Canad. Math. Bull. Vol. **44** (3), 2001 pp. 298–312

A Proof of Casselman-Shahidi's Conjecture for Quasi-split Classical Groups

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Abstract. In this paper the author prove that standard modules of classical groups whose Langlands quotients are generic are irreducible. This establishes a conjecture of Casselman and Shahidi for this important class of groups.

Introduction

The purpose of this paper is to prove that standard modules of classical groups whose Langlands quotients are generic (*i.e.*, having a Whittaker model) are irreducible (Theorem 1.1). This establishes a conjecture of Casselman and Shahidi [CS] for this important class of groups. Together with Theorem 4.1 of [CS], such a result plays an important role in determining poles of local intertwining operators (*cf.* [CS, Proposition 5.3] and [Z, Theorem 4], for example), a type of result which plays a significant role in establishing holomorphy of automorphic L-functions [KSh], as well as in determination of residual spectrums.

The main tool in proving Theorem 1.1 is our characterization of generic discrete series in terms of γ-factors (Theorem 3.1). Theorem 3.1 has several other applications which we now explain. We first point out that for symplectic and odd special orthogonal groups, Theorem 3.1 was proved earlier in [M, Theorem 4.1] and was used in studying discrete series having generic supercuspidal supports [M, Theorems 3.1 and 3.3], [M1, Proposition 2.1]. Furthermore, it is possible to study (generic) supercuspidal supports containing discrete series based on Theorem 3.1, for all classical groups. Finally, Theorem 3.1 is important in studying the first occurrence indices of generic discrete series in symplectic-orthogonal towers [MS].

Here is an outline of the paper. In the first section we give the definition of a standard module and recall certain results from [CS]. Also, we state Theorem 1.1 in Section 1. In the second section we recall the definition of classical groups, and we describe a set of standard parabolic subgroups. Also, we recall certain results on Plancherel measures and γ-factors from [Sh1], [Si]. In the third section we state Theorem 3.1 and we complete the proof of Theorem 1.1. The proof of Theorem 3.1 is given in Sections 4 and 5.

I would like to thank S. Rallis, G. Savin, F. Shahidi, and M. Tadić for useful suggestions.

Received by the editors May 20, 1999.
The author is partially supported by a grant from NSF.
AMS subject classification: 22E35.

1 Standard Modules

Let F be a nonarchimedean field of characteristic zero. Let q be the number of elements in the residue field of F. We fix a nontrivial additive character ψ_F of F. Let $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the ring of rational integers, the field of real numbers, and the field of complex numbers, respectively. If $\mathbf{H}$ is an algebraic group defined over F, then we shall write H for its group of F-points.

Throughout this section $\mathbf{G}$ denotes an arbitrary quasi-split connected reductive algebraic group over F. Fix a Borel subgroup $\mathbf{B}$ and write $\mathbf{B} = \mathbf{TU}$, where $\mathbf{T}$ is a maximal split torus and $\mathbf{U}$ denotes the unipotent radical of $\mathbf{B}$.

Next, fix a parabolic subgroup $\mathbf{P} = \mathbf{MN}$ of $\mathbf{G}$, defined over F, with $\mathbf{N} \subset \mathbf{U}$ and $\mathbf{T} \subset \mathbf{M}$. Let $X(\mathbf{M})_F$ be the group of F-rational characters of $\mathbf{M}$. Set

$$\mathfrak{a}^* = X(\mathbf{M})_F \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad \mathfrak{a}^*_{\mathbb{C}} = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}.$$

Let $(\mathfrak{a}^*)^+$ be the positive Weyl chamber that corresponds to the positive restricted roots of T in G determined by B. Let H_M be the homomorphism

$$H_M \colon M \longrightarrow \mathfrak{a} = \operatorname{Hom}\big(X(\mathbf{M})_F, \mathbb{R}\big),$$

defined by

$$q^{\langle \chi, H_P(m)\rangle} = |\chi(m)|_F$$

for all $\chi \in X(\mathbf{M})_F$. We denote by χ_ν the character of M defined by $\chi_\nu(m) = q^{\langle \nu, H_M(m)\rangle}$, for any $\nu \in \mathfrak{a}^*_{\mathbb{C}}$. Finally, put $\pi_\nu = \chi_\nu \otimes \pi$, for any irreducible representation π of M and $\nu \in \mathfrak{a}^*_{\mathbb{C}}$.

