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INTERSECTION DENSITIES OF NONSTATIONARY POISSON PROCESSES OF HYPERSURFACES

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Abstract

Intersection densities are introduced for a large class of nonstationary Poisson processes of hypersurfaces and inequalities for them are proved. In doing so, similar results from both Wieacker (1986) and Schneider (2003) are summarized in one theorem and the concept of an associated zonoid of a Poisson process of hypersurfaces is generalized to a nonstationary setting.

Keywords: Poisson process; intersection process; hypersurface; associated zonoid; Steiner convex set

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1. Introduction

Starting with Matheron's introduction of his Steiner convex set in [5], the idea of using an associated convex body to analyse a process of geometric objects has proven itself to be a very fruitful concept. To get an impression of the variety of problems to which this technique has been applied see, for example, the introductions of [11] and [7] or [8, Section 4.5] and the references given therein.

While the case of stationary processes has been treated by Wieacker in a very general way in [9], [10], and [11], some of the results for hyperplane processes have been generalized by Schneider to a nonstationary setting in [7]. In the present paper, we undertake the humble effort to extend a few of these findings.

After introducing some basic notation in Section 2, we define, in Section 3, intersection densities and associated zonoids for certain processes of hypersurfaces, i.e. cylinders with an $(\mathcal{H}^k, k)$-rectifiable set as a basis. Then a connection between intrinsic volumes of associated zonoids and intersection densities is derived. We conclude Section 3 by exploiting this relationship to prove inequalities for intersection densities. The final section is devoted to intersections of these processes with affine subspaces.

We also want to mention that questions regarding measurability and the proofs of some auxiliary results have been moved to the appendix to make this paper more readable.

2. Preliminaries and basic notation

Throughout this paper we will work in d-dimensional Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$, equipped with the canonical structures, S^{d-1} being its unit sphere, $\mathcal{B}(\mathbb{R}^d)$ the Borel σ-algebra, and λ_d the

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d-dimensional Lebesgue measure. More generally, for a topological space S, we will always denote the Borel σ-algebra by $\mathcal{B}(S)$.

Let $k \in \{0, \dots, d\}$. The k-dimensional Hausdorff measure will be denoted by $\mathcal{H}^k$ and the k-dimensional Lebesgue measure on a k-dimensional affine subspace E of $\mathbb{R}^d$ by λ_E. Note that any Borel subset of $\mathbb{R}^d$ is $\mathcal{H}^k$-measurable.

A subset M of $\mathbb{R}^d$ is called k-rectifiable if and only if either $k = 0$ and M is finite or if $k \geq 1$ and there exists a Lipschitz map of some bounded subset of $\mathbb{R}^k$ onto M. Here, M is called $(\mathcal{H}^k, k)$-rectifiable if and only if $\mathcal{H}^k(M) < \infty$ and $\mathcal{H}^k$-almost all of M can be covered by some countable family of k-rectifiable sets. Finally, M is called $\mathcal{H}^k$-rectifiable if $M \cap C$ is $(\mathcal{H}^k, k)$-rectifiable for all compact sets $C \subseteq \mathbb{R}^d$.

In the following, the tangential properties of $(\mathcal{H}^k, k)$-rectifiable sets play an important role. Let M be a $(\mathcal{H}^k, k)$-rectifiable subset of $\mathbb{R}^d$ with $0 < k < d$ and let $\text{Nor}^k(M, x)$ denote the cone of all approximate normal vectors of M at x. By Theorem 3.2.19 of [2], the latter is a $(\mathrm{d} - k)$-dimensional linear subspace of $\mathbb{R}^d$ for $\mathcal{H}^k$-almost all $x \in M$. For basic notions from geometric measure theory, we refer the reader to [2].

Let $\mathcal{L}_k^d$ be the Grassmannian of all k-dimensional linear subspaces of $\mathbb{R}^d$. For linear subspaces $L_1, \dots, L_k$ of $\mathbb{R}^d$ with

$$\dim L_1 + \cdots + \dim L_k = m \leq d,$$

we choose an orthonormal basis in each space L_i (the empty set if $\dim L_i = 0$) and define the determinant $[L_1, \dots, L_k]$ to be the m-dimensional volume of the parallelepiped spanned by these m vectors. For a linear subspace L, its orthogonal complement will be denoted by $L^\perp$ and the orthogonal projection onto L by p_L.

The space $\mathcal{F}'$ of all nonempty closed subsets of $\mathbb{R}^d$ will be endowed with the Fell topology and the σ-algebra $\mathcal{B}(\mathcal{F}')$. The subspace $\mathcal{K}' \subseteq \mathcal{F}'$ of all nonempty, compact convex sets (convex bodies) will be equipped with the Hausdorff metric and the σ-algebra induced by $\mathcal{B}(\mathcal{F}')$. For all basic notions from convex geometry, we refer the reader to [6]. Finally, let $\mathcal{F}^{(k)} \subseteq \mathcal{F}'$ be the subspace of all nonempty, closed $(\mathcal{H}^k, k)$-rectifiable subsets of $\mathbb{R}^d$. All required concepts from stochastic geometry (e.g. point processes, intensity measures, or Campbell's theorem) can be found in [8].

