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INFINITELY MANY SOLUTIONS FOR A CLASS OF SYSTEMS OF DIFFERENTIAL INCLUSIONS

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Dedicated to Professor Vicenţiu Rădulescu on the occasion of his 50th birthday

Abstract Using a non-smooth version (due to Marano and Motreanu) of a variational principle of Ricceri we prove the existence of infinitely many solutions for certain systems of differential inclusions with various types of boundary conditions.

Keywords: locally Lipschitz function; generalized directional derivative; systems of differential inclusions; Sobolev spaces

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1. Introduction

In recent years there has been an increasing interest in studying p-Laplacian systems with various types of boundary conditions. We mention here the contributions of Gasinski and Papageorgiu [**5**], Jebelean and Moroşanu [**7**, **8**], Manásevich and Mawhin [**10**, **11**] and Mawhin [**14**]. The methods used in these papers to prove the existence of (multiple, but finitely many) solutions of p-Laplacian systems are based on degree theory, on minimax results, on fixed-point theorems or on continuation methods of Leray–Schauder type. Using the symmetric version of a mountain-pass-type theorem, Jebelean proved in [**6**] that, under certain assumptions, ordinary p-Laplacian systems with nonlinear boundary conditions have infinitely many solutions.

In [**17**] Ricceri developed a general variational principle which can be applied to prove that, for example, certain Neumann-type problems have infinitely many solutions (see also [**18**]). This new method of Ricceri has been extended to the non-smooth case by Marano and Motreanu [**12**]. Applications of Ricceri's, and Marano and Motreanu's methods can be found, for example, in [**4**] and [**9**], respectively. In this sense we also mention Di Falco's contributions in proving (using Ricceri's method) the existence of infinitely many solutions for p, q-Laplacian-type problems with Neumann [**2**] and Dirichlet [**3**] boundary conditions, respectively.

The aim of the present paper is to study systems of differential inclusions with various types of boundary conditions using Marano and Motreanu's non-smooth version of Ricceri's method. The main results of the paper ensure the existence of infinitely many solutions of these systems, thus completing the results obtained in the papers mentioned at the beginning of this section. The price we have to pay to get infinitely many solutions is that we impose more restrictive boundary conditions than in these papers. On the other hand, we do not require (as in [**6**]) that the nonlinear term should satisfy certain symmetry properties, but it has to have an oscillating behaviour.

2. Prerequisites

Definition 2.1. Let (X, d) be a metric space and $f\colon X \to \mathbb{R}$ be a real-valued function. We say that

(i) f satisfies the *Lipschitz condition on a subset M of X* provided there exists a real number $L > 0$ (depending on M) such that

$$|f(x) - f(y)| \leqslant Ld(x, y) \quad \text{for all } x, y \in M.$$

(ii) f is *locally Lipschitz* if, for every $x \in X$, there exists a neighbourhood U of x such that f satisfies the Lipschitz condition on U.

Remark 2.2. A standard argument yields that, in the case of a locally compact metric space X, a function $f\colon X \to \mathbb{R}$ is locally Lipschitz if and only if f satisfies the Lipschitz condition on every compact subset of X.

For the rest of this section assume that $(X, \|\cdot\|)$ is a real normed space and $X^\star$ is its topological dual.

Definition 2.3. Let $f\colon X \to \mathbb{R}$ be a locally Lipschitz function. The *generalized directional derivative* of f at the point $x \in X$ in the direction $y \in X$ is defined as

$$f^\circ(x; y) := \limsup_{z \to x, \tau \to 0^+} \frac{f(z + \tau y) - f(z)}{\tau}.$$

The *generalized gradient* of f at $x \in X$ is the set

$$\partial f(x) := \{x^\star \in X^\star : x^\star(y) \leqslant f^\circ(x; y) \text{ for all } y \in X\}.$$

Remark 2.4. Let $f\colon X \to \mathbb{R}$ be a locally Lipschitz function and let $x \in X$. It can be shown that $f^\circ(x; y) \in \mathbb{R}$ for every $y \in X$. Also, the functional $f^\circ(x, \cdot)\colon X \to \mathbb{R}$ is subadditive and positively homogeneous, and there exists a real number $L > 0$ such that $|f^\circ(x; y)| \leqslant L\|y\|$ for every $y \in X$. Thus, due to the Hahn–Banach Theorem, the set $\partial f(x)$ is non-empty.

Consider now $h\colon X \to \mathbb{R}$ to be locally Lipschitz, $j\colon X \to]-\infty, +\infty]$ to be convex, proper and lower semicontinuous, and define $\Phi\colon X \to]-\infty, +\infty]$ as $\Phi = h + j$. The following type of critical point was introduced in [**16**] (in fact, it is the non-smooth analogue of the notion of the critical point introduced by Szulkin in [**19**]).

Definition 2.5. A point $x \in X$ is a *critical point* of Φ if

$$h^\circ(x; y-x) + j(y) - j(x) \geqslant 0 \quad \text{for all } y \in X.$$

Define $D(j) := \{x \in X \mid j(x) < +\infty\}$. A straightforward argument yields the following result (see also [**13**, Proposition 2.1]).