A standard module is an induced representation of the form

(1.1) $$I_P(\nu, \pi) = \operatorname{Ind}^G_{MN}(\pi_\nu), \quad \nu \in (\mathfrak{a}^*)^+, \quad \pi \text{ is tempered.}$$

(We consider the normalized induction.) This representation has a unique irreducible quotient denoted by $J(\nu, \pi)$. Next, assume that w_l and $w_{l,M}$ are the longest element of the Weyl group W_G of T in G and the longest element of the Weyl group W_M of T in M. Put $w = w_l \cdot w_{l,M}$. We write $\tilde{w}$ for a representative of w taken as in [Sh1, Section 3].

Assume that $\mathbf{M}$ is generated by a subset θ of simple roots Δ of $\mathbf{T}$ in $\mathbf{U}$. Then $\theta' = w(\theta) \subset \Delta$. Let $\mathbf{P}' = \mathbf{M}'\mathbf{N}'$ be the corresponding standard parabolic subgroup. Also $\mathbf{N}' = \mathbf{U} \cap w\overline{\mathbf{N}}w^{-1}$, where $\overline{\mathbf{N}}$ is unipotent subgroup opposed to $\mathbf{N}$. Now, under the assumptions of (1.1), the long intertwining operator

$$A(\nu, \pi, \tilde{w}) f(g) = \int_{N'} f(\tilde{w}^{-1} n g)\, dn \quad (g \in G)$$

is holomorphic and intertwines the induced representations $I_P(\nu, \pi)$ and $I_{P'}\big(w(\nu), w(\pi)\big)$. Its image is isomorphic to $J(\nu, \pi)$.

Next, we shall fix a non-degenerate character ψ of $U = \mathbf{U}(F)$. (This means that ψ is non-trivial on each simple root subgroup of U.) Set $\psi_M = \psi \mid U \cap M$, and assume π is ψ_M-generic. Changing the splitting in **U** we may assume that ψ and $\tilde{w}$ are compatible [Sh1], and we can consider a ψ-generic functional [Sh4] $\lambda_\psi(\nu,\pi)$ on $I_P(\nu,\pi)$:

$$\lambda_\psi(\nu,\pi)(R_u f) = \psi(u)\lambda_\psi(\nu,\pi)(f),$$

where f is in the space of $I_P(\nu,\pi)$, and R_g denotes the action by right translations of G on that space. By [Sh4, Theorem 3.1], there exists a complex meromorphic function $C_\psi(\nu,\pi,\tilde{w})$ such that

$$\lambda_\psi(\nu,\pi) = C_\psi(\nu,\pi,\tilde{w})\lambda_\psi\big(w(\nu),w(\pi)\big)A(\nu,\pi,\tilde{w}).$$

The Langlands quotient $J(\nu,\pi)$ is ψ-generic if and only if $C_\psi(\nu,\pi,w) \neq \infty$ [CS, Proposition 5.4]. This condition can be interpreted in terms of local γ-factors [Sh1, Theorem 3.5] as follows. First, let $\hat{M}(\mathbb{C})$ be the dual complex group of **M**. Let r be the adjoint action of $\hat{M}(\mathbb{C})$ on the Lie algebra $\hat{\mathfrak{n}}$ of the standard parabolic subgroup of $\hat{G}^0(\mathbb{C})$ with a Levi factor $\hat{M}^0(\mathbb{C})$. (Here, for example, $\hat{G}^0(\mathbb{C})$ means connected component of $\hat{G}(\mathbb{C})$. A set of standard parabolic subgroups of $\hat{G}^0(\mathbb{C})$ is completely determined by that of **G**.) Next, we write $r = \bigoplus_j r_j$ for a decomposition into irreducible constituents. Then, $C_\psi(\nu,\pi,w)$ is, up to a non-zero constant (independent of ν and π), equal to $\prod_j \gamma(1,\pi_\nu,r_j,\psi_F)$. Finally, we can state the following lemma.

Lemma 1.1 *The Langlands quotient $J(\nu,\pi)$ is ψ-generic if and only if*

$$\prod_j \gamma(1,\pi_\nu,r_i,\psi_F) \neq \infty. \tag{1.2}$$

In particular, if **G** *is a classical group (see Section 2), then the L-functions $L(s,\pi,r_i)$ (π tempered) are holomorphic for $\mathrm{Re}(s) > 0$, and (1.2) is equivalent to*

$$\prod_j L(1,\tilde{\pi}_{-\nu},r_i) \neq \infty. \tag{1.3}$$

(Here $\tilde{\pi}$ stands for the contragredient representation of π.)

