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ASYMPTOTIC BEHAVIOUR OF NONOSCILLATORY EQUATIONS

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1. Introduction. For nonlinear equations of the form

$$\text{(I)} \qquad (r(t)\, x^{(n)})^{(n)} = -\, x f(t, x),$$

there has been considerable interest in determining the asymptotic forms of nonoscillatory solutions. We assume $r(t)$ is continuous and positive on $[0, \infty)$, and $f(t, x)$ is continuous on $[0, \infty) \times \mathbf{R}$, and $f(t, x) > 0$ for $x \neq 0$. For $n = 2$, equation (I) was studied by Kusano and Naito [**3**], who found necessary and sufficient conditions for the existence of minimal and maximal nonoscillatory solutions. The former are the bounded solutions, while the later are those asymptotic to the function

$$(1.1) \quad R(t) = \int_T^t \frac{(t - u)(u - T)}{r(u)}\, du.$$

Their method consisted of writing (I) in the form of an integral operator and applying the Schauder fixed point theorem. For arbitrary n, but for $r(t) \equiv 1$, Kreith [**2**] found necessary and sufficient conditions for the existence of maximal solutions. His method did not require the fixed point theorem, but rather consisted of "tuning" the initial conditions to obtain the desired asymptotic form. Finally, Edelson and Schuur [**1**] used a fixed point theorem for multi-valued functions to determine necessary and sufficient conditions for the existence of both maximal and minimal solutions to (I), for $r(t) > 0$, and $n \geqq 1$. The method of [**1**] also gives rise to uniqueness theorems, not obtained by other methods.

In this article we apply the technique of [**1**] to determine the asymptotic form of non-oscillatory solutions of (I) which are non-extremal, that is neither maximal nor minimal. We will make use of the function

$$(1.2) \quad R(t, s) = \int_s^t \frac{(t - u)^{n-1}(u - s)^{n-1}}{[(n-1)!]^2\, r(u)}\, du$$

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and as usual we say that (I) is sublinear (resp. superlinear) if $f(t, x)$ is nonincreasing (resp. nondecreasing) in x. A solution $x(t)$ of I is said to be *oscillatory* if it is not identically zero, and there exists a sequence t_n, $n = 0, 1, 2, \ldots$ such that $\lim_{n\to\infty} t_n = \infty$, and $x(t_n) = 0$.

In Section 2 we describe very generally the kinds of nonoscillatory asymptotic behaviour which can occur subject to the restriction

$$(1.3) \quad \int^{\infty} \frac{dt}{r(t)} = \infty.$$

Then in Section 3 we give necessary and sufficient conditions for the perturbed equation (I), to have solutions asymptotic to the solutions of the nonoscillatory unperturbed equation

$$(r(t)x^{(n)})^{(n)} = 0.$$

These solutions are given by

$$R_k(t) = \frac{(t)^{k-1}}{(k-1)!} \quad 1 \leqq k \leqq n$$

$$R_k(t) = \int_0^t \frac{(t-u)^{n-1}(u)^{k-n-1}}{(n-1)!(k-n-1)!r(u)}\,du \quad n+1 \leqq k \leqq 2n.$$

For $k = 2n$, our results are contained in [**1**].

In Section 4, subject to additional hypotheses on $r(t)$, we give necessary and sufficient conditions that all solutions of (I) be oscillatory.

2. The asymptotic form of eventually positive solutions. We define the function E_k by

$$(2.1) \quad E_k(t, x) = \begin{cases} (r(t)x^{(n)}(t))^{(k-n)} & n+1 \leqq k \leqq 2n \\ x^{(k)}(t) & 0 \leqq k \leqq n. \end{cases}$$

If there is no danger of confusion, we will write $E_k(t, x) = E_k(t)$.

The functions E_k satisfy

$$\frac{\partial E_k}{\partial t}(t, x) = E_{k+1}(t, x), \quad \text{for } k \neq n,$$

$$\frac{\partial}{\partial t}(r(t)E_n(t, x)) = E_{n+1}(t, x),$$

and if $x(t)$ is a solution of (I), then

$$E_{2n}(t, x) + x(t)f(t, x) = 0.$$

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LEMMA 1. *If $x(t)$ is an eventually positive solution of* (I), *and condition* (1.3) *is satisfied, then there exists a* j, $1 \leqq j \leqq n$, *such that for* t *sufficiently large,*

$$(2.2)\quad \begin{array}{ll} E_k(t, x) > 0 & \text{for } 0 \leqq k < 2j \\ (-1)^k E_k(t, x) < 0 & \text{for } 2j \leqq k \leqq 2n. \end{array}$$