Lemma 2.6. *Each local minimum of Φ lying in $D(j)$ is a critical point of Φ.*

Suppose now that X and Y are real Banach spaces such that X is compactly embedded in Y. Let $h_1 \colon Y \to \mathbb{R}$ and $h_2 \colon X \to \mathbb{R}$ be locally Lipschitz functions, and let $j_1 \colon X \to]-\infty, +\infty]$ be convex, proper and lower semicontinuous. Define the maps $\Phi \colon X \to]-\infty, +\infty]$ and $\Psi \colon X \to \mathbb{R}$ as

$$\Phi(x) = h_1(x) + j_1(x), \quad \Psi(x) = h_2(x) \quad \text{for all } x \in X.$$

Assume that

$$\Psi^{-1}(\,]-\infty, \rho[) \cap D(j_1) \neq \emptyset \quad \text{for all } \rho > \inf_X \Psi. \tag{2.1}$$

For every $\rho > \inf_X \Psi$ set

$$\phi(\rho) := \inf_{u \in \Psi^{-1}(\,]-\infty,\rho[)} \frac{\Phi(u) - \inf_{v \in \overline{(\Psi^{-1}(\,]-\infty,\rho[))}_w} \Phi(v)}{\rho - \Psi(u)}, \tag{2.2}$$

where $\overline{(\Psi^{-1}(\,]-\infty, \rho[))}_w$ is the weak closure of $\Psi^{-1}(\,]-\infty, \rho[)$,

$$\gamma := \liminf_{\rho \to +\infty} \phi(\rho), \tag{2.3}$$

$$\delta := \liminf_{\rho \to (\inf_X \Psi)^+} \phi(\rho). \tag{2.4}$$

The main results of this paper are obtained applying the following Critical Point Theorem of Marano and Motreanu (see [**12**, Theorem 1.1]). In fact, this theorem is a non-smooth version of a (smooth) critical point result of Ricceri [**17**, Theorem 2.5].

Theorem 2.7. *Suppose that X is reflexive, Ψ is weakly sequentially lower semicontinuous and coercive, and (2.1) is satisfied. Then the following assertions hold.*

(i) *For every $\rho > \inf_X \Psi$ and every $\lambda > \phi(\rho)$ the function $\Phi + \lambda\Psi$ has a critical point (more exactly, a local minimum) lying in $\Psi^{-1}(\,]-\infty, \rho[) \cap D(j_1)$.*

(ii) *If $\gamma < +\infty$, then, for each $\lambda > \gamma$, either*

  (a) *$\Phi + \lambda\Psi$ has a global minimum or*

  (b) *there is a sequence (u_n) of critical points (more exactly, local minima) of $\Phi + \lambda\Psi$ lying in $D(j_1)$ and such that $\lim_{n\to\infty} \Psi(u_n) = +\infty$.*

(iii) *If $\delta < +\infty$, then, for every $\lambda > \delta$, either*

  (a) *$\Phi + \lambda\Psi$ has a local minimum which is also a global minimum of Ψ or*

  (b) *there is a sequence (u_n) of pairwise distinct critical points (more exactly, local minima) of $\Phi + \lambda\Psi$ lying in $D(j_1)$ and such that (u_n) converges weakly to a global minimum of Ψ and $\lim_{n\to\infty} \Psi(u_n) = \inf_X \Psi$.*

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3. The main results

Let $n \geqslant 1$ be a natural number, and let T and p be reals such that $T > 0$ and $p > 1$. The space $\mathbb{R}^n$ is equipped with a fixed norm $|\cdot|$. If $x \in \mathbb{R}^n$, then $x_1, \dots, x_n$ will always denote the components of x, i.e. $x = (x_1, \dots, x_n)$. Similarly, if $u\colon M \to \mathbb{R}^n$ is a vector-valued map defined on a non-empty set M, then $u_1, \dots, u_n$ are the components of u, i.e. $u(x) = (u_1(x), \dots, u_n(x))$ for every $x \in M$. The zero vector in $\mathbb{R}^n$ is denoted by 0_n. Also, we define the following binary relation on $\mathbb{R}^n$: for $x, y \in \mathbb{R}^n$, let

$$x * y := (x_1 y_1, \dots, x_n y_n).$$

Throughout this section we assume the following.

(C1) $F\colon \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz.

(C2) $S \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is a closed convex set with $\{(x, x) \mid x \in \mathbb{R}^n\} \subseteq S$.

(C3) $\gamma_1, \dots, \gamma_n \in L^\infty(\,]0, T[\,, \mathbb{R})$ are such that $\operatorname{ess\,inf} \gamma_i > 0$, for $i = \overline{1, n}$; we set $\gamma := (\gamma_1, \dots, \gamma_n)$.

(C4) $\alpha \in L^1(\,]0, T[\,, \mathbb{R})$ is such that $\alpha(t) \geqslant 0$ a.e. in $]0, T[\,$,

(C5) X is the Sobolev space $W^{1,p}(\,]0, T[\,, \mathbb{R}^n)$ equipped with the usual norm

$$\|u\| = \left(\int_0^T |u(t)|^p \, \mathrm{d}t + \int_0^T |u'(t)|^p \, \mathrm{d}t \right)^{1/p}.$$

Recall that X is isomorphic to $(W^{1,p}(\,]0, T[\,, \mathbb{R}))^n$ [**15**, Paragraph 1.4].

