BULL. AUSTRAL. MATH. SOC. 65D05, 41A05
VOL. 20 (1979), 115-130.

Interpolation in triangles

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Several new methods of approximation which assume arbitrary values on the boundary of a triangular domain are presented. All of the methods are affine invariant and have optimal algebraic precision. Nine parameter discrete interpolants which result from these methods are also given.

1. Introduction

This report is concerned with bivariate approximations which assume arbitrary values on the entire boundary of a triangular domain. This type of approximation has application in such areas as finite element analysis and computer aided geometric design. Examples of these two applications are given in [1] and [15] respectively.

The first investigation into this subject was by Barnhill, Birkhoff, and Gordon [2]. Several other papers, [3], [4], [5], [6], [7], [8], [9], [11], [13], have dealt with this same general topic.

In this paper, several new interpolation schemes are defined. Rather than give an exhaustive list of interpolants, we have selected methods on the basis of the techniques used in their development. It is hoped that this approach will lead to the extension of these interpolants to the case of interpolation to both position and slope on the boundary.

For the most part, the standard triangle T with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$ is used. The three boundary functions are assumed to be evaluation of a given bivariate function f and the interpolation process is viewed as mapping of the data function f to the interpolant

Received 12 December 1978. The research of Professor Nielson was supported by the United States Office of Naval Research.

$P[f]$ with the property that

$$P[f](0, y) = f(0, y) \; ,$$

$$(1.1) \qquad P[f](x, 0) = f(x, 0) \; ,$$

$$P[f](x, 1-x) = f(x, 1-x) \; .$$

All of the methods are of the form

$$(1.2) \qquad P[f](x, y) = \sum_{i=1}^{N} \omega_i(x, y) f\big(\alpha_i(x, y), \beta_i(x, y)\big) \; .$$

The functions ω_i , $i = 1, \ldots, N$ are called the weight functions and $\big(\alpha_i(x, y), \beta_i(x, y)\big) \in \partial T$, $i = 1, \ldots, N$, are collectively referred to as the stencil. It is often useful to describe the stencil graphically. Figure 1 depicts the stencil of the trilinear interpolant of [2],

$$(1.3) \quad Q^*[f] = \tfrac{1}{2}\Big[\frac{1-x-y}{1-y} f(0, y) + \frac{1-x-y}{1-x} f(x, 0) + \frac{y}{1-x} f(x, 1-x) + \frac{x}{1-y} f(1-y, y) + \frac{x}{x+y} f(x+y, 0) + \frac{y}{x+y} f(0, x+y) - xf(1, 0) - (1-x-y)f(0, 0) - yf(0, 1)\Big] \; .$$

The affine transformations which relate T and an arbitrary triangle Δ with vertices $v_i = \big(x_i, y_i\big)$, $i = 1, 2, 3$, can be used to obtain what is called an affine equivalent interpolant. For F defined on Δ , this process is described by the formula

$$(1.4) \qquad P_\Delta[F](s, t) = P[\tilde{f}]\big(b_j(s, t), b_k(s, t)\big) \; , \quad (s, t) \in \Delta \; ,$$

where

$$\tilde{f}(x, y) = F\big(xx_i+yx_j+(1-x-y)x_k, xy_i+yy_j+(1-x-y)y_k\big) \; ,$$

(i, j, k) represents a permutation of $(1, 2, 3)$, and $b_i = b_i(s, t)$, $i = 1, 2, 3$, represent the barycentric coordinates defined by