Proof. We first observe that (1.3) implies that if $E_k(t, x)$ and $E_{k+1}(t, x)$ are eventually positive (resp. negative), then $E_l(t, x)$ is eventually positive (resp. negative) for every l, $0 \leqq l \leqq k + 1$. Indeed, if $E_k(t) > 0$ and $E_{k+1}(t) > 0$ for $t > T$, then if $k \neq n$ there exists a $\bar{t} > T$ such that $E_k(t) \geqq E_k(\bar{t}) = c > 0$, for all $t > \bar{t}$. Then

$$\int_{\bar{t}}^{t} E_k(s)ds \geqq c(t - \bar{t}) \uparrow +\infty.$$

This implies

$$E_{k-1}(t) \uparrow +\infty \quad \text{if } k \neq n + 1 \quad \text{and} \quad r(t)E_{k-1}(t) \uparrow +\infty \quad \text{if } k = n + 1,$$

so $E_{k-1}(t)$ is eventually positive. In the case $k = n$, there exists $\bar{t} \geqq T$ such that

$$r(t)E_n(t) \geqq r(\bar{t})E_n(\bar{t}) = d > 0.$$

Then

$$\int_{\bar{t}}^{t} E_n(s)ds \geqq d \int_{\bar{t}}^{t} ds/r(s) \uparrow +\infty,$$

so $E_{n-1}(t)$ is eventually positive.

We are considering eventually positive solutions, so from (I), $E_{2n}(t, x)$ is eventually negative. Then it follows that $E_{2n-1}(t, x)$ is eventually positive and $E_{2n-2}(t, x)$ is eventually of constant sign. If E_{2n-2} is eventually positive the lemma is proved for $j = n$. Proceeding in this way we find a j satisfying the conditions of the lemma.

Remark. If $E_k(t, x)$ is eventually negative, then

$$\int^{\infty} E_k(t, x)dt > -\infty.$$

This is immediate for $k \neq n + 1$, and follows from (1.3) for $k = n + 1$.

Definition. A solution $x(t)$ of (I) is *of type* $2j$, $1 \leqq j \leqq n$, if for sufficiently large t,

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$$E_k(t, x) > 0, \quad 0 \leqq k < 2j, \quad E_{2j}(t, x) < 0.$$

According to Lemma 1, every eventually positive solution of (I) is of type $2j$ for some j, $1 \leqq j \leqq n$, provided (1.3) is satisfied. We assume hereafter that this is the case.

The asymptotic form of solutions to (I) will be given in terms of the solutions to the corresponding (linear) homogeneous equation.

Definition.

$$(3.5) \quad R_k(t, s) = \begin{cases} \displaystyle\int_s^t \frac{(t-u)^{n-1}(u-s)^{k-n-1}}{(n-1)!(k-n-1)!r(u)}\, du & \text{for } n+1 \leqq k \leqq 2n \\ \displaystyle\frac{(t-s)^{k-1}}{(k-1)!} \quad \text{for } 1 \leqq k \leqq n, \end{cases}$$

$$R_k(t) = R_k(t, 0).$$

We are interested in finding necessary and sufficient conditions for (I), either sublinear or superlinear, to have a solution asymptotic to $R_k(t)$. It is clear that for $n = 2$, $R_{2n}(t)$ is the function $R(t)$ defined in [**3**], and that these functions satisfy

$$(2.3) \quad E_l(t, R_k) \begin{cases} > 0 & \text{for } 0 \leqq l < k \\ = 0 & \text{for } l = k \end{cases}$$

and

$$(2.4) \quad R_k(t, s) = \begin{cases} (-1)^k \dfrac{\partial^{2n-k}}{\partial s^{2n-k}} R_{2n}(t, s) & \text{for } n+1 \leqq k \leqq 2n \\ (-1)^{n-k} \dfrac{\partial^{n-k}}{\partial s^{n-k}} R_n(t, s) & \text{for } 1 \leqq k \leqq n. \end{cases}$$

LEMMA 2. *If $x(t)$ is a solution of* (I) *of type $2j$, then there exist positive constants α, β such that for sufficiently large t,*

$$(2.5) \quad \alpha R_{2j-1}(t) \leqq x(t) \leqq \beta R_{2j}(t).$$

Proof. For $2j \neq n + 1$, $E_{2j-1}(t, x)$ is positive and decreasing, so

$$\lim_{t\to\infty} E_{2j-1}(t, x) = c \geqq 0.$$

For $2j = n + 1$,

$$\lim_{t\to\infty} r(t)E_n(t, x) = c' \geqq 0.$$

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Therefore there exists a positive constant t_1, or t_1', such that

(i) $0 \leqq E_{2j-1}(t, x) \leqq c_1$ for $2j \neq n + 1$, and $t > t_1$

or

(ii) $0 \leqq E_n(t, x) \leqq c_1'/r(t)$ for $2j = n + 1$, and $t > t_1'$.