Remark 3.1. According to Remark 2.2, condition (C1) is equivalent to the fact that F satisfies the Lipschitz condition on every open (or closed) ball centred at 0_n, i.e.

$$\forall r > 0 \quad \exists L_r > 0\colon |F(x) - F(y)| \leqslant L_r |x - y| \quad \text{for all } x, y \in \mathbb{R}^n \text{ with } |x|, |y| < r. \tag{3.1}$$

Convention.

It is well known that for every $f \in W^{1,q}(\,]0, T[\,, \mathbb{R})$ (where $1 \leqslant q \leqslant +\infty$) there exists a unique continuous function $\tilde{f} \in C([0, T], \mathbb{R})$ such that $f = \tilde{f}$ a.e. in $]0, T[$. When necessary (e.g. to give meaning to $f(0)$ or $f(T)$) we will replace f by its continuous representative $\tilde{f}$; for simplicity we will denote this continuous representative of f with the same letter f.

Denote by $\xi_p\colon \mathbb{R}^n \to \mathbb{R}^n$ the isomorphism defined as

$$\xi_p(x) = \begin{cases} |x|^{p-2} x, & x \neq 0_n, \\ 0_n, & x = 0_n. \end{cases}$$

The aim of this section is to show that under certain hypotheses there exist infinitely many $u \in X$ with $(u(0), u(T)) \in S$ and such that u satisfies the following differential inclusion problem:

$$\left.\begin{aligned} -[\xi_p(u')]' + \gamma * \xi_p(u) &\in \alpha(t) \partial F(u), \\ (\xi_p(u')(0), -\xi_p(u')(T)) &\in N_S(u(0), u(T)), \end{aligned}\right\} \tag{P}$$

where $N_S(x,y)$ denotes the normal cone* of S at $(x,y) \in S$. Every such function $u \in X$ is called a *solution of (P)*.

Remark 3.2.

(i) By the proof of [**7**, Proposition 3.2], for $u \in W^{1,p}(\,]0,T[\,,\mathbb{R}^n)$, the map $\xi_p \circ u'$ belongs to $W^{1,1}(\,]0,T[\,,\mathbb{R}^n)$; thus, the function $(\xi_p \circ u')'$ in problem (P) exists.

(ii) Choosing $S = \{(x,x) \mid x \in \mathbb{R}^n\}$, we get periodic boundary conditions in (P): $u(0) = u(T)$ and $\xi_p(u')(0) = \xi_p(u')(T)$.

(iii) In the case when $S = \mathbb{R}^n \times \mathbb{R}^n$ we obtain Neumann-type boundary conditions $\xi_p(u')(0) = \xi_p(u')(T) = 0_n$.

The first step in the study of (P) is to establish the corresponding energy function. For this some preparation is needed.

- For every $i \in \{1,\dots,n\}$ denote by X_i the Sobolev space $W^{1,p}(\,]0,T[\,,\mathbb{R})$ equipped with the norm $\|\cdot\|_i$, where

$$\|f\|_i = \left(\int_0^T \gamma_i(t)|f(t)|^p\,\mathrm{d}t + \int_0^T |f'(t)|^p\,\mathrm{d}t\right)^{1/p}.$$

  Since $\operatorname{ess\,inf}\gamma_i > 0$ it follows that $\|\cdot\|_i$ is equivalent to the usual norm on $W^{1,p}(\,]0,T[\,,\mathbb{R})$.

- Let Y be the real Banach space $C([0,T],\mathbb{R}^n)$ endowed with the supremum norm $\|\cdot\|_{\mathrm{s}}$. For simplicity we will denote the supremum norm on every space $C([0,T],\mathbb{R}^m)$, $m \in \mathbb{N}$, by the same symbol $\|\cdot\|_{\mathrm{s}}$.

- Set

$$\Sigma := \{u \in X \mid (u(0),u(T)) \in S\}.$$

  Note that Σ is a closed convex subset of X containing the constant functions.

- Define $h_1 \colon Y \to \mathbb{R}$ by

$$h_1(u) = \int_0^T \alpha(t)F(u(t))\,\mathrm{d}t \quad \text{for every } u \in Y,$$

  $h_2 \colon X \to \mathbb{R}$ by

$$h_2(u) = \frac{1}{p}(\|u_1\|_1^p + \cdots + \|u_n\|_n^p) \quad \text{for every } u \in X$$

  and $j_1 \colon X \to \,]-\infty,+\infty]$ by

$$j_1(u) = \begin{cases} 0, & u \in \Sigma, \\ +\infty, & u \notin \Sigma. \end{cases}$$

* If A is a non-empty convex subset of $\mathbb{R}^m$ ($m \in \mathbb{N}$) and $a' \in A$, then the *normal cone of A at a'* is the set defined as

$$N_S(a) := \{x \in \mathbb{R}^m \mid \langle x, a' - a\rangle \geqslant 0 \text{ for all } a \in A\},$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^m$.