$$s = b_1x_1 + b_2x_2 + b_3x_3 \; ,$$

$$(1.5) \qquad t = b_1y_1 + b_2y_2 + b_3y_3 \; ,$$

$$1 = b_1 + b_2 + b_3 \; .$$

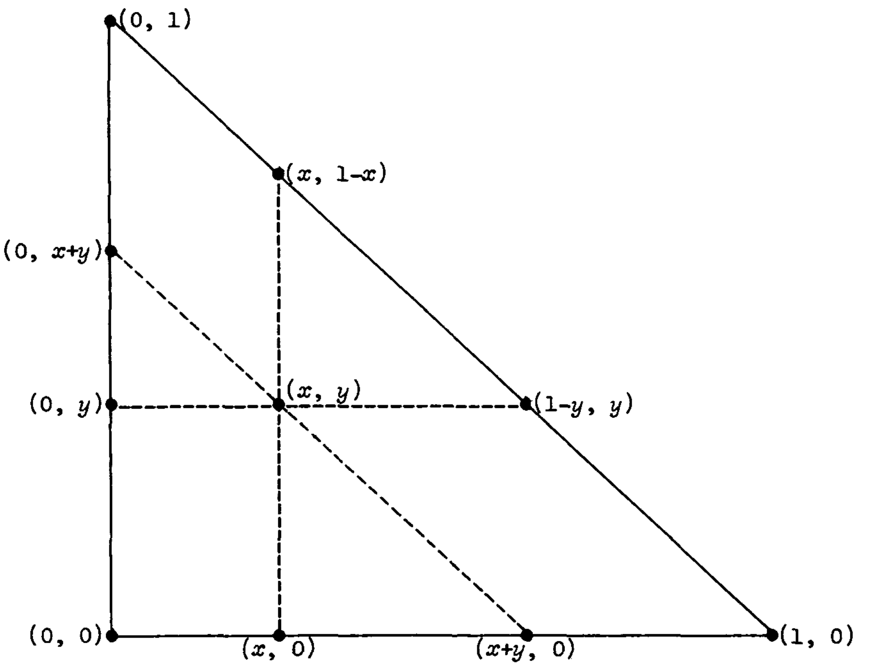

Figure 1

If each of the six affine transformations lead to the same interpolant, then the method is said to be affine invariant. As an example, we apply (1.4) to Q^* and obtain

$$Q^*[F] = \frac{1}{2} \sum_{\substack{i=1 \\ i\neq j\neq k\neq i}}^{3} \left[\frac{b_k F\left(b_i v_i + (1-b_i)v_k\right) + b_j F\left(b_i v_i + (1-b_i)v_j\right)}{1-b_i} - b_i F(v_i)\right] .$$

If an operator P satisfies $P[q] = q$ for

$$q \in \Pi^k = \{f : f(x, y) = x^i y^j ,\ 0 \le i+j \le k\} ,$$

then P is said to have algebraic precision of degree k. For interpolation operators of the form (1.2), it is clear that algebraic precision of degree two is optimal. This is due to the fact that the cubic polynomial $xy(1-x-y)$ is identically zero on ∂T and consequently will get mapped to zero.

2. Interpolation in triangles

It is easily verified that each of the following operators has quadratic precision and defines an affine invariant interpolant over T :

$$(2.1)\quad A^*[f](x, y) = (x+y)^2 f\left(\frac{x}{x+y}, \frac{y}{x+y}\right) + (1-y)^2 f\left(\frac{x}{1-y}, 0\right) + (1-x)^2 f\left(0, \frac{y}{1-x}\right) - x^2 f(1, 0) - y^2 f(0, 1) - (1-x-y)^2 f(0, 0) ;$$

$$(2.2)\quad \hat{B}[f](x, y) = \tfrac{1}{3}\big[(1-x+y)f(0, x+y)+(1+x-y)f(x+y, 0)+(2-2x-y)f(0, y) +(2-x-2y)f(x, 0)+(x+2y)f(x, 1-x)+(2x+y)f(1-y, y) -(1-x-y)f(0, x)-(1-x-y)f(y, 0)-yf(0, 1-x) -xf(1-y, 0)-yf(1-x-y, x+y)-xf(x+y, 1-x-y) -(4xy+1)(1-x-y)f(0, 0)-y\big(4x(1-x-y)+1\big)f(0, 1) -x\big(4y(1-x-y)+1\big)f(1, 0)+4xy(1-x-y)[f(\tfrac{1}{2}, 0)+f(0, \tfrac{1}{2})+f(\tfrac{1}{2}, \tfrac{1}{2})]\big] ;$$

$$(2.3)\quad B^*[f](x, y) = \tfrac{1}{4}[f(0, x+y)(1+2y-x)+f(x+y, 0)(1+2x-y) +f(0, y)(3-3x-2y)+f(x, 0)(3-2x-3y)+f(x, 1-x)(x+3y)+f(1-y, y)(y+3x) -f(0, x)(1-x-y)-f(y, 0)(1-x-y)-f(0, 1-x)y-f(1-y, 0)x -f(1-x-y, x+y)y-f(x+y, 1-x-y)x -2(1-x-y)f(0, 0)-2xf(1, 0)-2yf(0, 1)] ;$$

$$(2.4)\quad \overline{K}[f](x, y) = \tfrac{1}{6}\left[\frac{3-3x-2y}{1-x} f(x, 0) + \frac{3-2x-3y}{1-y} f(0, y) + \frac{1-x+2y}{1-x} f(x, 1-x) + \frac{1+2x-y}{1-y} f(1-y, y) - \frac{y}{1-x} f(0, 1-x) - \frac{x}{1-y} f(1-y, 0) + \frac{x+3y}{x+y} f(0, x+y) + \frac{3x+y}{x+y} f(x+y, 0) - \frac{1-x-y}{1-x} f(0, x) - \frac{1-x-y}{1-y} f(y, 0) - \frac{y}{x+y} f(1-x-y, x+y) - \frac{x}{x+y} f(x+y, 1-x-y) - \frac{(1-x-y)(2-x-y)}{(1-x)(1-y)} f(0, 0) - \frac{y(1-y)}{(1-x)(x+y)} f(0, 1) - \frac{x(1-x)}{(1-y)(x+y)} f(1, 0)\right]$$

$$(2.5)\quad C^*[f](x, y) = \frac{4x(1-x-y)}{(2x+y)(2-2x-y)} f\left(x + \frac{y}{2}, 0\right) + \frac{4y(1-x-y)}{(x+2y)(2-x-2y)} f\left(0, \frac{x}{2} + y\right) + \frac{4xy}{(1+x-y)(1-x+y)} f\left(\frac{1+x-y}{2}, \frac{1-x+y}{2}\right) - \frac{3xy(1-x-y)}{(x+2y)(2x+y)} f(0, 0) - \frac{3xy(1-x-y)}{(1+x-y)(2-x-2y)} f(0, 1) - \frac{3xy(1-x-y)}{(1-x+y)(2-2x-y)} f(1, 0) .$$

The stencils for these five operators are shown in Figures 2, 3, 4, 5, and 6.