3. Intersection densities and associated zonoids

Let $l \in \{1, \dots, d\}$. Throughout this paper, X_l will always be a point process on $\mathcal{F}'$ with locally finite nontrivial intensity measure Θ of the form

$$\Theta(\mathcal{A}) = \int_{\mathcal{L}_{d-l}^d} \int_{\mathcal{F}^{(l-1)}} \int_{L^\perp} \mathbf{1}_{\{\mathcal{A}\}}(M + L + x) f(M + L, x) \lambda_{L^\perp}(\mathrm{d}x)\, \mathrm{P}(L, \mathrm{d}M) \Phi(\mathrm{d}L), \quad (3.1)$$

where $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function and $\mathcal{A} \in \mathcal{B}(\mathcal{F}')$. Here,

$$f \colon \mathcal{F}' \times \mathbb{R}^d \to [0, \infty)$$

denotes a measurable mapping with the additional property that $f(F, \cdot)$ is locally integrable for all $F \in \mathcal{F}'$; for $L \in \mathcal{L}_{d-l}^d$, $\mathrm{P}(L, \cdot)$ is a probability measure such that

$$\mathrm{P}(L, \{M \in \mathcal{F}^{(l-1)} \mid M \subseteq L^\perp\}) = 1$$

and $L \mapsto \mathrm{P}(L, \mathcal{A})$ is measurable for $\mathcal{A} \in \mathcal{B}(\mathcal{F}')$; Φ denotes a finite measure on $\mathcal{L}^d_{d-l}$. Here, X_l can be considered as a process of cylinders, where Φ describes the distribution of the direction spaces and $\mathrm{P}(L, \cdot)$ the distribution of the basis. Note that f, P, and Φ are not uniquely determined by Θ.

Let $m \in \{1, \dots, d\}$, $n_0, \dots, n_m \in \{1, \dots, d-1\}$, and $M_0, \dots, M_m \subseteq \mathbb{R}^d$. Furthermore, define

$$n := n_0 + \cdots + n_m.$$

We say the pairs $(M_0, n_0), \dots, (M_m, n_m)$ satisfy condition (I) if and only if M_i is $\mathcal{H}^{n_i}$-measurable and $(\mathcal{H}^{n_i}, n_i)$-rectifiable for $i \in \{1, \dots, m\}$ and $M_0 \times \cdots \times M_m$ is $\mathcal{H}^n$-measurable and $(\mathcal{H}^n, n)$-rectifiable. By Theorem 4.2 of [4], this is equivalent to the definition of condition (I) in [9]. Examples of pairs of sets satisfying condition (I) can be found in [9, p. 238].

For $i \in \{0, \dots, m\}$, let μ_i be a measure on $\mathcal{F}^{(n_i)}$. We say the pairs $(\mu_0, n_0), \dots, (\mu_m, n_m)$ satisfy condition (I) if and only if the pairs $(M_0, n_0), \dots, (M_m, n_m)$ satisfy condition (I) for $\mu_0 \otimes \cdots \otimes \mu_m$-almost all $(M_0, \dots, M_m) \in \mathcal{F}^{(n_0)} \times \cdots \times \mathcal{F}^{(n_m)}$.

Before introducing intersection densities for X_l, let us prove the following lemma.

Lemma 3.1. *Let $j, l \in \{1, \dots, d\}$, $L_1, \dots, L_j \in \mathcal{L}^d_{d-l}$, and $M_1, \dots, M_j \in \mathcal{F}^{(l-1)}$ such that*

$$M_1 \subseteq L_1^\perp, \dots, M_j \subseteq L_j^\perp \quad \text{and} \quad (M_1, l-1), \dots, (M_j, l-1)$$

satisfy condition (I). Furthermore, let

$$f_i : L_i^\perp \to [0, \infty)$$

be a measurable function for $i \in \{1, \dots, j\}$ and $B \in \mathcal{B}(\mathbb{R}^d)$ a bounded Borel set. Then

$$\begin{aligned}
\int_{L_j^\perp} \cdots \int_{L_1^\perp} & \mathcal{H}^{d-j}(B \cap (M_1 + L_1 + x_1) \cap \cdots \cap (M_j + L_j + x_j)) f_1(x_1) \cdots f_j(x_j) \\
& \times \lambda_{L_1^\perp}(\mathrm{d}x_1) \cdots \lambda_{L_j^\perp}(\mathrm{d}x_j) \\
= \int_{\mathbb{R}^d} \mathbf{1}_{\{B\}}(z) \int_{M_1} \cdots \int_{M_j} & f_1((z - p_{L_1}(z)) - x_1) \cdots f_j((z - p_{L_j}(z)) - x_j) \\
& \times [\mathrm{Nor}^{d-1}(M_1 + L_1, x_1), \dots, \mathrm{Nor}^{d-1}(M_j + L_j, x_j)] \\
& \times \mathcal{H}^{l-1}(\mathrm{d}x_j) \cdots \mathcal{H}^{l-1}(\mathrm{d}x_1) \lambda_d(\mathrm{d}z).
\end{aligned}$$

Proof. Since B is bounded, there exist cubes

$$W_1, \dots, W_j \in \mathcal{K}'$$

such that $W_i \subseteq L_i$ and

$$B \cap (M_i + L_i + x_i) = B \cap (M_i + \operatorname{int} W_i + x_i)$$

for all $x_i \in L_i^\perp$ and $i \in \{1, \dots, j\}$, where $\operatorname{int} W$ denotes the interior of W.