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Lemma 3.3. *The following assertions hold:*

(i) *the above defined maps $h_1\colon Y \to \mathbb{R}$ and $h_2\colon X \to \mathbb{R}$ are locally Lipschitz;*

(ii) *the map $j_1\colon X \to \,]-\infty, +\infty]$ is convex, proper and lower semicontinuous.*

Proof. (i) To show that h_1 is locally Lipschitz, pick an arbitrary function $u^0 \in Y$. According to (3.1) there is a constant $L \geqslant 0$ such that

$$|F(x) - F(y)| \leqslant L|x - y| \quad \text{for all } x, y \in \mathbb{R}^n \text{ with } |x|, |y| < 1 + \|u^0\|_{\mathrm{s}}. \tag{3.2}$$

If $u \in Y$ is so that $\|u - u^0\|_{\mathrm{s}} < 1$, then

$$|u(t)| \leqslant \|u\|_{\mathrm{s}} \leqslant \|u - u^0\|_{\mathrm{s}} + \|u^0\|_{\mathrm{s}} < 1 + \|u^0\|_{\mathrm{s}} \quad \text{for all } t \in [0, T]. \tag{3.3}$$

Consider now $u, v \in Y$ with $\|u - u^0\|_{\mathrm{s}} < 1$ and $\|v - u^0\|_{\mathrm{s}} < 1$. Then, according to (3.2) and (3.3), the following inequalities hold:

$$\begin{aligned} |h_1(u) - h_1(v)| &\leqslant \int_0^T |\alpha(t)|\,|F(u(t)) - F(v(t))|\,\mathrm{d}t \\ &\leqslant L \int_0^T |\alpha(t)|\,|u(t) - v(t)|\,\mathrm{d}t \\ &\leqslant L\|\alpha\|_{L^1}\|u - v\|_{\mathrm{s}}. \end{aligned}$$

Thus, h_1 is locally Lipschitz.

Since $p > 1$, the map $h_2\colon X \to \mathbb{R}$ is convex. Obviously, h_2 is bounded above on every ball; thus, h_2 is locally Lipschitz by [**1**, Proposition 2.2.6].

Assertion (ii) is straightforward. □

Define now $\Phi\colon X \to \,]-\infty, +\infty]$ and $\Psi\colon X \to \mathbb{R}$ by

$$\Phi(u) = h_1(u) + j_1(u), \quad \Psi(u) = h_2(u) \quad \text{for every } u \in X. \tag{3.4}$$

Proposition 3.4. *If $u \in \Sigma$ is a critical point (in the sense of Definition 2.5) of the map*

$$\Phi + \Psi = (h_1 + h_2) + j_1\colon X \to \,]-\infty, +\infty],$$

then u is a solution of problem (P).

Proof. See Proposition 3.2 of [**7**]. □

Our aim is to apply Theorem 2.7 to the maps Φ and Ψ defined in (3.4). For this we first observe that all assumptions required in Theorem 2.7 are satisfied:

- X, Y are real Banach spaces; X is reflexive and X is compactly embedded in Y;
- $h_1\colon Y \to \mathbb{R}$, $h_2\colon X \to \mathbb{R}$ are locally Lipschitz and j_1 is convex, proper and lower semicontinuous (according to Lemma 3.3);

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- $\Psi = h_2 \colon X \to \mathbb{R}$ is weakly sequentially lower semicontinuous (being convex and continuous) and coercive;
- since $\inf_X \Psi = 0$ and since Σ contains the constant functions, condition (2.1) is satisfied.

Furthermore, we have to introduce some suitable subsets of $\mathbb{R}^n$. For this, note that, since X_i can be embedded in $C([0,T],\mathbb{R})$, there exist $c_i > 0$, $i = \overline{1,n}$, such that

$$\|f\|_{\mathrm{s}} \leqslant c_i \|f\|_i \quad \text{for every } f \in X_i. \tag{3.5}$$

For every $r > 0$ let

$$\left.\begin{aligned} A(r) &:= \left\{ x \in \mathbb{R}^n \colon \frac{1}{p} \sum_{i=1}^n \frac{1}{c_i^p} |x_i|^p \leqslant r \right\}, \\ B(r) &:= \left\{ x \in \mathbb{R}^n \colon \frac{1}{p} \sum_{i=1}^n |x_i|^p \int_0^T \gamma_i(t)\,\mathrm{d}t \leqslant r \right\}. \end{aligned}\right\} \tag{3.6}$$

Remark 3.5.

(i) For every $r > 0$ the inclusion $B(r) \subseteq A(r)$ holds. To see this we observe first that (3.5) implies that

$$1 \leqslant c_i \left(\int_0^T \gamma_i(t)\,\mathrm{d}t \right)^{1/p} \quad \text{for } i = \overline{1,n}.$$

Now pick an arbitrary $x \in B(r)$. Then

$$\frac{1}{p} \sum_{i=1}^n \frac{1}{c_i^p} |x_i|^p \leqslant \frac{1}{p} \sum_{i=1}^n |x_i|^p \int_0^T \gamma_i(t)\,\mathrm{d}t \leqslant r;$$

hence $x \in A(r)$.