We now comment on the derivation of these interpolants.

The operator A^* is the triple boolean sum of the three operators

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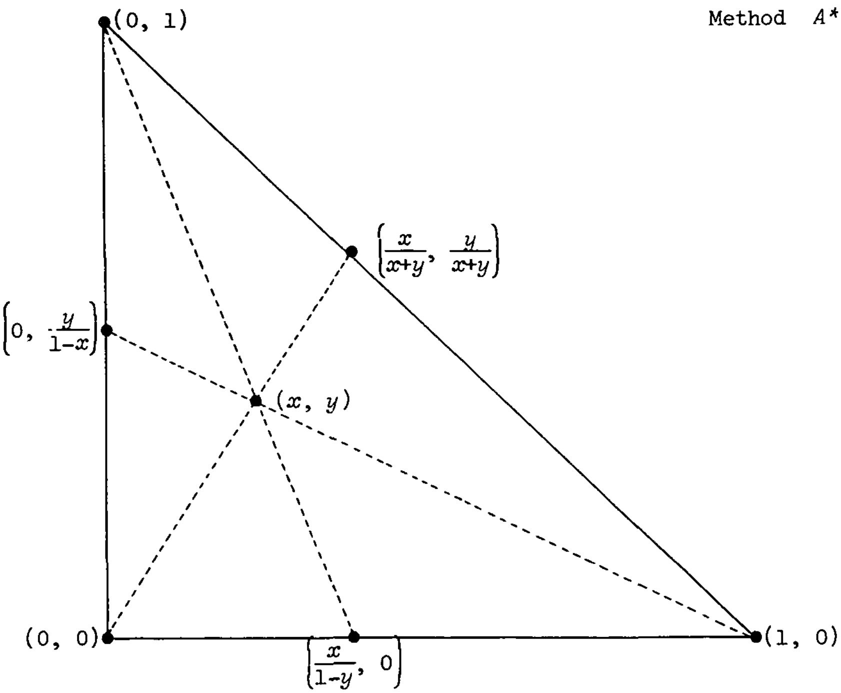