Integrating the above inequalities $2j - 1$ times gives (2.5). Indeed, for $2j = n + 1$, integrating (ii) from t_1' to t gives

$$E_{n-1}(t_1', x) \leqq E_{n-1}(t, x) \leqq E_{n-1}(t_1', x) + c_1' \int_{t_1'}^{t} \frac{ds}{r(s)}.$$

Integrating $n - 1$ times gives

$$\sum_{i=1}^{n} E_{n-i}(t_1', x) \frac{(t - t_1')^{n-1}}{(n - i)!} \leqq x(t)$$

$$\leqq \sum_{i=1}^{n} E_{n-i}(t_1', x) \frac{(t - t_1')^{n-i}}{(n - i)!} + c_1' \int_{t_1'}^{t} \int_{t_1'}^{s_1} \cdots \int_{t_1'}^{s_{n-1}} \frac{ds}{r(s)} ds_{n-1} \ldots ds_1$$

$$= \sum_{i=1}^{n} E_{n-i}(t_1', x) \frac{(t - t_1')^{n-i}}{(n-i)!} + c_1' \int_{t_1'}^{t} \frac{(t - s)^{n-1}}{n!\, r(s)} ds.$$

This last inequality implies (2.5) provided t is sufficiently large. The case $2j \neq n + 1$ is analogous.

COROLLARY 1. *$x(t)$ is asymptotic to $R_{2j}(t)$ if and only if the constant c (or c' in the case $2j = n + 1$) of Lemma 2 is strictly positive.*

Proof. For $c > 0$ (or $c' > 0$ if $2j = n + 1$) the inequalities of Lemma 2 become

(i) $c_2 \leqq E_{2j-1}(t, x) \leqq c_3$ for $2j \neq n + 1$

(ii) $\dfrac{c_2'}{r(t)} \leqq E_n(t, x) \leqq \dfrac{c_3'}{r(t)}$ for $2j = n + 1$.

It follows that there exist positive constants α, β such that

(2.6) $\alpha R_{2j}(t) \leqq x(t) \leqq \beta R_{2j}(t)$,

for t sufficiently large. Conversely, if $c = 0$, ($c' = 0$ if $2j = n + 1$) for every constant $c_1 > 0$, we have

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(i) $E_{2j-1}(t, x) < c_1$ for $2j \neq n + 1$
(ii) $E_n(t, x) < c_1/r(t)$ for $2j = n + 1$

for t sufficiently large. Integrating (i) and (ii) $2j - 1$ times we see that $x(t)$ cannot be asymptotic to $R_{2j}(t)$.

COROLLARY 2. *If $x(t)$ is of type $2j$, then $x(t)$ is asymptotic to $R_{2j-1}(t)$ if and only if $E_{2j-2}(t, x)$ is bounded.*

Proof. For $2j \neq n - 2$, $E_{2j-2}(t, x)$ is an eventually positive, increasing function so we have

$$\lim_{t\to\infty} E_{2j-2}(t, x) = c \leqq \infty.$$

(a) If $c < \infty$ there exists $c_1 > 0$ such that

$$c_1 \leqq E_{2j-2}(t, x) \leqq c,$$

and $x(t)$ is asymptotic to $R_{2j-1}(t)$.
(b) If $c = \infty$, then for every $c_1 > 0$ we have

$$E_{2j-2}(t, x) > c_1,$$

for t sufficiently large, and integrating $2j - 2$ times we see that $x(t)$ cannot be asymptotic to $R_{2j-1}(t)$.
For $2j = n - 2$, we have

$$\lim_{t\to\infty} r(t)E_n(t, x) = c' \leqq \infty.$$

(a) If $c' < \infty$, then

$$\frac{c_1'}{r(t)} \leqq E_n(t, x) \leqq \frac{c_2'}{r(t)}$$

for positive constants c_1', c_2'.
(b) If $c' = \infty$ then

$$E_n(t, x) > c_1/r(t) \quad \text{for every } c_1.$$

Integrating n times completes the proof.

For the special cases of solutions asymptotic to $R_{2n}(t)$ or $R_1(t)$ we use the following terminology (see **[2]**):

Definition. A solution of (I) is *maximal* if it is asymptotic to $R_{2n}(t)$. It is *minimal* if it is asymptotic to $R_1(t)$, i.e., bounded.

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3. Existence theorems. Now let $y(t)$ be any continuous function. Our principle results will be stated in terms of the functions $P_k:(\mathbf{R}^+)^2 \rightarrow \mathbf{R}^+$ defined by

$$(3.1)\quad P_k(t;\, y) = \begin{cases} y(t) & k = 0 \\ \int_t^{\infty} P_{k-1}(s;y)\, ds & 1 \leqq k \leqq n-1 \\ P_{k-n}\left(t, \frac{1}{r(\cdot)} \int_{\cdot}^{\infty} P_{n-1}(u;\, y)du\right) & n \leqq k \leqq 2n-1. \end{cases}$$

It is clear that

$$P_k^{(k)}(t, y) = (-1)^k\, y(t) \quad \text{for } 0 \leqq k \leqq n-1,$$

and

$$(r(t)\, P_k^{(k-n)}(t, y))^{(n)} = (-1)^k\, y(t) \quad \text{for } k \geqq n,$$

if these integrals exist.