(ii) Since the map

$$x \in \mathbb{R}^n \mapsto \sum_{i=1}^n |x_i|^p \int_0^T \gamma_i(t)\,\mathrm{d}t \in \mathbb{R}$$

is convex, we have that, for every $r > 0$,

$$\operatorname{Int} B(r) = \left\{ x \in \mathbb{R}^n \colon \frac{1}{p} \sum_{i=1}^n |x_i|^p \int_0^T \gamma_i(t)\,\mathrm{d}t < r \right\}.$$

Theorem 3.6. *If $r > 0$ is such that*

$$\min_{x \in A(r)} F(x) = \min_{x \in \operatorname{Int} B(r)} F(x),$$

then the following assertions hold:

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(i) $\phi(r) = 0$, *where* ϕ *is defined by (2.2);*

(ii) *problem (P) has a solution* $u \in \Sigma$ *satisfying the condition* $\Psi(u) < r$.

Proof. (i) We have $\phi(r) \geqslant 0$, by definition. To show the converse inequality choose $x^0 \in \operatorname{Int} B(r)$ so that

$$F(x^0) = \min_{x \in \operatorname{Int} B(r)} F(x) = \min_{x \in A(r)} F(x),$$

and let $u^0 \colon X \to \mathbb{R}^n$ be the function taking the constant value x^0. For every $i \in \{1, \dots, n\}$ we have that

$$\|u_i^0\|_i = |x_i^0| \biggl(\int_0^T \gamma_i(t) \, \mathrm{d}t \biggr)^{1/p};$$

thus,

$$\Psi(u^0) = \frac{1}{p} \sum_{i=1}^{n} |x_i^0|^p \int_0^T \gamma_i(t) \, \mathrm{d}t < r,$$

i.e. $u^0 \in \Psi^{-1}(\,]{-\infty}, r[)$. Since $\Psi^{-1}(\,]{-\infty}, r])$ is convex and closed (in the norm topology), it is closed also in the weak topology; hence,

$$\overline{(\Psi^{-1}(\,]{-\infty}, r[))}_w \subseteq \Psi^{-1}(\,]{-\infty}, r]).$$

Now pick an arbitrary element $v \in \overline{(\Psi^{-1}(\,]{-\infty}, r[))}_w$. Then $\Psi(v) \leqslant r$. Therefore, using also (3.5),

$$\frac{1}{p} \sum_{i=1}^{n} \frac{1}{c_i^p} |v_i(t)|^p \leqslant \frac{1}{p} \sum_{i=1}^{n} \frac{1}{c_i^p} \|v_i\|_{\mathrm{s}}^p \leqslant \frac{1}{p} \sum_{i=1}^{n} \|v_i\|_i^p = \Psi(v) \leqslant r \quad \text{for all } t \in [0, T].$$

We conclude that $v(t) \in A(r)$ for every $t \in [0, T]$. Hence,

$$F(x^0) \leqslant F(v(t)) \quad \text{for every } t \in [0, T].$$

It follows that

$$\Phi(u^0) = h_1(u^0) = \int_0^T \alpha(t) F(x^0) \, \mathrm{d}t \leqslant \int_0^T \alpha(t) F(v(t)) \, \mathrm{d}t = h_1(v) \leqslant \Phi(v).$$

Since $v \in \overline{(\Psi^{-1}(\,] - \infty, r[))}_w$ was chosen arbitrarily, we conclude that

$$\inf_{v \in \overline{(\Psi^{-1}(\,]-\infty, r[))}_w} \Phi(v) = \Phi(u^0).$$

This implies, according to the definition of ϕ in (2.2), that

$$\phi(r) \leqslant \frac{\Phi(u^0) - \Phi(u^0)}{r - \Psi(u^0)} = 0,$$

and hence $\phi(r) = 0$.

(ii) Since $\phi(r) = 0$, we can apply Theorem 2.7 (i) for $\lambda = 1$ and conclude that the map $\Phi + \Psi$ has a critical point u lying in Σ and such that $\Psi(u) < r$. The assertion follows now from Proposition 3.4. □

Theorem 3.7. *Assume that*

(i) *there exists a sequence* $(r_k)_{k\in\mathbb{N}}$ *of positive reals such that* $\lim r_k = +\infty$ *and*

$$\min_{x\in A(r_k)} F(x) = \min_{x\in \operatorname{Int} B(r_k)} F(x) \quad \text{for every } k \in \mathbb{N},$$

(ii) *the following inequality holds:*

$$\liminf_{|x|\to+\infty} \frac{F(x)\int_0^T \alpha(t)\,\mathrm{d}t}{\sum_{i=1}^n |x_i|^p \int_0^T \gamma_i(t)\,\mathrm{d}t} < -\frac{1}{p}.$$

Then problem (P) has an unbounded sequence of solutions.

Proof. Assumption (i) implies, according to Theorem 3.6 (i), that $\phi(r_k) = 0$ for every $k \in \mathbb{N}$. Let γ be defined as in (2.3). Since $\phi(r) \geqslant 0$ for every $r > 0$, we conclude that

$$\gamma = \liminf_{r\to+\infty} \phi(r) = 0.$$

Applying Theorem 2.7 (ii) for $\lambda = 1$, we conclude that either assertion (a) or assertion (b) of this part of the theorem must hold. Next we show that (a) of part (ii) is not satisfied, i.e. we prove that $\Phi + \Psi$ is unbounded below. For this fix a real number q such that

$$\liminf_{|x|\to+\infty} \frac{F(x)\int_0^T \alpha(t)\,\mathrm{d}t}{\sum_{i=1}^n |x_i|^p \int_0^T \gamma_i(t)\,\mathrm{d}t} < q < -\frac{1}{p}.$$