Figure 2

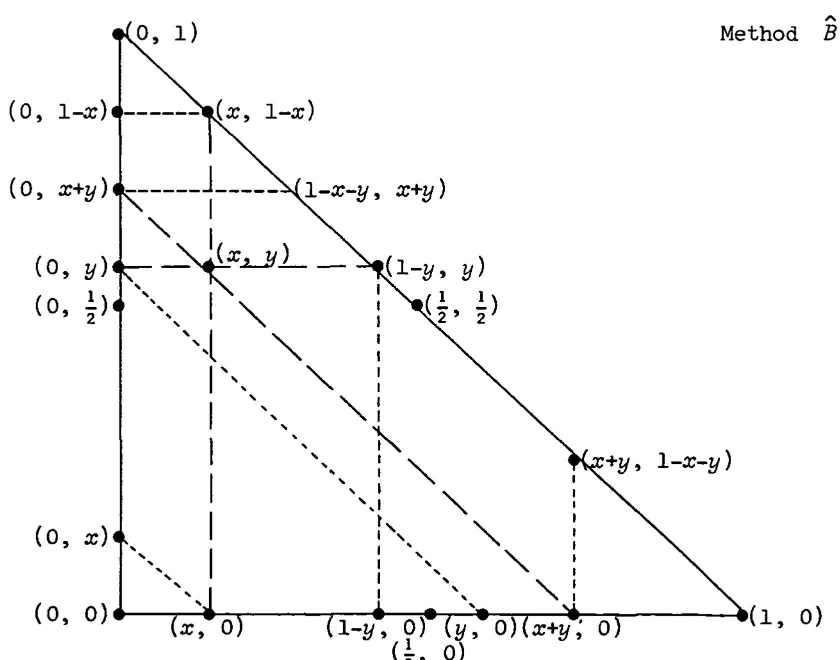

Figure 3

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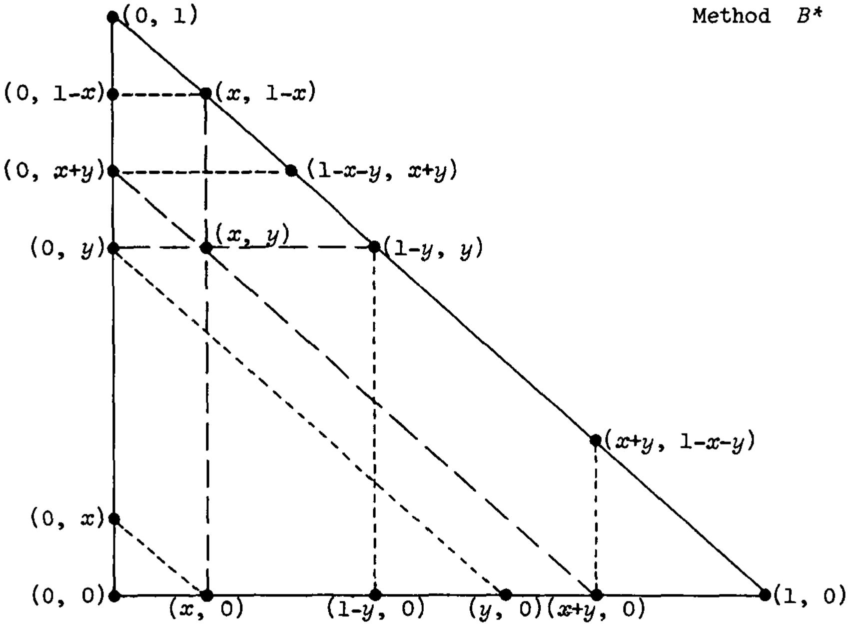