By Fubini's theorem, the following holds:

$$\begin{aligned}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} &\mathcal{H}^{d-j}(B\cap(M_1+W_1+x_1)\cap\cdots\cap(M_j+W_j+x_j))\\ &\times f_1(x_1-p_{L_1}(x_1))\cdots f_j(x_j-p_{L_j}(x_j))\lambda_d(\mathrm{d}x_1)\cdots\lambda_d(\mathrm{d}x_j)\\ =\ &\lambda_{L_1}(W_1)\cdots\lambda_{L_j}(W_j)\\ &\times\int_{L_j^\perp}\cdots\int_{L_1^\perp}\mathcal{H}^{d-j}(B\cap(M_1+L_1+x_1)\cap\cdots\cap(M_j+L_j+x_j))\\ &\qquad\times f_1(x_1)\cdots f_j(x_j)\lambda_{L_1^\perp}(\mathrm{d}x_1)\cdots\lambda_{L_j^\perp}(\mathrm{d}x_j).\end{aligned}$$

On the other hand, since $(M_1+W_1,d-1),\ldots,(M_j+W_j,d-1)$ satisfy the assumptions of Theorem A.1 in the appendix, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} &\mathcal{H}^{d-j}(B\cap(M_1+W_1+x_1)\cap\cdots\cap(M_j+W_j+x_j))\\ &\times f_1(x_1-p_{L_1}(x_1))\cdots f_j(x_j-p_{L_j}(x_j))\lambda_d(\mathrm{d}x_1)\cdots\lambda_d(\mathrm{d}x_j)\\ =\ &\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\mathcal{H}^{d-j}(B-x_1\cap(M_1+W_1)\cap(M_2+W_2+x_2)\cap\cdots\cap(M_j+W_j+x_j))\\ &\times f_1(x_1-p_{L_1}(x_1))f_2(x_2+x_1-p_{L_2}(x_2+x_1))\\ &\times\cdots f_j(x_j+x_1-p_{L_j}(x_j+x_1))\lambda_d(\mathrm{d}x_1)\cdots\lambda_d(\mathrm{d}x_j)\\ =\ &\int_{\mathbb{R}^d}\int_{M_1+W_1}\cdots\int_{M_j+W_j}\mathbf{1}_{\{B-x_1\}}(t_1)f_2(t_1-t_2+x_1-p_{L_2}(t_1-t_2+x_1))\\ &\qquad\times\cdots f_j(t_1-t_j+x_1-p_{L_j}(t_1-t_j+x_1))\\ &\qquad\times[\mathrm{Nor}^{d-1}(M_1+W_1,t_1),\ldots,\mathrm{Nor}^{d-1}(M_j+W_j,t_j)]\\ &\qquad\times\mathcal{H}^{d-1}(\mathrm{d}t_j)\cdots\mathcal{H}^{d-1}(\mathrm{d}t_1)f_1(x_1-p_{L_1}(x_1))\lambda_d(\mathrm{d}x_1)\\ =\ &\lambda_{L_1}(W_1)\cdots\lambda_{L_j}(W_j)\\ &\times\int_{\mathbb{R}^d}\int_{M_1}\cdots\int_{M_j}\mathbf{1}_{\{B-x_1\}}(t_1)f_2(t_1-t_2+x_1-p_{L_2}(t_1-t_2+x_1))\\ &\qquad\times\cdots f_j(t_1-t_j+x_1-p_{L_j}(t_1-t_j+x_1))\\ &\qquad\times[\mathrm{Nor}^{d-1}(M_1+L_1,t_1),\ldots,\mathrm{Nor}^{d-1}(M_j+L_j,t_j)]\\ &\qquad\times\mathcal{H}^{l-1}(\mathrm{d}t_j)\cdots\mathcal{H}^{l-1}(\mathrm{d}t_1)f_1(x_1-p_{L_1}(x_1))\lambda_d(\mathrm{d}x_1),\end{aligned}$$

where the last identity follows from Theorem 3.2.23 of [2]. Combined with another change of variable, this yields the assertion.

Let $j\in\{1,\ldots,d\}$. We introduce a Borel measure ν_j on $\mathbb{R}^d$ by

$$\nu_j(B):=\mathrm{E}\sum_{(M_1+L_1,\ldots,M_j+L_j)\in(X_l)^j_{\neq}}\mathcal{H}^{d-j}(B\cap(M_1+L_1)\cap\cdots\cap(M_j+L_j)),$$

$B\in\mathcal{B}(\mathbb{R}^d)$. Here, $(X_l)^j_{\neq}$ denotes the process of all j-tupels of j different particles of X_l. The following result holds.

