

ASYMPTOTICS OF THE DENSITY OF THE SUPREMUM OF A RANDOM WALK WITH HEAVY-TAILED INCREMENTS

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Abstract

Under some relaxed conditions, in this paper we obtain some equivalent conditions on the asymptotics of the density of the supremum of a random walk with heavy-tailed increments. To do this, we investigate the asymptotics of the first ascending ladder height of a random walk with heavy-tailed increments. The results obtained improve and extend the corresponding classical results.

Keywords: Random walk; heavy-tailed increment; first ascending ladder height; density of the supremum; asymptotics

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1. Introduction

It is well known that the asymptotics of the density of the supremum of a random walk with heavy-tailed increments is an important subject to which researchers pay close attention. Here we present some equivalent conditions on such asymptotics, under some relaxed conditions.

Let $\{\xi_i : i \geq 1\}$ be independent, identically distributed proper random variables with common distribution F on $(-\infty, \infty)$ and finite negative mean $\mu_F = -m$. Write

$$\bar{F}(x) = 1 - F(x), \quad x \in (-\infty, \infty).$$

Let $S_0 = 0$ and $S_n = \sum_{i=1}^n \xi_i$, $n \geq 1$, define a random walk. Then $M = \sup_{n \geq 0} S_n$, the supremum of the random walk, is almost surely a random variable, with distribution W such that $0 < p = \bar{W}(0) < 1$. If W is absolutely continuous, let w denote its density.

Because the asymptotics of W is related to the first ascending ladder height of the random walk, we give some concepts about, and notation for, the latter. Let

$$\tau_+ = \inf\{n \geq 1 : S_n > 0\}$$

be the first ascending ladder epoch of the above random walk. It is well known that, when $m = -\mu_F > 0$, τ_+ is a defective random variable, i.e. $P(\tau_+ < \infty) < 1$. Let $\{\eta_i : i \geq 1\}$ be independent proper random variables with common distribution G such that

$$G(x) = P(\eta_1 \leq x) = P(S_{\tau_+} \leq x \mid \tau_+ < \infty);$$

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here S_{τ_+} is (said to be) the first ascending ladder height of the above random walk. If G is absolutely continuous, let g denote its density.

Since the asymptotics of W is also related to F , we introduce some common distribution classes. To do so, we assume that, unless stated otherwise, all limiting relationships in this paper hold as $x \rightarrow \infty$. If $a(x)b(x)^{-1} \rightarrow 1$, we write $a(x) \sim b(x)$; if $\limsup_{x \rightarrow \infty} a(x)b(x)^{-1} < \infty$, we write $a(x) = O(b(x))$; and if $a(x) = O(b(x))$ and $b(x) = O(a(x))$, we write $a(x) \approx b(x)$. A nonnegative function l on $(-\infty, \infty)$ is said to be long tailed, written $l \in \mathcal{L}d$, if l is eventually positive and $l(x + y) \sim l(x)$ uniformly for $|y| \leq 1$. If

$$l \in \mathcal{L}d, \quad \int_0^\infty l(y) dy < \infty, \quad \text{and} \quad \int_0^x l(x - y)l(y) dy \sim 2 \int_0^\infty l(y) dy l(x),$$

then we write $l \in \mathcal{S}d$. For a proper distribution L on $(-\infty, \infty)$, if $1 - L = \bar{L} \in \mathcal{L}d$ then we write $L \in \mathcal{L}$; if $\bar{L} \in \mathcal{S}d$ then we write $L \in \mathcal{S}^*$; and if $\int_0^x \bar{L}(x - y)L(dy) \sim 2\bar{L}(x)$ then we write $L \in \mathcal{S}$. It is well known that \mathcal{S} and \mathcal{S}^* are the spaces introduced by Chistyakov (1964) and Klüppelberg (1988), respectively. They are two of the most important heavy-tailed distribution classes and have the following proper relationship: $\mathcal{S}^* \subset \mathcal{S} \subset \mathcal{L}$. Finally, for a proper distribution L on $(-\infty, \infty)$, let

$$\bar{L}^I(x) = \min \left\{ 1, \int_x^\infty \bar{L}(y) dy \right\}, \quad x \geq 0.$$