THEOREM 1. *A necessary and sufficient condition that* (I), *either sublinear or superlinear, have an eventually positive solution asymptotic to* $R_{2j}(t)$ *for* $2 \leqq 2j \leqq 2n$ *is that*

$$(3.2)\quad \int^{\infty} P_{2n-2j}(t;\, R_{2j}(\cdot) f(\cdot, cR_{2j}(\cdot)))\, dt < \infty$$

for some positive constant c.

Proof. (Necessity). Let $x(t)$ be of type $2j$ and suppose $x(t)$ is asymptotic to $R_{2j}(t)$. Then there exist positive constants α, β, T such that

$$\alpha R_{2j}(t) \leqq x(t) \leqq \beta R_{2j}(t) \quad \text{for } t \geqq T.$$

Then

$$x(t) f(t, x(t)) \geqq \alpha R_{2j}(t) f(t, \alpha R_{2j}(t))$$

if (I) is superlinear and

$$x(t) f(t, x(t)) \geqq \alpha R_{2j}(t) f(t, \beta R_{2j}(t))$$

if (I) is sublinear. It follows from the corollaries to Lemma 2 and the fact that

$$\lim_{t \to \infty} E_k(t, x) = 0 \quad \text{for } 2j \leqq k \leqq 2n,$$

that

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$$\infty > -\int^{\infty} E_{2j}(t, x)dt = \int^{\infty} P_{2n-2j}(t, x(\cdot)f(\cdot, x(\cdot)))dt$$
$$\geqq \alpha \int^{\infty} P_{2n-2j}(t, R_{2j}(s)f(\cdot, cR_{2j}(\cdot)))dt$$

where $c = \alpha$ if (I) is superlinear, and $c = \beta$ if (I) is sublinear.

(Sufficiency). Choose T so large that

$$\int_T^{\infty} P_{2n-2j}(t; R_{2j}f(\cdot, cR_{2j}))dt \leqq \frac{1}{2}.$$

We will apply the Schauder fixed point theorem in the space

$$C_{2j} = \{x \in C[T, \infty):\|x\| = \sup \{R_{2j}^{-2}(t)|x(t)|:T \leqq t\} < \infty\}.$$

Endowed with the norm topology, $(C_{2j},\|\cdot\|)$ is a Banach space. We define a closed subspace S of C_{2j} by

$$S = \{x \in C_{2j}:\alpha R_{2j}(t) \leqq x(t) \leqq 2\alpha R_{2j}(t)\}$$

where $\alpha = c/2$ if (I) is superlinear and $\alpha = c$ if (I) is sublinear. Let Φ be the operator defined by

$$(3.3) \quad [\Phi x](t) = 2\alpha R_{2j}(t) - \int_T^t R_{2j}(t, s)P_{2n-2j}(s; xf(\cdot, x))ds.$$

To apply the fixed point theorem we first observe that solutions of the operator equation $\Phi x = x$ also satisfy (I). In fact,

$$E_{2j}(t, \Phi x) = -P_{2n-2j}(t; xf(\cdot, x)),$$

so by the remarks following (3.1) x is a solution of I. We must now establish the following:

(i) $\Phi(S) \subset S$
(ii) Φ is continuous
(iii) $\Phi(S)$ is relatively compact.

Proof of (i). For $x \in S$,

$$\int_T^t R_{2j}(t, s)P_{2n-2j}(s; f(\cdot, x))ds$$
$$\leqq 2\alpha \int_T^t R_{2j}(t)P_{2n-2j}(s; R_{2j}f(\cdot, cR_{2j}))ds \leqq \alpha R_{2j}(t).$$

If follows that

$$[\Phi x](t) \geqq \alpha R_{2j}(t),$$

and the inequality

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$$[\Phi x](t) \leqq 2\alpha R_{2j}(t)$$

follows from the positivity of the integral.

Proof of (ii). Let $\{x_m\}$ be a sequence in S, converging to x. Then

$$|[\Phi x_m](t) - [\Phi x](t)|$$
$$\leqq R_{2j}(t) \int_T^\infty |P_{2n-2j}(s;\, x_m f(\cdot,\, x_m)) - P_{2n-2j}(s;\, x f(\cdot,\, x))|ds.$$

We also have

$$|P_{2n-2j}(s;\, x_m f(\cdot,\, x_m)) - P_{2n-2j}(s;\, x f(\cdot,\, x))|$$
$$\leqq 4\alpha P_{2n-2j}(s;\, R_{2j} f(\cdot,\, cR_{2j})),$$

and

$$\lim_{m\to\infty} |P_{2n-2j}(s;\, x_m f(\cdot,\, x_m)) - P_{2n-2j}(s;\, x f(\cdot,\, x))| = 0.$$

It follows from the Lebesgue dominated convergence theorem that

$$\lim_{m\to\infty} \|[\Phi x_m](t) - [\Phi x](t)\| = 0 .$$

Proof of (iii). This follows by an application of Ascoli's theorem. It suffices to show that the family $\mathscr{F} = \{R_{2j}^{-1}\varphi x\colon x \in S\}$ is uniformly bounded and equicontinuous. The uniform boundedness follows immediately from the definition of the space S. Equicontinuity will follow if we show that for $\epsilon > 0$, the interval $[T, \infty)$ can be subdivided into finitely many subintervals, on each of which, all functions in $\mathscr{F}$ have oscillation less than ϵ. The technique of the demonstration is identical to that used in [**3**], and the details will be omitted.