Figure 4

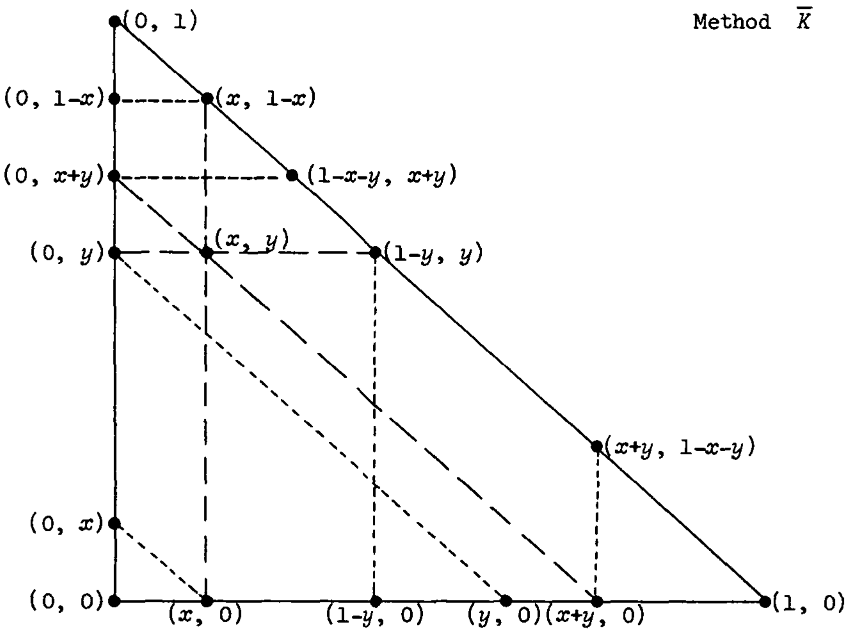

Figure 5

Method C^*

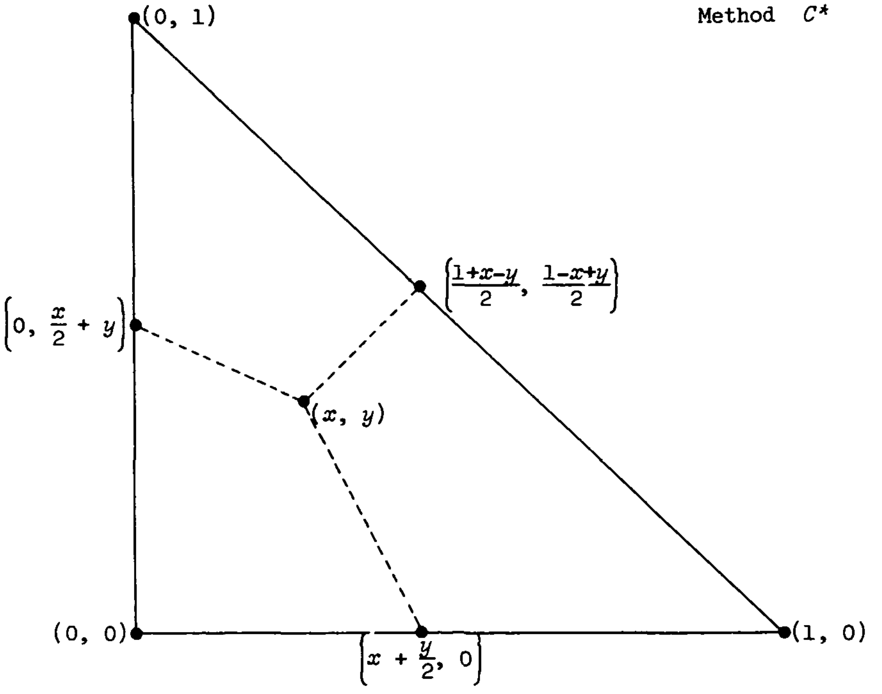

Figure 6

$$
\begin{aligned}
A_3^*[f](x, y) &= (x+y)^2 f\left(\frac{x}{x+y}, \frac{y}{x+y}\right) + (1-x-y)^2 f(0, 0) , \\
A_2^*[f](x, y) &= (1-y)^2 f\left(\frac{x}{1-y}, 0\right) + y^2 f(0, 1) , \qquad (2.6) \\
A_1^*[f](x, y) &= (1-x)^2 f\left(0, \frac{y}{1-x}\right) + x^2 f(1, 0) ;
\end{aligned}
$$

that is,

$$A^* = A_1^* \oplus A_2^* \oplus A_3^* = A_1^* + A_2^* + A_3^* - A_1^*A_2^* - A_1^*A_3^* - A_2^*A_3^* + A_1^*A_2^*A_3^* .$$

A precursor of A^* is the operator

$$A[F] = A_1 \oplus A_2 \oplus A_3 \qquad (2.7)$$

where

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$$A_1[f](x,\ y) = xf(1,\ 0) + (1-x)f\left(0,\ \frac{y}{1-x}\right)\ ,$$

(2.8)
$$A_2[f](x,\ y) = yf(0,\ 1) + (1-y)f\left(\frac{x}{1-y},\ 0\right)\ ,$$

$$A_3[f](x,\ y) = (1-x-y)f(0,\ 0) + (x+y)f\left(\frac{x}{x+y},\ \frac{y}{x+y}\right)\ .$$

These latter operators consist of linear interpolation along lines joining a vertex and its opposing side. The operator A only has linear precision and in general does not have continuous first order derivatives at the vertices of T . Not only does A^* have improved precision over A , it also has a higher degree of continuity.

THEOREM 2.1. *If* $f \in C^1(T)$, *then* $A^*[f] \in C^1(T)$.

Proof. It is clear that the only potential discontinuities of the first order derivatives of $A^*[f]$ are at the vertices. Since the other cases are similar, it is sufficient to consider only $\frac{\partial}{\partial x}$ at the origin:

$$\frac{A^*[f]}{\partial x}(x,\ y) = y\left[\frac{\partial f}{\partial x}\left(\frac{x}{x+y},\ \frac{y}{x+y}\right) - \frac{\partial f}{\partial y}\left(\frac{x}{x+y},\ \frac{y}{x+y}\right) + \frac{\partial f}{\partial y}\left(0,\ \frac{y}{1-x}\right)\right]$$
$$+ 2(x+y)f\left(\frac{x}{x+y},\ \frac{y}{x+y}\right) + (1-y)\frac{\partial f}{\partial x}\left(\frac{x}{1-y},\ 0\right) + 2(1-x)f\left(0,\ \frac{y}{1-x}\right)$$
$$+ 2(1-x-y)f(0,\ 0) - 2xf(1,\ 0)\ .$$

If $x = \alpha t$ and $y = \beta t$, it is clear that the limit as t approaches zero is independent of α and β , and equal to $\frac{\partial f}{\partial x}(0,\ 0)$.

This is the extent of the continuity of $A^*[f]$, regardless of the continuity of f . For example,

$$A^*[x^2y] = \frac{x^2y}{x+y}$$

does not have continuous second order partial derivatives at the vertices of T .

The development of $\hat{B}$ starts with the operator

$$B_1[f] = f(x,\ 0) + f(0,\ y) - f(0,\ 0)$$
$$+ y[f(x,\ 1-x)-f(0,\ 1-x)-f(x,\ 0)+f(0,\ 0)]$$
$$+ x[f(1-y,\ y)-f(0,\ y)-f(1-y,\ 0)+f(0,\ 0)]\ ,$$

which has the property that it uniquely minimizes the quantity

$$\int_T \int \left[\frac{\partial^2 f}{\partial x \partial y}(x, y)\right]^2 dxdy$$

subject to interpolation on the boundary of T. The reason for considering this type of characterization is the analogy to the widely known and utilized interpolant for a rectangular domain

$$\begin{aligned} M[f](x, y) = (1-y)f(x, 0) + yf(x, 1) + (1-x)f(0, y) + xf(1, y) \\ - (1-x)(1-y)f(0, 0) - x(1-y)f(1, 0) - y(1-x)f(0, 1) \\ - xyf(1, 1) , \quad (x, y) \in R = [0, 1] \times [0, 1] . \end{aligned}$$

This interpolant may be characterized as uniquely minimizing $\int_0^1 \int_0^1 \left[\frac{\partial^2 f}{\partial x \partial y}(x, y)\right]^2 dxdy$ subject to interpolation on the boundary of R.