Proof The criterion (1.2) is direct consequence of [CS, Proposition 5.4]. For the holomorphy of L-functions for classical **G** see [CS, Theorem 4.1] and [Sh1, Proposition 7.2]. (See also [M, Corollary 4.1].) ■

Remark 1.1 In Lemma 1.1 the roles of π and $\widetilde{\pi}$ are interchanged when compared to the usual definition of local factors [Sh1, Theorem 3.5]. This definition agrees with the definition given in [M] and [MSh].

Now, we are ready to state the first main result of this paper. It shows that the criterion (1.3) is a necessary and sufficient condition for reducibility of a standard module $I_P(\nu,\pi)$ for classical groups.

Theorem 1.1 *Assume that* **G** *is a quasi-split classical group defined over F (see Section 2). Then the standard module $I_P(\nu,\pi)$ is irreducible if and only if the Langlands quotient $J(\nu,\pi)$ is generic.*

Proof The proof is that of [M, Theorem 5.1] for symplectic and odd-orthogonal groups, and that of [MSh, Theorem 3.1] for Iwahori-spherical representations of a general split reductive group. For this method it is enough to prove the following result.

Propostion 1.1 *Assume that* $\mathbf{P} = \mathbf{MN}$ *is a standard parabolic subgroup of* $\mathbf{G}$ *defined over F. Let* ω *be a supercuspidal generic representation of M. Then, if the induced representation* $\mathrm{Ind}_{MN}^{G}(\omega)$ *contains a discrete series subquotient, then all generic irreducible subquotients are in discrete series.*

Proposition 1.1 will follow from a characterization of generic discrete series for quasi-split classical groups in terms of γ-factors. (See Theorem 3.1.) ■

2 Classical Groups

The purpose of this section is to recall certain facts about local factors attached to generic representations of classical groups.

First, all quasi-split classical groups can be described as follows. They fall into three different categories:

(i) The symplectic groups G_n. Let

$$V_n = e_1F \oplus \cdots \oplus e_nF \oplus e_{n+1}F \oplus \cdots \oplus e_{2n}F,$$

be a F-vector space. Then $G_n \subset \mathrm{GL}(V_n)$ is the group of all elements preserving a skew-symmetric form $\langle\ ,\ \rangle$ defined by $\langle e_i, e_{2n-j+1}\rangle = \delta_{ij}$, $i, j = 1, \ldots, n$. We let $\mathrm{Sp}(0, F) = \{1\}$.

(ii) The special orthogonal groups G_n. Let

$$V_n = e_1F \oplus \cdots \oplus e_nF \oplus V_0 \oplus e_{n+1}F \oplus \cdots \oplus e_{2n}F,$$

be a F-vector space. Then $G_n \subset \mathrm{GL}(V_n)$ is the group of all elements preserving a symmetric form $\langle\ ,\ \rangle$ and having determinant 1, defined by $\langle e_i, e_{2n-j+1}\rangle = \delta_{ij}$, $i, j = 1, \ldots n$, where V_0 is anisotropic space, $\langle\ ,\ \rangle$-orthogonal to each vector e_i. Since G_n is quasi-split, $\dim_F V_0 \leq 2$. If V_0 is one-dimensional, we can take $V_0 = F$, with the quadratic form $q(x) = x^2$. If V_0 is two-dimensional, then $V_0 = E$ (E quadratic extension of F), with the quadratic form $q(x) = N_{E/F}(x)$. Clearly, G_0 is well defined unless $V_0 = 0$. If $V_0 = 0$, we shall write $G_n = \mathrm{SO}(n, n)$, and define $\mathrm{SO}(0, 0) = \{1\}$.

(iii) Unitary groups G_n. Let E be the quadratic extension of F, and τ be the nontrivial element of the Galois group $\mathrm{Gal}(E/F)$. Let

$$V_n = e_1E \oplus \cdots \oplus e_nE \oplus V_0 \oplus e_{n+1}E \oplus \cdots \oplus e_{2n}E,$$

be a E-vector space. Then $G_n \subset \mathrm{GL}(V_n)$ is the group of all elements preserving a τ-hermitian form $\langle\ ,\ \rangle$, defined by $\langle e_i, e_{2n-j+1}\rangle = \delta_{ij}$, $i, j = 1, \ldots, n$, where V_0 is anisotropic space, $\langle\ ,\ \rangle$-orthogonal to each vector e_i. Since G_n is quasi-split, $\dim_E V_0 \leq 1$. If V_0 is one-dimensional, we can take $V_0 = E$, $\langle x, y\rangle = x\tau(y)$. G_0 is well defined unless $V_0 = 0$. If $V_0 = 0$, we define $G_0 = \{1\}$.