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Theorem 3.1. *Let $j, l \in \{1, \dots, d\}$ and let X_l be a Poisson point process with intensity measure as in (3.1) such that the pairs $(\mathrm{P}(L_1, \cdot), l-1), \dots, (\mathrm{P}(L_j, \cdot), l-1)$ satisfy condition (I) for $\Phi \otimes \cdots \otimes \Phi$-almost all $(L_1, \dots, L_j) \in \mathcal{L}^d_{d-l} \times \cdots \times \mathcal{L}^d_{d-l}$. Then the measure ν_j is absolutely continuous with respect to λ_d with the density given by*

$$\begin{aligned}\gamma_j(z) := \int_{\mathcal{L}^d_{d-l}} \cdots \int_{\mathcal{L}^d_{d-l}} \int_{\mathcal{F}^{(l-1)}} \cdots \int_{\mathcal{F}^{(l-1)}} \\ \times \int_{M_1} \cdots \int_{M_j} [\mathrm{Nor}^{d-1}(M_1 + L_1, x_1), \dots, \mathrm{Nor}^{d-1}(M_j + L_j, x_j)] \\ \times f(M_1 + L_1, (z - p_{L_1}(z)) - x_1) \\ \times \cdots f(M_j + L_j, (z - p_{L_j}(z)) - x_j) \\ \times \mathcal{H}^{l-1}(\mathrm{d}x_j) \cdots \mathcal{H}^{l-1}(\mathrm{d}x_1)\, \mathrm{P}(L_1, \mathrm{d}M_1) \cdots \mathrm{P}(L_j, \mathrm{d}M_j) \\ \times \Phi(\mathrm{d}L_1) \cdots \Phi(\mathrm{d}L_j), \qquad z \in \mathbb{R}^d.\end{aligned}$$

For $j \in 1, \dots, d$, we call γ_j the *jth intersection density of X_l*. Sometimes, γ_1 is called the surface area density of X_l.

Proof of Theorem 3.1. By Lemma A.1, Lemma A.2, and Campbell's and Fubini's theorems, and since X_l is Poisson, we have

$$\begin{aligned}\nu_j(B) = \int_{\mathcal{L}^d_{d-l}} \cdots \int_{\mathcal{L}^d_{d-l}} \int_{\mathcal{F}^{(l-1)}} \cdots \int_{\mathcal{F}^{(l-1)}} \\ \times \int_{L_j^\perp} \cdots \int_{L_1^\perp} \mathcal{H}^{d-j}(B \cap (M_1 + L_1 + x_1) \cap \cdots \cap (M_j + L_j + x_j)) \\ \times f(M_1 + L_1, x_1) \cdots f(M_j + L_j, x_j) \lambda_{L_1^\perp}(\mathrm{d}x_1) \cdots \lambda_{L_j^\perp}(\mathrm{d}x_j) \\ \times \mathrm{P}(L_1, \mathrm{d}M_1) \cdots \mathrm{P}(L_j, \mathrm{d}M_j) \Phi(\mathrm{d}L_1) \cdots \Phi(\mathrm{d}L_j).\end{aligned}$$

First, we assume that $B \in \mathcal{B}(\mathbb{R}^d)$ is bounded. Then, the application of Lemma 3.1 and Fubini's theorem yields

$$\begin{aligned}\nu_j(B) = \int_{\mathbb{R}^d} \mathbf{1}_{\{B\}}(z) \int_{\mathcal{L}^d_{d-l}} \cdots \int_{\mathcal{L}^d_{d-l}} \int_{\mathcal{F}^{(l-1)}} \cdots \int_{\mathcal{F}^{(l-1)}} \\ \times \int_{M_1} \cdots \int_{M_j} [\mathrm{Nor}^{d-1}(M_1 + L_1, x_1), \dots, \mathrm{Nor}^{d-1}(M_j + L_j, x_j)] \\ \times f(M_1 + L_1, (z - p_{L_1}(z)) - x_1) \\ \times \cdots f(M_j + L_j, (z - p_{L_j}(z)) - x_j) \mathcal{H}^{l-1}(\mathrm{d}x_j) \cdots \mathcal{H}^{l-1}(\mathrm{d}x_1) \\ \times \mathrm{P}(L_1, \mathrm{d}M_1) \cdots \mathrm{P}(L_j, \mathrm{d}M_j) \Phi(\mathrm{d}L_1) \cdots \Phi(\mathrm{d}L_j) \lambda_d(\mathrm{d}z).\end{aligned}$$

By monotone convergence, this is true for arbitrary $B \in \mathcal{B}(\mathbb{R}^d)$.

To introduce a class of zonoids associated with X_l, we define a Borel measure μ_z on S^{d-1} by

$$\begin{aligned}\mu_z(A) := \int_{\mathcal{L}^d_{d-l}} \int_{\mathcal{F}^{(l-1)}} \int_M \mathcal{H}^0(S^{d-1} \cap \mathrm{Nor}^{d-1}(M + L, x) \cap A) \\ \times f(M + L, (z - p_L(z)) - x) \\ \times \mathcal{H}^{l-1}(\mathrm{d}x)\, \mathrm{P}(L, \mathrm{d}M) \Phi(\mathrm{d}L), \qquad A \in \mathcal{B}(S^{d-1}).\end{aligned}$$