$B_1[f]$ is not affine invariant. Using (1.4) two additional operators which also have minimum pseudonorm properties may be obtained:

$$\begin{aligned} B_2[f](x, y) = f(x+y, 0) + f(1-y, y) - f(1, 0) \\ + y[f(0, x+y)-f(1-x-y, x+y)-f(x+y, 0)+f(1, 0)] \\ + (1-x-y)[f(0, y)-f(1-y, y)-f(y, 0)+f(1, 0)] \end{aligned}$$

and

$$\begin{aligned} B_3[f](x, y) = f(0, x+y) + f(x, 1-x) - f(0, 1) \\ + (1-x-y)[f(x, 0)-f(0, x)-f(x, 1-x)+f(0, 1)] \\ + x[f(x+y, 0)-f(0, x+y)-f(x+y, 1-x-y)+f(0, 1)] . \end{aligned}$$

THEOREM 2.2. *Let* $f \in C^4(T)$. *The approximations* $B_1[f]$, $B_2[f]$, *and* $B_3[f]$ *uniquely minimize the respective quantities*

$$\int_T \int \left[\frac{\partial^2 f}{\partial x \partial y}(x, y)\right]^2 dxdy$$

$$\int_T \int \left[\frac{\partial^2 f}{\partial x^2}(x, y) - \frac{\partial^2 f}{\partial x \partial y}(x, y)\right]^2 dxdy ,$$

$$\int_T \int \left[\frac{\partial^2 f}{\partial y^2}(x, y) - \frac{\partial^2 f}{\partial x \partial y}(x, y)\right]^2 dxdt$$

subject to the interpolation on the boundary of T .

Proof. Due to the affine equivalence of $B_i[f]$, $i = 1, 2, 3$, it is sufficient to consider only the case of $B_1[f]$. Let

$$\langle g, h\rangle = \int_T\int \frac{\partial^2 g}{\partial x\partial y}(x, y)\frac{\partial^2 h}{\partial x\partial y}(x, y)dxdy .$$

Integration by parts will yield

$$\begin{aligned}\langle g, h\rangle = g(0, 0)\frac{\partial^2 h}{\partial x\partial y}(0, 0)\\ &- \tfrac{1}{2}g(0, 1)\frac{\partial^2 h}{\partial x\partial y}(0, 1) - \tfrac{1}{2}g(1, 0)\frac{\partial^2 h}{\partial x\partial y}(1, 0)\\ &+ \int_0^1 g(0, y)\frac{\partial^3 h}{\partial x\partial y^2}(0, y)dy + \int_0^1 g(x, 0)\frac{\partial^3 h}{\partial x^2\partial y}(x, 0)dx\\ &- \int_0^1 g(x, 1-x)\tfrac{1}{2}\left[\frac{\partial^3 h}{\partial x^2\partial y}(x, 1-x) + \frac{\partial^3 h}{\partial x\partial y^2}(x, 1-x)\right]dx\\ &+ \int_0^1 \left[\frac{\partial g}{\partial x}(x, 1-x) + \frac{\partial g}{\partial y}(x, 1-x)\right]\tfrac{1}{2}\frac{\partial^2 h}{\partial x\partial y}(x, 1-x)dx\\ &+ \int_T\int g(x, y)\frac{\partial^4 h}{\partial x^2\partial y^2}(x, y)dxdt .\end{aligned}$$

Since

$$\frac{\partial^2 B_1[f]}{\partial x\partial y}(x, 1-x) = 0 , \quad 0 \le x \le 1 ,$$

and

$$\frac{\partial^4 B_1[f]}{\partial x^2\partial y^2}(x, y) = 0 , \quad (x, y) \in T ,$$

it must be the case that

$$\langle q-B_1[f], B_1[f]\rangle = 0$$

for any function q which interpolates to f on T . Therefore,

$$\begin{aligned}\langle q, q\rangle - \langle B_1[f], B_1[f]\rangle &= \langle q-B_1[f], q-B_1[f]\rangle + 2\langle q-B_1[f], B_1[f]\rangle\\ &= \langle q-B_1[f], q-B_1[f]\rangle \ge 0 ,\end{aligned}$$

which establishes the minimum property of $B_1[f]$. If Q is another minimizing interpolant, it must be the case that $\langle Q-B_1[f], Q-B_1[f]\rangle = 0$, which implies that

$$\frac{\partial^2[Q-B_1[f]]}{\partial x\partial y}(x, y) = 0 , \quad (x, y) \in T .$$

Any $h \in C^4(T)$ has the expansion

$$h(x, y) = h(x, 0) + h(0, y) - h(0, 0) + \int_0^y \int_0^x \frac{\partial^2 h}{\partial x\partial y}(s, t)dsdt ,$$

and so

$$\begin{aligned} Q(x, y) - B_1[f](x, y) &= Q(x, 0) - B_1[f](x, 0) + Q(0, y) - B_1[f](0, y) \\ &\qquad - Q(0, 0) - B_1[f](0, 0) \\ &= 0 , \quad (x, y) \in T . \end{aligned}$$

Therefore $Q = B_1[f]$ and the argument is complete.