Definition 2.1 Let (F', τ) be the pair consisting either of $F' = F$ and the identity map τ (see (i) and (ii)), or of $F' = E$ and τ as in (iii). Let $|\ \ |$ be the normalized absolute value of F'. Put $m(n) = 2n + 1$, $m(n) = 2n$, and $m(n) = \dim_{F'} V_n$ if G_n is a symplectic group, special odd-orthogonal group, and special even-orthogonal or unitary group, respectively.

We shall fix a Borel subgroup of G_n consisting of elements fixing the flag of isotropic subspaces

$$W_0 \subset W_1 \subset \cdots \subset W_n,$$

where $W_0 = 0$, and $W_i = e_1F \oplus \cdots \oplus e_iF$, $i = 1, \ldots, n$. In this way a set of standard parabolic subgroup is fixed. If $G_n \neq \mathrm{SO}(n, n)$, this set is parametrized by ordered partitions $\alpha = (l_1, \ldots, l_k)$ of $0 \leq m \leq n$. Let $P_\alpha = M_\alpha N_\alpha$ be the parabolic subgroup attached to α, where

$$M_\alpha = \mathrm{GL}(l_1, F') \times \cdots \times \mathrm{GL}(l_k, F') \times G_{n-m}. \tag{2.1}$$

If $G_n = \mathrm{SO}(n, n)$, a part of standard parabolic subgroups is parametrized by the ordered partitions of $0 \leq m \leq n$, $m \neq n - 1$. Finally, ϵ-conjugates of standard parabolic subgroups attached to partitions of n of type $\alpha = (l_1, \ldots, l_k)$, $l_k > 1$, describe the remainder of standard parabolic subgroups. (Here ϵ is an element of the corresponding orthogonal group having determinant -1.)

Next, assume that $n \geq 1$ and $G_n \neq \mathrm{SO}(1, 1)$. Let $P = P_{(m)}$ be the standard maximal parabolic subgroup described the above. Here $0 < m \leq n$, and if $G = \mathrm{SO}(n, n)$, then $m \neq n - 1$. Write $P = MN$ for its Levi decomposition. In particular, $M_{(m)} \cong \mathrm{GL}(m, F) \times G_{n-m}$. Let $\delta \otimes \mathcal{T} \in \mathrm{Irr}(M)$ be a discrete series representation. Let w be the element of the Weyl group W_{G_n} decribed in Section 1, and $\tilde{w}$ its representative.

Let $\mu(s, \delta \otimes \mathcal{T}, w)$ be the Plancherel measure [Sh1, Section 2], associated by Harish-Chandra to the induced representation $\mathrm{Ind}_P^{G_n}(|\det|^s\delta \otimes \mathcal{T})$, $s \in \mathbb{C}$, and $w \in W_{G_n}$. Further, we write $\gamma_{\tilde{w}}(G_n/P)$ for the positive real number introduced, for example, in [Sh1, p. 280], and put $\mu(s, \delta \otimes \mathcal{T}) = \mu(s, \delta \otimes \mathcal{T}, w)\gamma_{\tilde{w}}(G_n/P)^{-2}$. We will also call $\mu(s, \delta \otimes \mathcal{T})$ the Plancherel measure.

Lemma 2.1 *$\mu(s, \delta \otimes \mathcal{T})$ has at most a double zero at $s = 0$.*

Proof This follows from [Si, Theorem 5.4.2.1]. ■

Next, assuming that $\delta \otimes \mathcal{T}$ is generic, we can express the Plancherel measure in terms of γ-factors [Sh1, Corollary 3.6].