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Let $A \in \mathcal{B}(S^{d-1})$ and $B \in \mathcal{B}(\mathbb{R}^d)$. Then

$$\begin{aligned}
&\mathrm{E} \sum_{(M+L)\in X_l} \int_{M+L} \mathbf{1}_{\{B\}}(x)\mathcal{H}^0(S^{d-1} \cap \mathrm{Nor}^{d-1}(M+L, x) \cap A)\mathcal{H}^{d-1}(\mathrm{d}x) \\
&\quad = \int_{\mathcal{L}^d_{d-l}} \int_{\mathcal{F}^{(l-1)}} \int_{L^\perp} \\
&\quad\quad \times \int_{M+L+z} \mathbf{1}_{\{B\}}(x)\mathcal{H}^0(S^{d-1} \cap \mathrm{Nor}^{d-1}(M+L+z, x) \cap A)\mathcal{H}^{d-1}(\mathrm{d}x) \\
&\quad\quad\quad\quad \times f(M+L, z)\lambda_{L^\perp}(\mathrm{d}z)\, \mathrm{P}(L, \mathrm{d}M)\Phi(\mathrm{d}L) \\
&\quad = \int_{\mathbb{R}^d} \mathbf{1}_{\{B\}}(z) \int_{\mathcal{L}^d_{d-l}} \int_{\mathcal{F}^{(l-1)}} \\
&\quad\quad \times \int_M \mathcal{H}^0(S^{d-1} \cap \mathrm{Nor}^{d-1}(M+L, x) \cap A) \\
&\quad\quad\quad \times f(M+L, (z - p_L(z)) - x)\mathcal{H}^{l-1}(\mathrm{d}x)\, \mathrm{P}(L, \mathrm{d}M)\Phi(\mathrm{d}L)\lambda_d(\mathrm{d}z) \\
&\quad = \int_{\mathbb{R}^d} \mathbf{1}_{\{B\}}(z)\mu_z(A)\lambda_d(\mathrm{d}z).
\end{aligned}$$

Hence, for λ_d-almost all $z \in \mathbb{R}^d$, the measure μ_z is uniquely determined by Θ (and, thus, does not depend on the choice of f, $\mathrm{P}(L, \cdot)$, and Φ). Under the additional assumption that

$$B \mapsto \mathrm{E} \sum_{(M+L)\in X_l} \int_{M+L} \mathbf{1}_{\{B\}}(x)\mathcal{H}^{d-1}(\mathrm{d}x)$$

is a locally finite Borel measure on $\mathbb{R}^d$, a normalization of μ_z could be interpreted as the distribution of the normals of the particles of X_l in z whenever $\mu_z(S^{d-1}) > 0$. We call μ_z the *local mean normal measure of X_l at z*.

By Theorem 2.1 of [3] (and a classical result from convex geometry), for each $z \in \mathbb{R}^d$, there exists a unique zonoid $\Pi(X_l, z)$ whose support function is given by

$$h(\Pi(X_l, z), u) = \int_{S^{d-1}} |\langle u, v\rangle|\mu_z(\mathrm{d}v), \qquad u \in S^{d-1}.$$

From now on, we refer to $\Pi(X_l, z)$ as the *local associated zonoid of X_l at z*.

Let $\mathrm{lin}(x)$ denote the linear hull of a vector $x \in \mathbb{R}^d$. From Theorem 2.5 of [3] we obtain, for $j \in \{0, \dots, d\}$,

$$\begin{aligned}
&V_j(\Pi(X_l, z)) \\
&\quad = \frac{4^j}{j!} \int_{\mathcal{L}^d_{d-l}} \int_{\mathcal{F}^{(l-1)}} \cdots \int_{\mathcal{L}^d_{d-l}} \int_{\mathcal{F}^{(l-1)}} \\
&\quad\quad \times \int_{M_1} \cdots \int_{M_j} [\mathrm{Nor}^{d-1}(M_1 + L_1, x_1), \dots, \mathrm{Nor}^{d-1}(M_j + L_j, x_j)] \\
&\quad\quad\quad\quad \times f(M_1 + L_1, (z - p_{L_1}(z)) - x_1) \\
&\quad\quad\quad\quad \times \cdots f(M_j + L_j, (z - p_{L_j}(z)) - x_j)\mathcal{H}^{l-1}(\mathrm{d}x_j) \cdots \mathcal{H}^{l-1}(\mathrm{d}x_1) \\
&\quad\quad\quad\quad \times \mathrm{P}(L_1, \mathrm{d}M_1)\Phi(\mathrm{d}L_1) \cdots \mathrm{P}(L_j, \mathrm{d}M_j)\Phi(\mathrm{d}L_j).
\end{aligned}$$

We can exploit this last identity to prove the following theorem for the intersection densities.

Theorem 3.2. *Let $j, l \in \{1, \dots, d\}$ and let X_l be a Poisson point process with intensity measure as in (3.1) such that the pairs $(\mathrm{P}(L_1, \cdot), l-1), \dots, (\mathrm{P}(L_j, \cdot), l-1)$ satisfy condition (I) for $\Phi \otimes \cdots \otimes \Phi$-almost all $(L_1, \dots, L_j) \in \mathcal{L}^d_{d-l} \times \cdots \times \mathcal{L}^d_{d-l}$. For λ_d-almost all $z \in \mathbb{R}^d$, we then have*

$$\gamma_j(z) = \frac{j!}{4^j} V_j(\Pi(X_l, z)).$$

Moreover, if $j \geq 2$, we have

$$\gamma_j(z) \leq \left(j!\, \kappa_{d-1}^j \binom{d}{j} \Big/ d^j \kappa_d^{j-1} \kappa_{d-j} \right) \gamma_1(z)^j,$$

for λ_d-almost all $z \in \mathbb{R}^d$, where equality holds if and only if $\Pi(X_l, z)$ is a ball. For $k \in \mathbb{N}$, κ_k denotes the volume of the k-dimensional unit ball.