The average of these three affine equivalent interpolants will form the affine invariant interpolant

$$\bar{B}[f] = \frac{B_1[f]+B_2[f]+B_3[f]}{3} .$$

The operator $\bar{B}$ only has linear precision. We now proceed to improve upon the precision of $\bar{B}$. Let

$$\begin{aligned} Q[f](x, y) = f(0, 0) &+ x[4f(\tfrac{1}{2}, 0)-3f(0, 0)-f(1, 0)] \\ &+ y[4f(0, \tfrac{1}{2})-3f(0, 0)-f(0, 1)] + x^2[2f(0, 0)+2f(1, 0)-4f(\tfrac{1}{2}, 0)] \\ &+ xy[4f(\tfrac{1}{2}, \tfrac{1}{2})+4f(0, 0)-4f(0, \tfrac{1}{2})-4f(\tfrac{1}{2}, 0)] \\ &+ y^2[2f(0, 0)+2f(0, 1)-4f(0, \tfrac{1}{2})] . \end{aligned}$$

This operator represents the unique quadratic polynomial which interpolates to f at the three vertices and midpoints of T. Therefore it is clear that

$$\begin{aligned} B[f](x, y) &= \bar{B}[f-Q[f]](x, y) + Q[f](x, y) \\ &= \bar{B}[f](x, y) + \tfrac{4}{3}xy(1-x-y) \\ &\qquad \times [f(\tfrac{1}{2}, \tfrac{1}{2})+f(\tfrac{1}{2}, 0)+f(0, \tfrac{1}{2})-f(0, 0)-f(0, 1)-f(1, 0)] \end{aligned}$$

represents an interpolant with optimal algebraic precision.

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The operator B^* is developed by considering an interpolant with the same stencil as $\overline{B}$ and linear weights. The weights are then chosen so that the operator will be affine invariant and have quadratic precision.

The operator $\overline{K}$ is the average of six affine equivalent operators:

$$K_1[f](x, y) = f(x, 0) + f(0, y) - f(0, 0) + \frac{y}{1-x}[f(x, 1-x)-f(0, 1-x)-f(0, y)+f(0, 0)] ,$$

$$K_2[f](x, y) = f(x, 0) + f(0, y) - f(0, 0) + \frac{x}{1-y}[f(1-y, y)-f(1-y, 0)-f(0, y)+f(0, 0)] ,$$

$$K_3[f](x, y) = f(0, x+y) + f(x, 1-x) - f(0, 1) + \frac{(1-x-y)}{1-x}[f(x, 0)-f(0, x)-f(x, 1-x)+f(0, 1)] ,$$

$$K_4[f](x, y) = f(1-y, y) + f(x+y, 0) - f(1, 0) + \frac{y}{x+y}[f(0, x+y)-f(1-x-y, x+y)-f(x+y, 0)+f(1, 0)] ,$$

$$K_5[f](x, y) = f(x+y, 0) + f(1-y, y) - f(1, 0) + \frac{(1-x-y)}{1-y}[f(0, y)-f(y, 0)-f(1-y, y)+f(1, 0)] ,$$

and

$$K_6[f](x, y) = f(x, 1-x) + f(0, x+y) - f(0, 1) + \frac{x}{x+y}[f(x+y, 0)-f(x+y, 1-x-y)-f(0, x+y)+f(0, 1)] .$$

Each of interpolants $K_i[f]$, $i = 1, 2, 3, 4, 5, 6$, may be characterized as the unique interpolant in the kernel of a third order partial differential operator. These differential operators are respectively: $D_x(D_y)^2$, $(D_x)^2D_y$, $(D_y)^2(D_x-D_y)$, $(D_x)(D_x-D_y)^2$, $(D_x)^2(D_x-D_y)$, $(D_y)(D_x-D_y)^2$. This type of characterization is analogous to the characterization of $M[f]$ which also states that it is the unique interpolant in the kernel of $(D_x)^2(D_y)^2$. This attribute of $M[f]$ is discussed in [10].

Because of the affine equivalence of $K_i[f]$, $i = 1, 2, 3, 4, 5, 6$, it is sufficient to prove the following.

THEOREM 2.3. *If* $f \in C^3(T)$ *, then among all functions in* $C^3(T)$ *annihilated by* $D_x(D_y)^2$ *there exists a unique function* $K_1[f]$ *which interpolates to* f *on* ∂T *.*

Proof. It is a simple matter to verify that $D_x(D_y)^2$ annihilates $K_1[f]$. Therefore the existence portion of the argument is obvious once it is noted that the apparent singularities are removable and that indeed $K_1[f] \in C^3(T)$. In order to establish the uniqueness, assume the existence of another interpolant, say g , and consider the difference $E = K_1[f] - g$. Since $h \in C^3(T)$ admits the expansion

$$h(x, y) = h(x, 0) + h(0, y) - h(0, 0) + y\left[\frac{\partial h}{\partial y}(x, 0) - \frac{\partial h}{\partial y}(0, 0)\right] + \int_0^y \int_0^x (y-t) \frac{\partial^3 h}{\partial x \partial y^2}(s, t)\,ds\,dt$$

it must be the case that

$$E(x, y) = E(x, 0) + E(0, y) - E(0, 0) + y\left[\frac{\partial E}{\partial y}(x, 0) - \frac{\partial E}{\partial y}(0, 0)\right] .$$

This, along with the fact that E is identically zero on ∂T implies that $E = 0$ and so the argument is complete.