Now choose a sequence $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ such that $\lim |x^k| = +\infty$ and

$$\frac{F(x^k)\int_0^T \alpha(t)\,\mathrm{d}t}{\sum_{i=1}^n |x_i^k|^p \int_0^T \gamma_i(t)\,\mathrm{d}t} < q \quad \text{for every } k \in \mathbb{N}.$$

For every $k \in \mathbb{N}$ denote by $u^k\colon X \to \mathbb{R}^n$ the constant function taking the value x^k. Then the following relations hold for every $k \in \mathbb{N}$:

$$\begin{aligned}\Phi(u^k) + \Psi(u^k) &= F(x^k)\int_0^T \alpha(t)\,\mathrm{d}t + \frac{1}{p}\sum_{i=1}^n |x_i^k|^p \int_0^T \gamma_i(t)\,\mathrm{d}t \\ &< \left(q + \frac{1}{p}\right)\sum_{i=1}^n |x_i^k|^p \int_0^T \gamma_i(t)\,\mathrm{d}t.\end{aligned}$$

Since $|x^k| \to +\infty$,

$$\int_0^T \gamma_i(t)\,\mathrm{d}t > 0 \quad \text{for every } i \in \{1, \dots, n\}$$

and $q + 1/p < 0$, we conclude that $\lim_{k\to\infty}(\Phi(u^k) + \Psi(u^k)) = -\infty$. Thus, $\Phi + \Psi$ is unbounded below. The assertion follows now from part (b) of Theorem 2.7 (ii), the definition of Ψ and Proposition 3.4. □

Theorem 3.8. *Assume that*

(i) *there exists a sequence $(r_k)_{k\in\mathbb{N}}$ of positive reals such that $\lim r_k = 0$ and*

$$\min_{x\in A(r_k)} F(x) = \min_{x\in \operatorname{Int} B(r_k)} F(x) \quad \text{for every } k \in \mathbb{N},$$

(ii) *the following inequality holds:*

$$\liminf_{x\to 0_n} \frac{F(x)\int_0^T \alpha(t)\,\mathrm{d}t}{\sum_{i=1}^n |x_i|^p \int_0^T \gamma_i(t)\,\mathrm{d}t} < -\frac{1}{p}.$$

Then problem (P) has a sequence of pairwise distinct solutions which converges strongly to the zero function $\theta_X \in X$.

Proof. We first observe that θ_X is the only global minimum of Ψ. Assumption (i) and Theorem 3.6 (i) imply that $\phi(r_k) = 0$ for every $k \in \mathbb{N}$. Let δ be defined as in (2.4). Since $\phi(r) \geqslant 0$ for every $r > 0$, we conclude that

$$\delta = \liminf_{r\to 0^+} \phi(r) = 0.$$

Applying Theorem 2.7 (iii) for $\lambda = 1$, we conclude that either assertion (a) or assertion (b) of this part of the theorem must hold. Next we show that part (a) of Theorem 2.7 (iii) is not satisfied, i.e. we prove that θ_X is not a local minimum of $\Phi + \Psi$. For this fix a real number q such that

$$\liminf_{x\to 0_n} \frac{F(x)\int_0^T \alpha(t)\,\mathrm{d}t}{\sum_{i=1}^n |x_i|^p \int_0^T \gamma_i(t)\,\mathrm{d}t} < q < -\frac{1}{p}.$$

Now choose a sequence $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ such that $\lim x^k = 0_n$ and

$$\frac{F(x^k)\int_0^T \alpha(t)\,\mathrm{d}t}{\sum_{i=1}^n |x_i^k|^p \int_0^T \gamma_i(t)\,\mathrm{d}t} < q \quad \text{for every } k \in \mathbb{N}.$$

In particular, the above inequalities imply that $F(0_n) = \lim_{k\to\infty} F(x^k) = 0$, and hence $\Phi(\theta_X) = 0$.

For every $k \in \mathbb{N}$ denote by $u^k\colon X \to \mathbb{R}^n$ the constant function taking the value x^k. Then the following relations hold for every $k \in \mathbb{N}$:

$$\begin{aligned}\Phi(u^k) + \Psi(u^k) &= F(x^k)\int_0^T \alpha(t)\,\mathrm{d}t + \frac{1}{p}\sum_{i=1}^n |x_i^k|^p \int_0^T \gamma_i(t)\,\mathrm{d}t \\ &< \left(q + \frac{1}{p}\right)\sum_{i=1}^n |x_i^k|^p \int_0^T \gamma_i(t)\,\mathrm{d}t \leqslant 0 \\ &= \Phi(\theta_X) + \Psi(\theta_X).\end{aligned}$$

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We have that $\lim \|u^k\| = 0$, and thus θ_X is not a local minimum of $\Phi + \Psi$. Part (b) of Theorem 2.7 (iii) and Proposition 3.4 imply now the existence of a sequence $(\tilde{u}^k)$ of pairwise distinct solutions of (P) such that $\lim \Psi(\tilde{u}^k) = 0$. Since Ψ is a norm on X which is equivalent to the norm $\|\cdot\|$, we conclude that $(\tilde{u}^k)$ converges strongly to θ_X in X. $\square$