Remark. For stationary processes of hypersurfaces, more general results can be found in [10, Section 4] (see e.g. [10, Corollary 2]).

Proof of Theorem 3.2. The first equation follows from Theorem 3.1 and the above formula for $V_j(\Pi(X, z))$. By Equation (7.28) of [8, p. 307], we have

$$\left(\frac{\kappa_{d-1}}{d} V_1(K) \right)^j \geq \kappa_d^{j-1} \left(\kappa_{d-j} \Big/ \binom{d}{j} \right) V_j(K),$$

for $K \in \mathcal{K}'$ and $j \geq 2$, with equality if and only if K is a ball.

Example. Let $l \in \{1, \dots, d\}$ and $k \in \{0, \dots, l\}$. For $L \in \mathcal{L}^d_{d-l}$, let $\widetilde{\mathrm{P}}(L, \cdot)$ be a probability measure such that $\widetilde{\mathrm{P}}(L, \{K \in \mathcal{K}' \mid \dim K = k, K \subseteq L^{\perp}\}) = 1$ and $L \mapsto \mathrm{P}(L, \mathcal{A})$ is measurable for $\mathcal{A} \in \mathcal{B}(\mathcal{F}')$. The dimension of a convex body is defined as the dimension of its affine hull.

Let X_l be a Poisson process with intensity measure as in (3.1). Furthermore, we assume that $\mathrm{P}(L, \cdot)$ is the image measure of $\widetilde{\mathrm{P}}(L, \cdot)$ under the mapping $K \mapsto \operatorname{rel\,bd} K$, where $\operatorname{rel\,bd} K$ denotes the relative boundary of K, i.e. the boundary of K in its affine hull. Then X_l satisfies the assumptions of Theorem 3.1 and Theorem 3.2.

For $l = 1$ and $k = 0$, X_l is a Poisson hyperplane process. In this case, Theorem 3.1 and Theorem 3.2 give part of the results of Theorem 2 of [7].

Next, assume that $l = k = d$. By Theorem 2.2.4 of [6], for $\mathcal{H}^{d-1}$-almost all boundary points of a convex body K with nonempty interior, there exists a unique outer normal vector $\sigma_K(x)$ of K at x. Because $\operatorname{lin}(\sigma_K(x)) = \operatorname{Nor}^{d-1}(\operatorname{bd} K, x)$, the local mean normal measure μ_z can be written as

$$\mu_z(A) = \int_{\mathcal{K}'} \int_{\operatorname{bd} K} (\mathbf{1}_{\{A\}}(\sigma_K(x)) + \mathbf{1}_{\{-A\}}(\sigma_K(x))) f(K, z-x) \mathcal{H}^{d-1}(\mathrm{d}x)\, \mathrm{P}_0(\mathrm{d}K),$$

where $\operatorname{bd} K$ denotes the boundary of K and $A \in \mathcal{B}(S^{d-1})$, with $\mathrm{P}_0(\cdot) = \widetilde{\mathrm{P}}(\{0\}, \cdot)$.

For any $K \in \mathcal{K}'$, the measure $\mathcal{H}^{d-1}(\operatorname{bd} K \cap \cdot)$ is equal to the $(d-1)$th curvature measure of K. Thus, by Theorem 4.2 of [1], $B \mapsto \mathrm{E} \sum_{K \in X} \int_{\operatorname{bd} K} \mathbf{1}_{\{B\}}(x) \mathcal{H}^{d-1}(\mathrm{d}x)$ is a locally finite Borel measure on $\mathbb{R}^d$ and, hence, μ_z is a finite measure for λ_d-almost all $z \in \mathbb{R}^d$. If we additionally assume that $f \equiv \gamma > 0$ is a constant function, Theorem 3.1 and Theorem 3.2 yield Equations (4.54) and (4.55) of [8, p. 164].

Furthermore, for all $l \in \{1, \dots, d\}$ and $k \in \{1, \dots, l\}$, the above considerations show that the local mean normal measure is of the form

$$\mu_z(A) = \int_{\mathcal{L}^d_{d-l}} \int_{\mathcal{K}'} \int_{\text{rel bd } K} (\mathbf{1}_{\{A\}}(\sigma_{K+L}(x)) + \mathbf{1}_{\{-A\}}(\sigma_{K+L}(x))) \times f(K + L, (z - p_L(z)) - x) \times \mathcal{H}^{l-1}(\mathrm{d}x)\widetilde{\mathrm{P}}(L, \mathrm{d}K)\Phi(\mathrm{d}L), \qquad A \in \mathcal{B}(S^{d-1}).$$

Note that, since $K + L$ has nonempty interior, $\sigma_{K+L}(x)$ can be defined as for convex bodies; $\sigma_{K+L}(x)$ coincides with $\sigma_K(x)$ considered in the affine hull of K.