The operator C^* is defined by

$$C^*[f](x, y) = P(x, y)$$

where

$$P(u, v) = a_0 + a_1 u + a_2 v + a_3 u^2 + a_4 uv + a_5 v^2$$

is the unique quadratic with the interpolatory properties

$$P(0, 0) = f(1, 0) ,$$

$$P(0, 1) = f(0, 1) ,$$

$$P(1, 0) = f(1, 0) ,$$

$$P\left(x + \frac{y}{2}, 0\right) = f\left(x + \frac{y}{2}, 0\right) ,$$

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$$P\left(0,\ \frac{x}{2}+y\right) = f\left(0,\ \frac{x}{2}+y\right) ,$$

$$P\left(\frac{1+x-y}{2},\ \frac{1-x+y}{2}\right) = f\left(\frac{1+x-y}{2},\ \frac{1-x+y}{2}\right) .$$

The stencil for C^* was chosen on the basis that it is invariant under affine transformations and has linear arguments.

Table 1 contains a summary of the type of weights and stencil for each method.

Table 1

Method	Number of Stencil points	Stencil	Weights
Q^*	9	polynomial	rational
A^*	6	rational	polynomial
$\hat{B}$	18	polynomial	polynomial
B^*	15	polynomial	polynomial
$\overline{K}$	15	polynomial	rational
C^*	6	polynomial	rational

In Table 2 we have given the coefficients for the nine parameter discrete interpolants which result from each of these methods. These interpolants are obtained by using cubic Hermite interpolation along each edge of Δ . For this table, we assume $i \neq j \neq k \neq i$. The barycentric coordinates b_i , $i = 1, 2, 3$, are defined by (1.5) and

$$\frac{\partial f}{\partial e_j}(v_i) = (x_j - x_i)\frac{\partial f}{\partial x}(x_i, y_i) + (y_j - y_i)\frac{\partial f}{\partial y}(x_i, y_i) , \quad i \neq j .$$

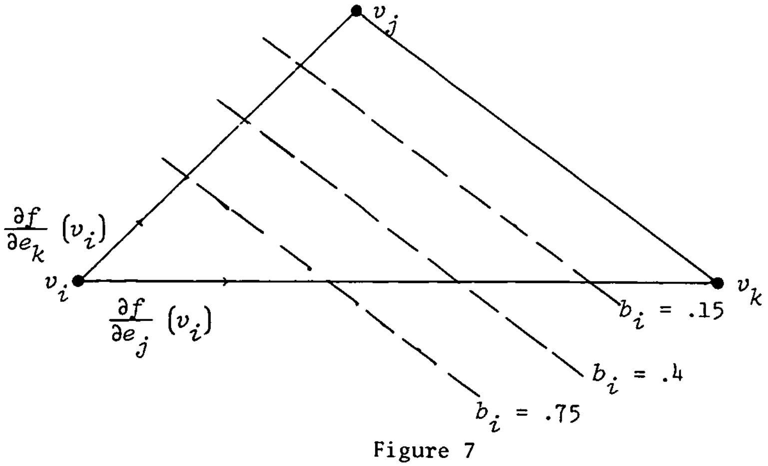

Figure 7

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Table 2

	$f(v_i)$	$\frac{\partial f}{\partial e_j}(v_i)$
Q^*	$b_i^2(3-2b_i) + 2b_ib_jb_k$	$b_ib_k\left(b_i + \frac{b_j}{2}\right)$
A^*	$b_i^2\left[\frac{3b_k+b_i}{b_i+b_k} + \frac{3b_j+b_i}{b_i+b_j} - 1\right]$	$\frac{b_i^2b_k}{b_i+b_k}$
$\hat{B}$	$b_i^2(3-2b_i) + 2b_ib_jb_k$	$b_ib_k\left(b_i + \frac{b_j}{2}\right)$
B^*	$b_i^2(3-2b_i) + \frac{3}{2}b_ib_jb_k(b_i+1)$	$b_ib_k\left(b_i+b_j\left(\frac{5}{2}+b_i-b_k\right)\right)$
$\overline{K}$	$b_i^2(3-2b_i) + 2b_ib_jb_k$	$b_ib_k\left(b_i + \frac{b_j}{2}\right)$
C^*	$b_i^2(3-2b_i) + 2b_ib_jb_k$	$b_ib_k\left(b_i + \frac{b_j}{2}\right)$

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