Lemma 2.2 *Assume that $\delta \otimes \mathcal{T}$ is generic. Then, along $\mathrm{Re}(s) = 0$, the Plancherel measure $\mu(s, \delta \otimes \mathcal{T})$ is equal to the square of the ordinary absolute value of $\gamma(s, \delta \times \mathcal{T}, \psi_F)\gamma(2s, \delta, \psi_F, r)$, and it is holomorphic there.*

The γ-factor

$$\gamma(s, \delta \times \mathcal{T}, \psi_F) = \epsilon(s, \delta \times \mathcal{T}, \psi_F)L(1 - s, \tilde{\delta} \times \tilde{\mathcal{T}})L(s, \delta \times \mathcal{T})^{-1},$$

is defined in [Sh1, Theorem 3.5], and further studied in [Sh2], [CS, Section 4], and [M]. Here, the L-factor, attached to $\delta \otimes \mathcal{T}$, $L(s, \delta \times \mathcal{T})$ is defined as $P(q^{-s})^{-1}$, where P is a polynomial, such that $P(q^{-s})$ has the same zeroes as $\gamma(s, \delta \times \mathcal{T}, \psi_F)$. Next, $\epsilon(s, \delta \times \mathcal{T}, \psi_F)$ is a unit in the ring $\mathbb{C}[q^s, q^{-s}]$.

Next, let ρ_m be the standard representation of $\mathrm{GL}(m, \mathbb{C})$. Further, the dual complex group of $R_{E/F}\,\mathrm{GL}(m)$ (E quadratic extension of F) is isomorphic to $\big(\mathrm{GL}(m, \mathbb{C}) \times \mathrm{GL}(m, \mathbb{C})\big) \rtimes \mathrm{Gal}(E/F)$, where τ acts on $\mathrm{GL}(m, \mathbb{C}) \times \mathrm{GL}(m, \mathbb{C})$ as follows:

$$(g_1, g_2)^\tau = \big((g_2^t)^{-1}, (g_1^t)^{-1}\big).$$

(Here g^t denotes the transpose matrix of a matrix g.) Finally, we have the following decomposition into irreducible subrepresentations [G, Lemma 5.3]:

$$\mathrm{Ind}_{\mathrm{GL}(m,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C})}^{\mathrm{GL}(m,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C}) \rtimes \mathrm{Gal}(E/F)} (\rho_m \otimes \tilde{\rho}_m) = \Psi \oplus \eta \otimes \Psi.$$

Here, η is a character of $F^\times$ attached to the extension $F \subset E$ via class-field theory. (This means, trivial on $N_{E/F}(E^\times)$, and in particular a character of $\mathrm{Gal}(E/F) \cong F^\times / N_{E/F}(E^\times)$.) Also, we shall assume that Ψ is given as in [G, p. 77]. Now, we shall introduce an irreducible representation r of the complex dual group of $R_{F'/F}\,\mathrm{GL}(m)$ as follows:

(2.2)

$$\begin{cases} r = \wedge^2 \rho_m, & \text{if } G_n \text{ is symplectic group or special even orthogonal group} \\ r = \mathrm{Sym}^2 \rho_m, & \text{if } G_n \text{ is special odd-orthogonal group} \\ r = \Psi, & \text{if } G_n \text{ is unitary group, } V_0 = 0 \\ r = \eta \otimes \Psi, & \text{if } G_n \text{ is unitary group, } V_0 \neq 0. \end{cases}$$

Finally, the γ-factor

$$\gamma(s, \delta, \psi_F, r) = \epsilon(s, \delta, \psi_F, r) L(1 - s, \tilde{\delta}, r) L(s, \delta, r)^{-1}$$

is defined again in [Sh1], and further studied in [Sh3] and [G].

We shall end this section by introducing more notation. We write Irr for a disjoint union of all $\mathrm{Irr}\big(\mathrm{GL}(m, F')\big)$, $m \geq 1$. As before, let $\tilde{\delta}$ be the contragredient representation of $\delta \in \mathrm{Irr}$. Clearly, we have the continuous automorphism $g \longmapsto \tau(g)$, where, for $g = (g_{ij}) \in \mathrm{GL}(n, F')$, we put $\tau(g) = \big(\tau(g_{ij})\big)$ (that is, τ acts on each entry of a matrix g). It acts on Irr as follows: $\delta^\tau(g) = \delta\big(\tau(g)\big)$. Now, we shall make the following definition.

Definition 2.2 Let $\delta \in \mathrm{Irr}$, then we shall write $\delta^* = \tilde{\delta}^\tau$.