4. Applications

We now specialize some of the data from the previous section in order to obtain applications of Theorems 3.7 and 3.8. Throughout this section let $n \geqslant 1$ be a natural number, $T = 1$, $p > 1$ be a real number, $S \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be a set satisfying condition (C2), γ_i be the function taking the constant value 1, for every $i \in \{1, \dots, n\}$, and let $\alpha \in L^1(]0, T[, \mathbb{R})$ be such that

$$\int_0^1 \alpha \, \mathrm{d}t > 1$$

and $\alpha(t) \geqslant 0$ a.e. in $]0, 1[$. In this case every norm $\|\cdot\|_i$, $i = \overline{1, n}$, reduces to the usual norm on $W^{1,p}(]0, 1[, \mathbb{R})$, and all the constants c_i, $i = \overline{1, n}$, in (3.5) can be considered to be equal to a suitable real number $c > 0$. Furthermore, we assume in this section that $\mathbb{R}^n$ is endowed with the p-norm

$$|x| = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

Thus, for $r > 0$, the sets $A(r)$ and $B(r)$ defined in (3.6) become

$$A(r) = \left\{ x \in \mathbb{R}^n : \frac{1}{p} \cdot \frac{1}{c^p} |x|^p \leqslant r \right\} \quad \text{and} \quad B(r) = \left\{ x \in \mathbb{R}^n : \frac{1}{p} |x|^p \leqslant r \right\}.$$

Example 4.1. We now give an application of Theorem 3.7. In order to define $F \colon \mathbb{R}^n \to \mathbb{R}$, we consider a function $f \colon [0, +\infty[\to [0, +\infty[$ with the following properties:

(i) f is surjective;

(ii) f is strictly increasing;

(iii) $\lim_{t \to +\infty} f(t + \pi)/f(t) > c^p$;

(iv) f^{-1} is locally Lipschitz.

Note that the above properties imply that $f(0) = 0$, that f^{-1} is also strictly increasing and that

$$\lim_{t \to +\infty} f(t) = \lim_{t \to +\infty} f^{-1}(t) = +\infty.$$

An example of a differentiable function satisfying properties (i)–(iv) is the function

$$t \in [0, +\infty[\mapsto a^t - 1 \in [0, +\infty[,$$

where a real $a > 1$ is chosen so that $a^\pi > c^p$.

Fix a real number $q \in [0, 1[$ and define $F\colon \mathbb{R}^n \to \mathbb{R}$ by

$$F(x) = \frac{1}{p}|x|^p \min\{q, \sin(f^{-1}(|x|^p))\} \quad \text{for every } x \in \mathbb{R}^n.$$

A straightforward argument yields that F is locally Lipschitz (i.e. it satisfies condition (C1)).

We next show that the assumptions (i) and (ii) of Theorem 3.7 are satisfied: to verify Theorem 3.7 (i), observe that (by (iii) above), for every sufficiently large $k \in \mathbb{N}$,

$$\frac{f((2k+1)\pi)}{f(2k\pi)} > c^p. \tag{4.1}$$

For these values of k set

$$r_k := \frac{f((2k+1)\pi)}{pc^p}.$$

Then $\lim r_k = +\infty$. Furthermore,

$$\min_{x \in A(r_k)} F(x) = \min_{|x|^p \leqslant f((2k+1)\pi)} F(x). \tag{4.2}$$

If $f(2k\pi) \leqslant |x|^p \leqslant f((2k+1)\pi)$, then

$$2k\pi \leqslant f^{-1}(|x|^p) \leqslant (2k+1)\pi;$$

hence,

$$\sin(f^{-1}(|x|^p)) \geqslant 0,$$

and thus $F(x) \geqslant 0$. Taking into account that $F(0_n) = 0$, it follows that

$$\min_{|x|^p \leqslant f((2k+1)\pi)} F(x) = \min_{|x|^p \leqslant f(2k\pi)} F(x). \tag{4.3}$$

On the other hand, if $x \in \mathbb{R}^n$ is such that $|x|^p \leqslant f(2k\pi)$, then, in view of (4.1), we have

$$\frac{1}{p}|x|^p \leqslant \frac{1}{p} f(2k\pi) < \frac{f((2k+1)\pi)}{pc^p} = r_k. \tag{4.4}$$

Using (4.2)–(4.4), we conclude that

$$\min_{x \in A(r_k)} F(x) = \min_{x \in \operatorname{Int} B(r_k)} F(x),$$

i.e. assumption (i) of Theorem 3.7 is satisfied. For assumption (ii) of this theorem note that

$$\liminf_{|x| \to +\infty} \frac{F(x) \int_0^1 \alpha(t)\,\mathrm{d}t}{\sum_{i=1}^n |x_i|^p} = \liminf_{|x| \to +\infty} \frac{1}{p} \sin(f^{-1}(|x|^p)) \int_0^1 \alpha(t)\,\mathrm{d}t = -\frac{1}{p} \int_0^1 \alpha(t)\,\mathrm{d}t < -\frac{1}{p}.$$

Theorem 3.7 now yields that problem (P) has an unbounded sequence of solutions.