4. Intersections with affine subspaces

Let $l \in \{1, \dots, d\}$ and let X_l be a point process with intensity measure as in (3.1). In the last section, we introduced associated zonoids for such processes and connected their intrinsic volumes to intersection densities. In this section, we study intersections of X_l with affine subspaces. For example, X_l induces a (nonstationary) point process onto any affine line and we will show that the intensity function of such a process can be expressed in terms of the support functions of the associated zonoids.

Let $k \in \{1, \dots, d-1\}$, $z \in \mathbb{R}^d$, and $U \in \mathcal{L}^d_k$. We want to find an expression for the mean $(k-1)$th Hausdorff measure of the intersections of the particles of X_l with $z+U$ in an arbitrary Borel set $B \subseteq \mathcal{B}(U + z)$. Therefore, we define the following measure:

$$\nu_{U+z}(B) := \mathrm{E} \sum_{(M+L) \in X_l} \mathcal{H}^{k-1}(B \cap (z + U) \cap (M + L)), \qquad B \in \mathcal{B}(U + z).$$

Remark. For $k = 1$, $\nu_{U+z}(B)$ coincides with the mean number of intersection points in B of the line $z + U$ with the particles of X_l. If X_l is stationary and $\lambda_{U+z}(B) = 1$, then $\nu_{U+z}(B)$ is the intensity of the stationary point process induced by X_l onto $U + z$ which, by stationarity, is independent of z.

The main result of this section is the following theorem.

Theorem 4.1. *Let $k \in \{1, \dots, d-1\}$, $l \in \{1, \dots, d\}$, and let X_l be a point process with intensity measure as in (3.1). Furthermore, let $U \in \mathcal{L}^d_k$ and $z \in \mathbb{R}^d$. Then ν_{U+z} is absolutely continuous with respect to λ_{U+z} with the density given by*

$$\gamma_{U+z}(y) := 2 \int_{S^{d-1}} [\mathrm{lin}(u), U^{\perp}] \mu_y(\mathrm{d}u), \qquad y \in U + z.$$

Proof. By Campbell's theorem,

$$\nu_{U+z}(B) = \int_{\mathcal{L}^d_{d-l}} \int_{\mathcal{F}^{(l-1)}} \int_{L^{\perp}} \mathcal{H}^{k-1}(B \cap (U + z) \cap (M + L + x)) \times f(M + L, x)\lambda_{L^{\perp}}(\mathrm{d}x)\, \mathrm{P}(L, \mathrm{d}M)\Phi(\mathrm{d}L).$$

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Analogous to the proofs of Lemma 3.1, Lemma A.1, and Lemma A.2, we obtain

$$\nu_{U+z}(B) = \int_{z+U} \mathbf{1}_{\{B\}}(y) \int_{\mathcal{L}_{d-l}^d} \int_{\mathcal{F}^{(l-1)}} \int_M [\mathrm{Nor}^{d-1}(M+L,x), U^\perp] \times f(M+L,(y-p_L(y))-x)\mathcal{H}^{l-1}(\mathrm{d}x) \times \mathrm{P}(L,\mathrm{d}M)\Phi(\mathrm{d}L)\lambda_{U+z}(\mathrm{d}y)$$
$$= \int_{z+U} \mathbf{1}_{\{B\}}(y) 2 \int_{S^{d-1}} [\mathrm{lin}(u), U^\perp]\mu_y(\mathrm{d}u)\lambda_{U+z}(\mathrm{d}y).$$

Example. Let $k=1$ and $U := \{su \mid s \in \mathbb{R}\}$ for some $u \in S^{d-1}$. In this case, we have

$$\gamma_{U+z}(z+su) = 2\int_{S^{d-1}} |\langle v,u\rangle| \mu_{z+su}(\mathrm{d}v) = 2h(\Pi(X_l, z+su), u),$$

for $s \in \mathbb{R}$. This is a generalization of Equation (4.50) of [8, p. 159].

Appendix A. Auxiliary and measurability results

In this section, we collect auxiliary and measurability results needed in Section 3. Respective results implicitly used in Section 4 follow analogously.

The main tool in the proof of Lemma 3.1 is the next result which is a slight generalization of Theorem 3 of [9] and follows directly from the proof of the latter.

Theorem A.1. *Let $m \in \{1,\dots,d\}$, $n_0,\dots,n_m \in \{1,\dots,d-1\}$, $B \in \mathcal{B}(\mathbb{R}^d)$, and let $M_i \subseteq \mathbb{R}^d$ be an $\mathcal{H}^{n_i}$-measurable set for $i \in \{0,\dots,m\}$. Furthermore, define $n := n_0+\dots+n_m$ and let $f\colon \mathbb{R}^d \times \dots \times \mathbb{R}^d \to [0,\infty)$ be a measurable mapping. We assume that the pairs $(M_0,n_0),\dots,(M_m,n_m)$ satisfy condition (I) and that $n \geq md$. Then*