Theorem 1.1. Assume that $F \in \mathcal{L}$ has density $f(x) = f_1(x)f_2(x)$, $x \in (-\infty, \infty)$, where $f_1 \in \mathcal{L}d$ and f_2 is an eventually positive, nonincreasing function. Then G and W are absolutely continuous and the following statements are equivalent: (i) $F \in \mathcal{S}^*$; (ii) $g \in \mathcal{S}d$; (iii) $w \in \mathcal{S}d$; (iv) $w(x) \sim m^{-1}\bar{F}(x)$.

Remark 1.1. Under the conditions that $F \in \mathcal{S}^*$ and $f \in \mathcal{L}d$, Theorem 4 of Asmussen *et al.* (2003) stated part (iv) of our Theorem 1.1. Obviously, the latter theorem improves and extends this result. However, we must point out that the idea of the proof comes from Asmussen *et al.* (2003) and Asmussen *et al.* (2002). The main difference is that we divide f into two factors.

In Section 2 we will prove Theorem 1.1. For this, we need a lemma on the asymptotics of the first ascending ladder height. In Section 3 we will extend the lemma of Section 2 and show that it is independently significant and can be used to give new proofs of some classical results.

2. Proof of Theorem 1.1

We use the following lemma to prove Theorem 1.1.

Lemma 2.1. Under the conditions of Theorem 1.1 we have

$$\begin{aligned} I(x) &:= \sum_{n=1}^\infty E f(x - S_n) \mathbf{1}(A_n) \\ &\sim (1 - p)m^{-1}\bar{F}(x), \end{aligned} \tag{2.1}$$

where $A_n = \{S_j \leq 0: j = 0, \dots, n\}$, $n \geq 0$.

Proof. We consider the taboo renewal function $H(B) = \sum_{n=0}^\infty H_n(B)$, $B \subset (-\infty, 0]$, where $H_0(B) = \mathbf{1}(0 \in B)$ and $H_n(B) = P(A_n, S_n \in B)$, $n \geq 1$. According to Blackwell's

renewal theorem, for each $a \in (-\infty, \infty)$ we have

$$H(-x + a) - H(-x) \rightarrow m^{-1}a(1 - p). \tag{2.2}$$

For sufficiently large x , since $f_1 \in \mathcal{L}d$ and f_2 is eventually nonincreasing, we have

$$\begin{aligned} I(x) &= - \sum_{j=0}^{\infty} \int_j^{j+1} f_1(x+z)f_2(x+z)H(-dz) \\ &\sim - \sum_{j=0}^{\infty} f_1(x+j) \int_j^{j+1} f_2(x+z)H(-dz) \\ &\leq \sum_{j=0}^{\infty} f(x+j)(H(-j) - H(-j-1)) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} I(x) &\sim - \sum_{j=0}^{\infty} f_1(x+1+j) \int_j^{j+1} f_2(x+z)H(-dz) \\ &\geq \sum_{j=0}^{\infty} f(x+1+j)(H(-j) - H(-j-1)). \end{aligned} \tag{2.4}$$

Since $f_1 \in \mathcal{L}d$ we furthermore know that

$$\begin{aligned} \mathcal{H}(f_1) := \{h(x) : [0, \infty) \rightarrow [0, \infty) : h(x) \uparrow \infty, x^{-1}h(x) \rightarrow 0, \text{ and} \\ f_1(x-y) \sim f_1(x) \text{ uniformly for } |y| \leq h(x)\} \\ \neq \emptyset, \end{aligned}$$

and if $h \in \mathcal{H}(f_1)$ then, for any $h_1 : [0, \infty) \rightarrow [0, \infty)$, if $h_1(x) \uparrow \infty$ and $h_1(x) \leq h(x)$, we have $h_1 \in \mathcal{H}(f_1)$ (see Foss and Zachary (2003), for example). We thus can choose an integer-valued function $h \in \mathcal{H}(f_1)$ such that

$$f(x)h(x) = o(\overline{F}(x)) \quad \text{and} \quad h(x) = O(h(x-1)). \tag{2.5}$$