Having shown that Φ has a fixed point in S, we will show that this solution has the desired asymptotic properties. In fact, for $1 \leqq k \leqq 2j - 1$, applying l'Hospital's rule to (3.3) we obtain

$$\lim_{t\to\infty} R_{2j}^{-1}(t)\int_T^t R_{2j}(t,\, s)P_{2n-2j}(s;\, xf(\cdot,\, x))ds$$
$$= \lim_{t\to\infty} E_k^{-1}(t,\, R_{2j})\int_T^t E_k(t,\, R_{2j}(\cdot,\, s))P_{2n-2j}(s,\, xf(\cdot,\, x))ds.$$

Therefore, if $x(t)$ satisfies $\Phi x = x$, for $k = 2j - 1$ we have

$$\lim_{t\to\infty} R_{2j}^{-1}(t)x(t) = 2\alpha - P_{2n-2j+1}(T;\, xf(\cdot,\, x))$$

which is non-zero for the given choice of T.

THEOREM 2. *A necessary and sufficient condition that* (I), *either sublinear or superlinear, have an eventually positive solution asymptotic to* $R_{2j-1}(t)$ *for* $2 \leqq 2j \leqq 2n$ *is that*

$$(3.4) \quad \int^{\infty} P_{2n-2j+1}(t;\ R_{2j-1} f(\cdot, cR_{2j-1}))\, dt < \infty$$

for some positive constant c.

Proof. (Necessity). Let $x(t)$ be of type $2j$ and suppose x is asymptotic to R_{2j-1}. The necessity follows from the corollaries to Lemma 2 by a proof analogous to that of Theorem 1.

(Sufficiency). Let T be as in Theorem 1, and define

$$C_{2j-1} = \{x \in C[T, \infty): \quad \|x\| = \sup\{R_{2j-1}^{-2}(t)|x(t)|: T \leqq t\} < \infty\}$$

and

$$S = \{x \in C_{2j-1}: \quad \alpha R_{2j-1}(t) \leqq x(t) \leqq 2\alpha R_{2j-1}(t)\}\ .$$

If we define the operator $\Phi: S \to S$ by

$$[\Phi x](t) = \alpha R_{2j-1}(t) + \int_T^t R_{2j-1}(t, s)\, P_{2n-2j+1}(s;\ x f(\cdot, x))ds,$$

the proof of sufficiency is analogous to that of Theorem 1.

Theorems 1 and 2 can be combined, with considerable economy of notation, to give the following result:

THEOREM 3. *A necessary and sufficient condition that* (I), *either sublinear or superlinear, have an eventually positive solution asymptotic to* $R_k(t)$, $k = 1, 2, \ldots, 2n$, *is*

$$(3.5) \quad \int^{\infty} R_{2n-k+1}(t)\, R_k(t) f(t, cR_k(t))dt < \infty$$

for some constant $c > 0$.

Proof. For every k, we define

$$Q_{h,k}(t, s; f) = \begin{cases} R_k(s, t) f(s, cR_k(s, t)) & h = 0 \\ \int_t^s Q_{n-1,k}(t, s; f)dt & 1 \leqq h \leqq n-1 \\ Q_{h-n,k}(t, s;\ r^{-1} \int^s Q_{n-1,k}(w, s; f)dw) & n \leqq h \leqq 2n-1. \end{cases}$$

Integrating by parts we obtain

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$$\int_t^s R_{2n-k+1}(v, t)R_k(v, t)f(v, cR_k(v, t))dv = \int_t^s Q_{2n-k,k}(v, s; f)dv.$$

Letting $t = 0$ and $s \to \infty$, we see from the definition of P_{2n-k} that

$$\int_0^\infty R_{2n-k+1}(v)\, R_k(v)f(v, cR_k(v))dv = \int_0^\infty P_{2n-k}(u;\, R_k f(\cdot, R_k))dv,$$

and the theorem is an immediate consequence.

We observe also, that for every k,

$$R_{2n}(t) \leqq R_k(t)\, R_{2n-k+1}(t),$$

so that if (I) is sublinear, then

$$\int_0^\infty R_{2n-k+1}(v)\, R_k(v)f(v, cR_k(v))dv \geqq \int_0^\infty R_{2n}(v)f(v, cR_{2n}(v))dv.$$

Similarly, if (I) is superlinear, then

$$\int_0^\infty R_{2n-k+1}(v)\, R_k(v)f(v, cR_k(v))dv \geqq \int_0^\infty R_{2n}(v)f(v, c)dv.$$

The following corollaries follow from these observations.