3 A Characterization of Discrete Series

Assume that $\mathcal{T}$ is a generic irreducible representation of $G = G_n$ ($n \geq 1$) and $G \neq \mathrm{SO}(1, 1)$. Choose a standard parabolic subgroup $P = MN$, with $M \cong M_\alpha$ (see (2.1)),

https://doi.org/10.4153/CMB-2001-030-4 Published online by Cambridge University Press

and a supercuspidal representation $\omega = \omega_1 \otimes \cdots \otimes \omega_k \otimes \sigma$ of M such that $\mathcal{T}$ is a subquotient of $\mathrm{Ind}_P^G(\omega)$. Each ω_i can be uniquely written as $\omega_i = |\det|^{e(\omega_i)}\omega_i^u$, where $e(\omega_i) \in \mathbb{R}$ and ω_i^u is unitary. Then we say that $\mathcal{T}$ satisfies property $(*)$ if

$$(*) \qquad (\omega_i^u)^* \cong \omega_i^u \quad \text{and} \quad 2e(\omega_i) \in \mathbb{Z}, \quad \text{for all } i = 1, \ldots, k.$$

Theorem 3.1 *The representation $\mathcal{T}$ is generic discrete series if and only if the following holds.*

(i) The property $(*)$ holds.
(ii) The function defined by $\gamma(s, \delta \times \mathcal{T}, \psi_F)\gamma(2s, \delta, \psi_F, r)$ is holomorphic at $s = 0$, and there has at most a simple zero, for all discrete series representations $\delta \in$ Irr. In fact, it is enough to consider only discrete series of $\mathrm{GL}(m, F')$'s (F' is given by Definition 2.1), where m is less than or equal to the dimension of the standard representation of $\hat{G}_n^0(\mathbb{C})$.

Tadić [T] first observed that $(*)$ is a necessary condition for the existence of a square integrable subquotient in $\mathrm{Ind}_P^G(\omega)$ for G symplectic or special odd-orthogonal groups. His proof carries directly to all quasi-split classical groups described in Section 2, except $\mathrm{SO}(n,n)$. Ban [Ba] extended his result to $\mathrm{SO}(n,n)$ ($n \geq 2$). (See also [CS, p. 573].)

Now, we are ready to prove Proposition 1.1.

Proof of Proposition 1.1 Assume that $\mathcal{T}_0$ is a discrete series subquotient of $\mathrm{Ind}_P^G(\omega)$. In fact we shall assume that $\mathcal{T}_0$ is a subrepresentation. Let $\mathcal{T}$ be any irreducible generic subquotient of $\mathrm{Ind}_P^G(\omega)$. Then the multiplicative properties of γ-factors [Sh2, Corollary 5.6] (see Remark 1.1) and the multiplicative properties of Plancherel measures [Si] imply

$$\begin{cases} \gamma(s, \delta \times \mathcal{T}, \psi_F) = \gamma(s, \delta \times \sigma, \psi_F) \prod_i \gamma(s, \delta \times \tilde{\omega}_i, \psi_{F'})\gamma(s, \tilde{\delta}^* \times \omega_i, \psi_{F'}) \\ \mu(s, \delta \otimes \mathcal{T}_0) = \mu(s, \delta \otimes \sigma) \prod_i \mu(s, \delta \otimes \omega_i)\mu(s, \omega_i \otimes \delta^*). \end{cases}$$

Here the γ-factors $\gamma(s, \delta \times \tilde{\omega}_i, \psi_F)$ and $\gamma(s, \tilde{\delta}^* \times \omega_i, \psi_F)$ are γ-factors attached to pairs of representations in Irr [JPSS], [Sh1]. Next, $\psi_{F'} = \psi_F \circ Tr_{F'/F}$. (Actually, in [Sh2, Corollary 5.6] is required that $\mathcal{T}$ is a subrepresentation of $\mathrm{Ind}_P^G(\omega)$, but it is easy to see that the same formula holds if $\mathcal{T}$ is just a subquotient.) Next, we have [Sh4].

Lemma 3.1 *Assume that δ_1 and δ_2 are essentially square integrable representations in* Irr. *Then the (modified) Plancherel measure $\mu(s, \delta_1 \otimes \delta_2)$ is equal to*

$$\gamma(s, \delta_1 \times \tilde{\delta}_2, \psi_{F'})\gamma(-s, \tilde{\delta}_1 \times \delta_2, \bar{\psi}_{F'}).$$

Now, it is not hard to see that $\mu(s, \delta \otimes \mathcal{T}_0)$ is equal to the square of the ordinary absolute value of $\gamma(s, \delta \times \mathcal{T}, \psi_F)\gamma(2s, \delta, \psi_F, r)$. We leave details to the reader. Now, Theorem 3.1 and Lemma 2.1 complete the proof of Proposition 1.1. ■

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4 Proof of Theorem 3.1

Assume that $\mathcal{T}$ is (generic) discrete series. Then, by Lemma 2.1 and Lemma 2.2, (ii) is a necessary condition. We noted after the statement of Theorem 3.1 that (i) is also a necessary condition. Thus, let us prove that if (i) and (ii) hold, then $\mathcal{T}$ is discrete series.

