let $x \in X$ and $n \in \mathbb{N}$. Since $\mathcal{V}$ is compatible with $\mathcal{J}(\mathcal{U})$, there is a transitive entourage $T \in \mathcal{V}$ such that $T(x) \subset U_n(x)$. Moreover, there is an $m \in \mathbb{N}$ such that $U_m \subset T$, because $\mathcal{V} \subset \mathcal{U}$. Therefore, $U_m^\infty \subset T$ and $U_m^\infty(x) \subset U_n(x)$.

COROLLARY: *If $\mathcal{U}$ is a local quasi-uniformity with a countable base on a non-empty set X such that $\mathcal{J}(\mathcal{U})$ has a subbase of compact open sets, then $\mathcal{J}(\mathcal{U})$ is non-Archimedeanly quasi-pseudo-metrizable.*

PROOF: If $\mathcal{U}$ is a compatible local quasi-uniformity with a countable base $\{U_n | n \in \mathbb{N}\}$ on X, then $\mathcal{U}$ contains the compatible transitive quasi-uniformity $\mathcal{V}$ generated by $\{[(X \backslash G) \times X] \cup [X \times G] | G \text{ is open and compact}\}$. For, if G is compact and open, then there is an $n \in \mathbb{N}$ such that $U_n(G) = G$ (see [5], Theorem 7.18). It follows that $U_n \subset [(X \backslash G) \times X] \cup [X \times G]$.

REMARK: By the Corollary every zero-dimensional locally compact γ-space is non-Archimedeanly quasi-metrizable. We do not know whether every locally compact γ-space is (non-Archimedeanly) quasi-metrizable.

Note that a submetacompact locally compact γ-space is non-Archimedeanly quasi-metrizable: Since a compact γ-space is developable, the space under consideration is locally developable. Hence, as a submetacompact space, it is developable. By ([5], Theorem 7.26), it is quasi-metrizable and by ([5], Theorems 6.19 and 6.21c), it is transitive.

COROLLARY: *A regular $\delta\theta$-refinable locally compact γ-space X is a non-Archimedeanly quasi-metrizable Moore space.*

PROOF: By ([11], Theorem 18) X is subparacompact, because a compact γ-space is second countable.

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