COROLLARY 3. *If* (I) *is sublinear, then the nonexistence of a maximal solution implies the nonexistence of solutions asymptotic to* $R_k(t)$ *for every* $k \leqq 2n$.

Recall that a solution of I is *asymptotic* to $R_1(t)$ if and only if it is bounded.

COROLLARY 4. *If* (I) *is superlinear, then the existence of a solution asymptotic to* $R_k(t)$ *for some k, implies the existence of bounded solutions.*

With additional hypotheses on $r(t)$, it is possible to derive conditions sufficient to guarantee that for every $k = 1, 2, \ldots, 2n$, there exists a solution of (I) asymptotic to $R_k(t)$.

LEMMA 3. *If there exist constants H and H′ such that* $H' \leqq r(t) \leqq H$ *for* $t \geqq T$, *then for every* $h = 1, 2, \ldots, n - 1$, *there exists a positive constant* λ_h *such that*

(3.6) $R_{2n}(t) \geqq \lambda_h R_h(t) R_{2n-h+1}(t).$

Proof. For $T \leqq s < t$ we have

$$R_{n+1}(t, s) = \int_s^t \frac{(t-u)^{n-1}}{(n-1)!r(u)} du \geqq \frac{1}{H} \frac{(t-s)^n}{n!} = \frac{(t-s)}{n\,H} R_n(t, s).$$

and

$$\int_s^t R_n(t, u) du \geqq H' R_{n+1}(t, s).$$

Therefore,

$$R_{n+2}(t, s) = \int_s^t R_{n+1}(t, u) du \geqq \frac{1}{n\,H} \int_s^t (t-u) R_n(t, u) du$$

$$\geqq \frac{H'}{nH} (t-s) R_{n+1}(t, s) - \frac{1}{n} R_{n+2}(t, s),$$

and this implies

$$R_{n+2}(t, s) \geqq \frac{(t-s)H'}{(n+1)\,H} R_{n+1}(t, s).$$

Integrating $n - 1$ times we have

$$R_{2n}(t, s) \geqq \frac{(t-s)H'}{(2n-1)\,H} R_{2n-1}(t, s) \geqq \dots$$

$$\geqq \frac{(t-s)^{h-1}(H')^{h-1}}{(2n-1)(2n-2)\dots(2n-h+1)\,H^{h-1}} R_{2n-h+1}(t, s)$$

$$\geqq \frac{(h-1)!(H')^{h-1}}{(2n-1)\dots(2n-h+1)\,H^{h-1}} \cdot R_h(t, s) R_{2n-h+1}(t, s)$$

and this implies (3.6), by setting $s = 0$.

THEOREM 4. *Let $r(t)$ be bounded and* (I) *sublinear. If there exists a bounded solution of* (I), *then for every $k = 1, 2, \dots, 2n$, there exists a solution asymptotic to $R_k(t)$.*

The boundedness of $r(t)$ is essential. This is demonstrated by the equation $(tx')' = x/t$, having general solution $c_1 t^{-1} + c_2 t$.

THEOREM 5. *Let $r(t)$ be bounded and* (I) *superlinear. If there exists a maximal solution of* (I), *then for every $k = 1, 2, \dots, 2n$, there exists a solution asymptotic to $R_k(t)$.*

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The proofs of Theorems 4 and 5 are immediate consequences of Theorem 3 and Lemma 3.

Remark. These theorems show that the equation $x^{(10)} + kt^{-10}x$, introduced by G.D. Jones as an example of an equation of type (I) having both oscillatory and nonoscillatory solutions, has the property that no nonoscillatory solution can be asymptotic to t^k.

We should point out that the preceding results remain true for eventually negative solutions of (I). In this case, the constants α, β of Lemma 2, and the constants c of Theorems 1, 2, and 3 must be negative.

As application of the preceding theorems we consider the equations

$$(3.7) \quad (r(t)x^{(n)}(t))^{(n)} = -p(t)(|x|^\gamma \operatorname{sgn} x)$$

where $r(t)$, $p(t)$ are positive and continuous on $[0, \infty)$, $\gamma > 0$, and (1.3) is satisfied.

COROLLARY 5. *Equation* (3.7) *has a maximal solution if and only if*

$$\int_T^\infty p(t)\, R_{2n}(t)^\gamma dt < \infty.$$

COROLLARY 6. *Equation* (3.7) *has a bounded, nonoscillatory solution if and only if*

$$\int_T^\infty p(t) R_{2n}(t)\, dt < \infty.$$

It should be pointed out that these corollaries contain as special cases the corresponding results of Kusano and Naito [**3**], for $n = 2$.

4. Oscillation theorems. Throughout this section we suppose that $r(t)$ is a nondecreasing function satisfying (1.3), and let $R(t) = R_{2n}(t)$.