Assume that $\mathcal{T}$ is not square integrable. Then, by the Langlands quotient theorem [BW] and [Ze], $\mathcal{T}$ can be realized as the unique irreducible quotient of a standard module $\mathrm{Ind}_P^G(\delta_1 \otimes \cdots \otimes \delta_m \otimes \mathcal{T}_1)$, where

(a) $P = MN$ is a standard parabolic subgroup, $M \cong \mathrm{GL}(l_1, F') \times \cdots \times \mathrm{GL}(l_m, F') \times G'$ (G' is a classical group of the same type as G, but of a smaller rank);

(b) $\delta_i \in \mathrm{Irr}\big(\mathrm{GL}(l_i, F')\big)$, is an essentially square integrable representation, attached by Bernstein, to the segment $\Delta_i = [|\det|^{-n_i+\alpha_i}\rho_i, |\det|^{n_i+\alpha_i}\rho_i]$, where ρ_i is a unitary supercuspidal representation, $\alpha_i = e(\delta_i)$ is real, and $n_i \geq 0$ is a half-integer, $1 \leq i \leq m$. Next, we must have

$$\begin{cases} e(\delta_1) \geq \cdots \geq e(\delta_m) > 0, & \text{unless} \\ e(\delta_1) \geq \cdots \geq e(\delta_{m-1}) > |e(\delta_m)|, & \text{if } G = \mathrm{SO}(n,n), G' = \mathrm{SO}(0,0), l_m = 1. \end{cases}$$

(c) $\mathcal{T}_1$ is a (generic) tempered representation of G'.
(Here we also allow $m = 0$.) These data must satisfy the following constraints:

(d) Applying the assumption (i) of Theorem 3.1, $\rho_i \cong \rho_i^*$, and $\alpha_i = e(\delta_i)$ is half-integer, $i = 1, \ldots, m$. In particular, $(\delta_i^u)^* \cong \delta_i^u$, $i = 1, \ldots, m$.

(e) Finally, since $\mathcal{T}$ is generic we have the following (see Lemma 1.2):

$$\begin{cases} L(1, \tilde{\delta}_i \times \delta_j)^{-1} \neq 0 \text{ and } L(1, \delta_i^* \times \tilde{\delta}_j)^{-1} \neq 0, 1 \leq i < j \leq m \\ L(1, \tilde{\delta}_i \times \tilde{\mathcal{T}}_1)^{-1} \neq 0 \text{ and } L(1, \tilde{\delta}_i, r)^{-1} \neq 0, i = 1, \ldots, m. \end{cases}$$

To work with these constraints we need the following two lemmas.

Lemma 4.1 *Suppose that $\Delta = [|\det|^{-n}\rho, |\det|^{n}\rho]$ and $\Delta' = [|\det|^{-n'}\rho', |\det|^{n'}\rho']$ are segments, such that ρ and ρ' are unitary, and $\alpha \geq 0$ is a real number. Write $\delta(\Delta)$ and $\delta(\Delta')$ for discrete series attached to Δ and Δ', respectively. Then $L\big(s, \delta(\Delta) \times \delta(\Delta')\big)$ is holomorphic for $\mathrm{Re}(s) > 0$, and it has a pole at $s = -\alpha$ if and only if*

$$\rho' \cong \tilde{\rho}, \quad n' + \alpha \geq n \geq |\alpha - n'| \quad \text{and} \quad n + n' - \alpha \in \mathbb{Z}.$$

Proof This is [M, Corollary 2.1]. ■

Lemma 4.2 *Let Δ be as in Lemma 4.1. Assume $\rho \cong \rho^*$. Let $\alpha > 0$ be a real number. If r is defined as in (2.2), then $L\big(s, \delta(\Delta), r\big)$ is holomorphic for $\mathrm{Re}(s) > 0$, and it has a pole at $s = 1 - 2\alpha$ if and only if one of the following holds*

(i) $L(0, \rho, r) = \infty$, $n + \alpha \in \frac{1}{2} + \mathbb{Z}$, *and* $\alpha \leq n + 1/2$.