By (2.2), Toeplitz’s lemma (see Stout (1974, p. 120)), and (2.5), when x is sufficiently large we have

$$\begin{aligned} \sum_{j=0}^{h(x)} f(x+j)(H(-j) - H(-j-1)) &\sim f_1(x) \sum_{j=0}^{h(x)} f_2(x+j)(H(-j) - H(-j-1)) \\ &\leq f(x)h(x)h(x)^{-1} \sum_{j=0}^{h(x)} (H(-j) - H(-j-1)) \\ &\sim (1-p)m^{-1}f(x)h(x) \\ &= o(\overline{F}(x)). \end{aligned} \tag{2.6}$$

Furthermore, by (2.2), (2.5), and $F \in \mathcal{L}$, when x is sufficiently large we have

$$\begin{aligned} & \sum_{j=h(x)+1}^{\infty} f(x+j)(H(-j) - H(-j-1)) \\ & \sim (1-p)m^{-1} \sum_{j=h(x)+1}^{\infty} f_1(x+j)f_2(x+j) \\ & \leq (1-p)m^{-1} \sum_{j=h(x)+1}^{\infty} f_1(x+j) \int_j^{j+1} f_2(x-1+y) dy \\ & \sim (1-p)m^{-1} \sum_{j=h(x)+1}^{\infty} \int_j^{j+1} f(x-1+y) dy \\ & \leq (1-p)m^{-1} \int_0^{\infty} f(x-1+y) dy. \end{aligned} \tag{2.7}$$

It follows from (2.3), (2.6), (2.7), and $F \in \mathcal{L}$ that

$$\limsup_{x \rightarrow \infty} I(x)\bar{F}(x)^{-1} \leq (1-p)m^{-1}. \tag{2.8}$$

Similarly, using (2.4) we can prove that

$$\liminf_{x \rightarrow \infty} I(x)\bar{F}(x)^{-1} \geq (1-p)m^{-1}. \tag{2.9}$$

Thus, (2.1) follows from (2.8) and (2.9).

Proof of Theorem 1.1. By definition of τ_+ , we have

$$\begin{aligned} \bar{G}(x) &= P(S_{\tau_+} > x \mid \tau_+ < \infty) \\ &= p^{-1} \sum_{n=1}^{\infty} P(S_n > x, A_{n-1}) \\ &= p^{-1} \sum_{n=1}^{\infty} \int_{-\infty}^0 \int_{x-y}^{\infty} f(z) dz dP(S_{n-1} \leq y, A_{n-1}). \end{aligned}$$

Thus, G is absolutely continuous and, by Lemma 2.1, we know that its density is

$$\begin{aligned} g(x) &= p^{-1} \sum_{n=1}^{\infty} \int_{-\infty}^0 f(x-y) dP(S_{n-1} \leq y, A_{n-1}) \\ &= p^{-1} \int_{-\infty}^0 f(x-y)H(dy) \\ &= p^{-1}I(x) \\ &\sim (1-p)m^{-1}p^{-1}\bar{F}(x). \end{aligned} \tag{2.10}$$

It follows from (2.10) that (i) \iff (ii). Also by (2.10), we know that $g \in \mathcal{L}d$. Thus, by Theorem 3.2 of Klüppelberg (1989), we have (ii) \iff (iii) \iff (iv), completing the proof.

3. Extension of Lemma 2.1 and applications

Lemma 2.1 is the key to proving Theorem 1.1. In fact, Lemma 2.1 is of independent significance. In order to explain this, we extend Lemma 2.1 and obtain a general result. For any given function u and some T , $0 < T \leq \infty$, let $v(x) := u(x) - u(x + T)$, $U(x) := \int_x^\infty u(y) dy$, and $V(x) := U(x) - U(x + T)$, for $x \in (-\infty, \infty)$. We then have the following proposition.