LEMMA 4. *If* $x(t)$ *is an eventually positive solution of* (I), *then for all sufficiently large* T *there exists a constant* λ *such that*

$$(4.1) \quad x(t) \geqq \lambda\{R(t)[r(t)x^{(n)}(t)]^{(n-1)} + \int_T^t R(s)x(s)f(s, x(s))ds\}.$$

Proof. Consider first the case x of type $2j \leqq n$. Integrating by parts $2n - 2j + 1$ times we have

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$$\int_T^t R(s)x(s)f(s, x(s))ds = \sum_{i=1}^{n}(-1)^i R^{(i-1)}(t)[r(t)x^{(n)}(t)]^{(n-i)}$$
$$+ \sum_{i=1}^{n-2j+1} (-1)^{n+i} \frac{(t-T)^{n-i}}{(n-i)!} x^{(n-i)}(t)$$
$$+ \int_T^t \frac{(s-T)^{2j-2}}{(2j-2)!} x^{(2j-1)}(s)ds.$$

From this we conclude

$$\text{(i)} \quad \int_T^t R(s)x(s)f(s, x(s))ds + R(t)[r(t)x^{(n)}(t)]^{(n-1)} \leqq \int_T^t \frac{(s-T)^{2j-2}}{(2j-2)!} x^{(2j-1)}(s)ds.$$

$x^{(2j-1)}(t)$ is positive and decreasing, so we have

$$x^{(2j-2)}(t) \geqq \int_T^t x^{(2j-1)}(s)ds \geqq x^{(2j-1)}(t)(t-T).$$

Furthermore, for $k \leqq 2j-2$,

$$x^{(k)}(t) \geqq \int_T^t x^{(k+1)}(s)ds \geqq \frac{(t-T)^{2j-k-1}}{(2j-k-1)!} x^{(2j-1)}(t).$$

From this we conclude

$$\text{(ii)} \quad x(t) \geqq \int_T^t x^{(1)}(s)ds \geqq \int_T^t \frac{(s-T)^{2j-2}}{(2j-2)!} x^{(2j-1)}(s)ds.$$

Then (4.1) is an immediate consequence of (i) and (ii).

Now consider the case $2j \geqq n+1$. Integrating by parts we have

$$\int_T^t R(s)x(s)f(s, x(s))ds$$
$$= \sum_{i=1}^{n-2j+1} (-1)^i R^{(i-1)}(t)[r(t)x^{(n)}(t)]^{(n-i)}$$
$$- \sum_{i=1}^{n-2j+1} (-1)^i R^{(i-1)}(T)[r(T)x^{(n)}(T)]^{(n-i)}$$
$$+ \int_T^t R^{(2n-2j+1)}(s)[rx^{(n)}(s)]^{(2j-n-1)}ds.$$

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It follows from the definition of "type $2j$" that all terms appearing in the summations on the right are $\geqq 0$, and therefore

$$\text{(i}'\text{)} \quad \int_T^t R(s)x(s)f(s, x(s))ds + R(t)[r(t)x^{(n)}(t)]^{(n-1)} \leqq \int_T^t R^{(2n-2j+1)}(s)[r(s)x^{(n)}(s)]^{(2j-n-1)}ds.$$

The function $(rx^{(n)})^{(2j-n-1)}$ is positive and decreasing, so

$$[r(t)x^{(n)}(t)]^{(k)} \geqq \frac{(t-T)^{2j-n-1-k}}{(2j-n-1-k)!}[r(t)x^{(n)}(t)]^{(2j-n-1)}.$$

From this we conclude

$$\text{(ii}'\text{)} \quad x(t) \geqq \int_T^t x^{(1)}(s)ds \geqq \int_T^t R_{2j}^{(1)}(s)[r(s)\, x^{(n)}(s)]^{(2j-n-1)}ds.$$

Since $r(t)$ is non decreasing

$$R_{2j}^{(n-1)}(t) = \int_T^t \frac{(v-T)^{2j-n-1}}{(2j-n-1)!r(v)}dv \geqq \frac{(t-T)^{2j-n}}{(2j-n)!r(t)} = \frac{(t-T)}{(2j-n)}R_{2j}^{(n)}(t).$$

Integrating $n-k$ times, the last inequality gives

$$R_{2j}^{(k-1)}(t) \geqq \frac{(t-T)}{2j-k}R_{2j}^{(k)}(t), \quad \text{for } k = 1, 2, \dots, n,$$

so we have

$$R_{2j}^{(1)}(t) \geqq \frac{(t-T)}{(2j-2)}R_{2j}^{(2)}(t) \geqq \dots \geqq \frac{(2(2j-n-1))!}{(2j-2)!}(t-T)^{2n-2j}R_{2j}^{(2n-2j+1)}(t).$$

From this we conclude

$$\text{(iii}'\text{)} \quad R^{(2n-2j+1)}(s) \leqq \frac{(2j-n-1)!}{(n-1)!}(s-T)^{2n-2j}R_{2j}^{(2n-2j+1)} \leqq \frac{(2j-n-1)!}{(n-1)!} \cdot \frac{(2j-2)!}{[2(2j-n-1)]!}R_{2j}^{(1)}(s) = \frac{1}{\lambda}R_{2j}^{(1)}(s).$$

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Then (4.1) follows from (i′), (ii′), and (iii′), which completes the proof.