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Example 4.2. To get an application of Theorem 3.8, consider a function $f\colon\ ]0,+\infty[\ \to\ ]0,+\infty[$ with the following properties:

(i) f is surjective;

(ii) f is strictly increasing;

(iii) f is differentiable on $]0,+\infty[$;

(iv) the map $t \in\ ]0,+\infty[\ \mapsto (f'(1/t)/t) \in \mathbb{R}$ is bounded on every interval $]0,a[\,$, $a > 0$;

(v) $\lim_{t\to+\infty} f^{-1}(t+\pi)/f^{-1}(t) > c^p$.

Observe that it follows from above that f^{-1} is also strictly increasing, and that $\lim_{t\to+\infty} f^{-1}(t) = +\infty$. An example of a function satisfying properties (i)–(v) is the function

$$t \in\ ]0,+\infty[\ \mapsto \log_a(t+1) \in\ ]0,+\infty[\,,$$

where a real $a > 1$ is chosen so that $a^\pi > c^p$.

Define $g\colon [0,+\infty[\ \to \mathbb{R}$ by

$$g(t) = \begin{cases} t\sin\left(f\left(\dfrac{1}{t}\right)\right), & t \neq 0, \\ 0, & t = 0. \end{cases}$$

This map has the following properties.

- g is continuous on $[0,+\infty[$.
- g is differentiable on $]0,+\infty[$; g is not differentiable at 0.
- g' is bounded on every interval $]0,a[\,$, $a > 0$. This follows from property (iv) of f and the fact that for every $t > 0$ we have

$$g'(t) = \sin\left(f\left(\frac{1}{t}\right)\right) - \cos\left(f\left(\frac{1}{t}\right)\right)\left(\frac{1}{t}f'\left(\frac{1}{t}\right)\right).$$

Now define $F\colon \mathbb{R}^n \to \mathbb{R}$ by

$$F(x) = \frac{1}{p}g(|x|^p) \quad \text{for every } x \in \mathbb{R}^n.$$

To show that F is locally Lipschitz, let $r > 0$ and $x, y \in \mathbb{R}^n$ with $|x|, |y| < r$. By the Mean-Value Theorem there exists $\xi \in\ ]0,r[$ such that

$$|F(x) - F(y)| = \xi^{p-1}|g'(\xi^p)| \cdot \big||x| - |y|\big| \leqslant r^{p-1}|g'(\xi^p)| \cdot |x-y|.$$

Since g' is bounded on the interval $]0,r^p[\,$, it follows that F satisfies the Lipschitz condition on the open ball centred at 0_n and with radius r. Since $r > 0$ was chosen arbitrarily, it follows that F is locally Lipschitz.

We next show that F satisfies conditions (i) and (ii) of Theorem 3.8. To verify (i), observe that (by (v) above), for every sufficiently large $k \in \mathbb{N}$,

$$\frac{f^{-1}((2k+1)\pi)}{f^{-1}(2k\pi)} > c^p. \tag{4.5}$$

For these values of k set

$$r_k := \frac{1}{pc^p} \cdot \frac{1}{f^{-1}(2k\pi)}.$$

Then $\lim r_k = 0$. Furthermore,

$$\min_{x \in A(r_k)} F(x) = \min_{|x|^p \leqslant 1/f^{-1}(2k\pi)} F(x). \tag{4.6}$$

If

$$\frac{1}{f^{-1}((2k+1)\pi)} \leqslant |x|^p \leqslant \frac{1}{f^{-1}(2k\pi)},$$

then

$$2k\pi \leqslant f\left(\frac{1}{|x|^p}\right) \leqslant (2k+1)\pi;$$

hence,

$$\sin\left(f\left(\frac{1}{|x|^p}\right)\right) \geqslant 0,$$

and thus $F(x) \geqslant 0$. Taking into account that $F(0_n) = 0$, it follows that

$$\min_{|x|^p \leqslant 1/f^{-1}(2k\pi)} F(x) = \min_{|x|^p \leqslant 1/f^{-1}((2k+1)\pi)} F(x). \tag{4.7}$$

On the other hand, if $x \in \mathbb{R}^n$ is such that $|x|^p \leqslant 1/f^{-1}((2k+1)\pi)$, then, in view of (4.5), we have that

$$\frac{1}{p}|x|^p \leqslant \frac{1}{p}\frac{1}{f^{-1}((2k+1)\pi)} < \frac{1}{pc^p} \cdot \frac{1}{f^{-1}(2k\pi)} = r_k. \tag{4.8}$$

Using (4.6)–(4.8), we conclude that

$$\min_{x \in A(r_k)} F(x) = \min_{x \in \operatorname{Int} B(r_k)} F(x),$$

i.e. assumption (i) of Theorem 3.8 is satisfied. For assumption (ii) of this theorem note that

$$\liminf_{x \to 0_n} \frac{F(x)\int_0^1 \alpha(t)\,\mathrm{d}t}{\sum_{i=1}^n |x_i|^p} = \liminf_{x \to 0_n} \frac{1}{p}\sin\left(f\left(\frac{1}{|x|^p}\right)\right)\int_0^1 \alpha(t)\,\mathrm{d}t = -\frac{1}{p}\int_0^1 \alpha(t)\,\mathrm{d}t < -\frac{1}{p}.$$

According to Theorem 3.8, problem (P) has a sequence of pairwise distinct solutions which converges strongly to the zero function $\theta_X \in X$.

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