Proposition 3.1. *Assume that $U(0) < \infty$ and $V \in \mathcal{L}d$; that u on $(-\infty, \infty)$ is an eventually positive, nonincreasing function for $0 < T < \infty$; and that $u(x) = u_1(x)u_2(x)$, $x \in (-\infty, \infty)$, for $T = \infty$, where $u_1 \in \mathcal{L}d$ and u_2 is an eventually positive, nonincreasing function. Then*

$$I(x) := \sum_{n=1}^\infty E v(x - S_n) \mathbf{1}(A_n) \sim (1 - p)m^{-1}V(x),$$

where $A_n = \{S_j \leq 0: j = 0, \dots, n\}$, $n \geq 0$.

If we take $v = f$ and $T = \infty$ in Proposition 3.1, then we recover Lemma 2.1. If we take $u = \bar{F}$ in Proposition 3.1, then we obtain the following result.

Corollary 3.1. *Let $V(x) = \bar{F}^I(x) - \bar{F}^I(x + T)$, $x \in (-\infty, \infty)$, for some T , $0 < T \leq \infty$. Assume that $V \in \mathcal{L}d$. Then*

$$\bar{G}(x) - \bar{G}(x + T) \sim (1 - p)p^{-1}m^{-1}(\bar{F}^I(x) - \bar{F}^I(x + T)). \tag{3.1}$$

In order to prove Proposition 3.1 we need the following lemmas. We omit their proofs.

Lemma 3.1. *Assume that $u(x) = u_1(x)u_2(x)$, $x \in (-\infty, \infty)$, where $u_1 \in \mathcal{L}d$ and u_2 is an eventually positive, nonincreasing function. For $0 < T < \infty$, we have $u \in \mathcal{L}d \iff V \in \mathcal{L}d$, each side of which implies that $V(x) \sim Tu(x)$. For $T = \infty$, we have $U \in \mathcal{L}d \iff u(x) = o(U(x))$.*

Lemma 3.2. *Assume that $U(0) < \infty$ and that u satisfies the condition of Lemma 3.1. Then $u(x) \rightarrow u(\infty) = v(\infty) = U(\infty) = V(\infty) = 0$ and $V(x) = U(x)$, for $T = \infty$.*

Proof of Proposition 3.1. For $T = \infty$, the proof is similar to the proof of Lemma 2.1. For $0 < T < \infty$, we can prove the proposition using Lemma 3.1 and Lemma 3.2, following the lines of the proof of Lemma 3 of Asmussen *et al.* (2002). We omit the details.

In Corollary 3.1, from (3.1) and Lemma 3.1 we have

$$\bar{G}(x) - \bar{G}(x + T) \sim T(1 - p)p^{-1}m^{-1}\bar{F}(x) \tag{3.2}$$

for $0 < T < \infty$; (3.2) is Lemma 3 of Asmussen *et al.* (2002). Furthermore, if $F \in \mathcal{S}^*$ then, using a standard method, it is easy to prove that

$$\bar{W}(x) - \bar{W}(x + T) \sim m^{-1}T\bar{F}(x),$$

which is Theorem 1 of Asmussen *et al.* (2002).

In Corollary 3.1, from Lemma 3.2, if $U = V = F^I \in \mathcal{L}$ then

$$\bar{G}(x) \sim (1 - p)p^{-1}m^{-1}\bar{F}^I(x) \tag{3.3}$$

for $T = \infty$. Thus, we obtain another known result, Theorem 10 of Borovkov (1976, Chapter 4), by a different method. Furthermore, if $F^I \in \mathcal{S}$ then, using (3.3) and a standard method, it is easy to prove that

$$\overline{W}(x) \sim m^{-1}\overline{F}(x). \quad (3.4)$$

It is well known that (3.4) is Theorem 2(B) of Veraverbeke (1977).

We expect that for different choices of u we can obtain other results.

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