We are now able to give necessary and sufficient conditions for (I) to be oscillatory, i.e, have all solutions oscillatory.

Definition. (I) is *strongly superlinear* (*sublinear*) if for some $\sigma > 0$, $x^{-\sigma}f(t, x)$ is nondecreasing ($x^{\sigma}f(t, x)$ is nonincreasing) in x.

THEOREM 6. *Let* (I) *be strongly sublinear and assume that* $r(t)$ *is nondecreasing. A necessary and sufficient condition that* (I) *be oscillatory is that for all* $c \neq 0$,

$$(4.2) \quad \int^{\infty} R(t)f(t, cR(t))dt = \infty.$$

Proof. The necessity is an immediate consequence of Theorem 3. For the sufficiency, let $x(t)$ be a nonoscillatory solution of (I). Without loss of generality we may assume $x(t)$ eventually positive. By Lemmas 2 and 3, there exist positive constants λ, β, such that

$$\text{(i)} \quad \lambda R(t)[r(t)x^{(n)}(t)]^{(n-1)} \leqq x(t) \leqq \beta R(t), \quad \text{for } t \geqq T.$$

The strong sublinearity implies that there exists a $\sigma > 0$ such that

$$\text{(ii)} \quad x^{\epsilon}(t)f(t, x(t)) \geqq \beta^{\epsilon}\mathrm{R}^{\epsilon}(t)f(t, \beta R(t))$$

for every $\epsilon \in (0, \sigma]$.

From (i) and (ii) we see

$$\begin{aligned} x(t)f(t, x(t)) &= x^{1-\epsilon}(t)x^{\epsilon}(t)f(t, x(t)) \\ &\geqq \lambda^{1-\epsilon}R^{1-\epsilon}(t)\{[r(t)x^{(n)}(t)]^{(n-1)}\}^{1-\epsilon}\beta^{\epsilon}R^{\epsilon}(t)f(t, \beta, R(t)). \end{aligned}$$

We then have

$$\begin{aligned} &(-[(r(t)x^{(n)}(t))^{(n-1)}]^{\epsilon})' \\ &= \epsilon[(r(t)x^{(n)}(t))^{(n-1)}]^{\epsilon-1}x(t)f(t, x(t)) \\ &\geqq \epsilon[(r(t)x^{(n)}(t))^{(n-1)}]^{\epsilon-1}\, x^{1-\epsilon}(t)\beta^{\epsilon}R^{\epsilon}(t)f(t, \beta R(t)) \\ &\geqq \epsilon\, \beta^{\epsilon}\, \lambda^{1-\epsilon}R(t)\, f(t, \beta R(t)) \end{aligned}$$

for arbitrarily small ϵ. This implies

$$\int^{\infty} R(t)f(t, \beta R(t))dt < \infty,$$

contradicting the hypothesis.

THEOREM 7. *Let* (I) *be strongly superlinear and assume that* $r(t)$ *is nondecreasing. A necessary and sufficient condition that* (I) *be oscillatory is that for all* $c \neq 0$,

$$(4.3) \quad \int^{\infty} R(t)f(t, c)dt = \infty.$$

Proof. The necessity is an immediate consequence of Theorem 3. Assume (4.3) and let $x(t)$ a positive solution of (I) for $t \geqq T$. By Lemmas 2 and 3, there exist α, λ, τ such that

$$\text{(i)} \quad x(t) \geqq \lambda \int_T^t R(s)x(s)f(s, x(s))ds$$
$$x(t) \geqq \alpha$$

for $t \geqq \tau \geqq T$. The strong superlinearity implies

$$\text{(ii)} \quad x(s)f(s, x(s))$$
$$= x(s)^{1+\epsilon}x(S)^{-\epsilon}f(s, x(s)) \geqq \alpha^{-\epsilon}x(s)^{1+\epsilon}f(s, \alpha).$$

From (i) and (ii) we conclude

$$\text{(iii)} \quad x(t)^{-1-\epsilon} \leqq (\lambda \int_T^t \alpha^{-\epsilon}x(s)^{1+\epsilon}f(s, \alpha)R(s)ds)^{-1-\epsilon}.$$

Multiplying (iii) by $x(t)^{1+\epsilon}R(t)f(t, \alpha)$ and integrating over $[\tau, t]$ we obtain

$$\text{(iv)} \quad \int_\tau^t R(s)f(s, \alpha)ds$$
$$\leqq -\left(\frac{\lambda}{\alpha^{\epsilon}}\right)^{-1-\epsilon}\left(\int_\tau^u x(s)^{1+\epsilon}f(s, \alpha)R(s)ds\right)^{-\epsilon}\Bigg|_{u=\tau}^{u=t}.$$

Letting $t \to \infty$, (iv) implies

$$\int^{\infty} R(s)f(s, \alpha)ds < \infty,$$

which contradicts the hypothesis. Since a similar argument holds if $x(t)$ is eventually negative, this completes the proof.

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