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathcal{H}^{n-md}(B \cap M_0 \cap (M_1+x_1) \cap \dots \cap (M_m+x_m)) f(x_1,\dots,x_m) \times \lambda_d(\mathrm{d}x_1)\cdots\lambda_d(\mathrm{d}x_m)$$
$$= \int_{M_0} \cdots \int_{M_m} \mathbf{1}_{\{B\}}(t_0) f(t_0-t_1,\dots,t_0-t_m)[\mathrm{Nor}^{n_0}(M_0,t_0),\dots,\mathrm{Nor}^{n_m}(M_m,t_m)] \times \mathcal{H}^{n_m}(\mathrm{d}t_m)\cdots\mathcal{H}^{n_0}(\mathrm{d}t_0).$$

The measurability of the mappings

$$(t_1,\dots,t_m) \mapsto [\mathrm{Nor}^{n_0}(M_0,t_0),\dots,\mathrm{Nor}^{n_m}(M_m,t_m)]$$

and

$$(x_1,\dots,x_m) \mapsto \mathcal{H}^{n-md}(B \cap M_0 \cap (M_1+x_1) \cap \dots \cap (M_m+x_m))$$

was proved in [9].

The following two auxiliary results were needed in the proof of Theorem 3.1.

Lemma A.1. *Let $k \in \{1,\dots,d\}$, $B \in \mathcal{B}(\mathbb{R}^d)$, and*

$$\mathcal{M} := \{(M_1',\dots,M_k') \in (\mathcal{F}')^k \mid M_1',\dots,M_k' \text{ is } (\mathcal{H}^{d-1}, d-1)\text{-rectifiable}, M_1' \cap \dots \cap M_k' \text{ is a } \mathcal{H}^{d-k}\text{-rectifiable set}\}.$$

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Then both $\mathcal{M}$ and the mapping

$$(M_1, \dots, M_k) \mapsto \mathcal{H}^{d-k}(M_1 \cap \cdots \cap M_k \cap B)\, \mathbf{1}_{\{\mathcal{M}\}}(M_1, \dots, M_k)$$

are measurable.

Proof. By Theorem 2.1.3 of [12] and as, by Theorem 1.1.6 of [8], the mapping $(F_1, F_2) \mapsto F_1 \cap F_2,\ F_1, F_2 \in \mathcal{F}'$, is measurable, $\mathcal{M}$ is a measurable set. The result then follows from Theorem 2.1.3, Theorem 2.2.1, and Corollary 2.1.4 of [12].

Lemma A.2. *Let k and $\mathcal{M}$ be as in Lemma A.1 and X_l as in the assumptions of Theorem 3.1. Then*

$$\mathrm{E} \sum_{(M_1+L_1,\dots,M_k+L_k)\in(X_l)^k_{\neq}} \mathbf{1}_{\{\mathcal{M}^c\}}(M_1 + L_1, \dots, M_k + L_k) = 0,$$

i.e. $(M_1 + L_1) \cap \cdots \cap (M_k + L_k)$ is almost surely an $\mathcal{H}^{d-k}$-rectifiable set for all $(M_1 + L_1, \dots, M_k + L_k) \in (X_l)^k_{\neq}$.

Proof. First, let $M_1, \dots, M_k, L_1, \dots, L_k$ be fixed. From Theorem 3.2.23 of [2] and Theorem 1.4.1 of [12] we obtain that $(x_1 + M_1 + L_1) \cap \cdots \cap (x_k + M_k + L_k)$ is $\mathcal{H}^{d-k}$-rectifiable for λ_{kd}-almost all $(x_1, \dots, x_k) \in \mathbb{R}^{kd}$.

Since X_l is assumed to be Poisson, this, together with Lemma A.1, Campbell's theorem, and Corollary 3.1.6 of [8], yields

$$\begin{aligned}
&\mathrm{E} \sum_{(M_1+L_1,\dots,M_k+L_k)\in(X_l)^k_{\neq}} \mathbf{1}_{\{\mathcal{M}^c\}}(M_1 + L_1, \dots, M_k + L_k) \\
&\quad = \int_{\mathcal{L}^d_{d-l}} \cdots \int_{\mathcal{L}^d_{d-l}} \int_{\mathcal{F}^{(l-1)}} \cdots \int_{\mathcal{F}^{(l-1)}} \\
&\qquad \times \int_{L_k^{\perp}} \cdots \int_{L_1^{\perp}} \mathbf{1}_{\{\mathcal{M}^c\}}(x_1 + M_1 + L_1, \dots, x_k + M_k + L_k) \\
&\qquad\qquad \times f(M_1 + L_1, x_1) \cdots f(M_k + L_k, x_k) \lambda_{L_1^{\perp}}(\mathrm{d}x_1) \cdots \lambda_{L_k^{\perp}}(\mathrm{d}x_k) \\
&\qquad\qquad \times \mathrm{P}(L_1, \mathrm{d}M_1) \cdots \mathrm{P}(L_k, \mathrm{d}M_k) \Phi(\mathrm{d}L_1) \cdots \Phi(\mathrm{d}L_k) \\
&\quad = 0.
\end{aligned}$$

Measurability of the mapping

$$M + L \mapsto \int_{M+L} \mathbf{1}_{\{B\}}(x) \mathcal{H}^0(S^{d-1} \cap \mathrm{Nor}^{d-1}(M + L, x) \cap A) \mathcal{H}^{d-1}(\mathrm{d}x)$$

follows analogously to [9, p. 235].

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