(ii) $L(0, \rho, r) \neq \infty$, $n + \alpha \in \mathbb{Z}$, *and* $\alpha \leq n$.

Finally, $L\big(s, \delta(\tilde{\Delta}), r\big) = \overline{L\big(\bar{s}, \delta(\Delta), r\big)}$.

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Proof This follows directly from [Sh2, Proposition 8.1] and [G, Theorem 5.6]. The last formula follows from [Sh1, Proposition 7.8]. ∎

Next, we shall show the following lemma.

Lemma 4.3 *Assume that* $G = \mathrm{SO}(n,n)$, $G' = \mathrm{SO}(0,0)$, *and* $l_m = 1$. *Then* $\alpha_m \neq 0$.

Proof Assume that $\alpha_m = 0$. Thus, δ_m is a unitary character of $F^\times$, $\delta_m^2 = 1$. Then $\gamma(s, \delta_m \times \mathcal{T}_1, \psi_F)$ and $\gamma(2s, \delta_m, \psi_F, r)$ are trivial. Thus, the assumption (ii) of Theorem 3.1 must be applied to $\gamma(s, \delta_m \times \mathcal{T}, \psi_F)$. Next, this factor is, by definition [Sh1], equal to

$$\gamma(s, \delta_m \times \delta_m, \psi_F)^2 \prod_{i=1}^{m-1} \gamma(s, \delta_m \times \delta_i, \psi_F)\gamma(s, \delta_m \times \tilde{\delta}_i, \psi_F).$$

Now, Lemma 4.1 implies that $\gamma(0, \delta_m \times \delta_m, \psi_F) = 0$. Then, by the assumption (ii) of Theorem 3.1, there exists i, $1 \leq i \leq m-1$, such that $\gamma(0, \delta_m \times \delta_i, \psi_F) = \infty$ or $\gamma(0, \delta_m \times \tilde{\delta}_i, \psi_F) = \infty$. By Lemma 4.1, $\gamma(0, \delta_m \times \delta_i, \psi_F) = \infty$ is not possible. Thus, $\gamma(0, \delta_m \times \tilde{\delta}_i, \psi_F) = \infty$. This contradicts (e). ∎

Remark 4.1 In the remainder of the proof, we shall assume $\alpha_m > 0$. Otherwise, we can conjugate the standard module of $\mathcal{T}$ by $\epsilon \in O(n,n)$, $\det \epsilon = -1$, to accomplish that.

Now, we are ready to unfold the assumption (ii) of Theorem 3.1.

Lemma 4.4 *For any discrete series representation* $\delta \in \mathrm{Irr}$, $\delta \cong \delta^*$, *the function defined by* $L(s, \delta \times \mathcal{T}_1)L(2s, \delta, r) \cdot \psi(s, \delta)$, *where*

$$\psi(s, \delta) = \prod_{i=1}^{m} L(-\alpha_i + s, \delta \times \tilde{\delta}_i^u) \cdot L(1 - \alpha_i - s, \delta \times \tilde{\delta}_i^u)^{-1},$$

is non-zero at $s = 0$ *and it can have at most a simple pole there.*

Proof It follows from the definition [Sh1, Theorem 3.5 and Section 7]

$$\gamma(s, \delta \times \mathcal{T}, \psi_F) = \gamma(s, \delta \times \mathcal{T}_1, \psi_F) \prod_{i=1}^{m} \gamma(s, \delta \times \tilde{\delta}_i, \psi_{F'})\gamma(s, \tilde{\delta}^* \times \delta_i, \psi_{F'}).$$

Now, the assumption (ii) of Theorem 3.1 and (d) imply the lemma. In this paper, all L-functions attached to tempered representations are holomorphic for $\mathrm{Re}(s) > 0$. (See Lemma 1.1.) ∎

Lemma 4.5 $\mathcal{T}_1$ *is square integrable.*

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Proof This has exactly the same proof as Step 3 on [M, p. 721]. ■

Now, we shall fix a supercuspidal representation $\rho \in \mathrm{Irr}$, $\rho \cong \rho^*$, and $u \in \{0, 1/2\}$. Then we can consider the set S of all i, $i = 1, \ldots, m$, such that

$$\rho_i \cong \rho \quad \text{and} \quad n_i + \alpha_i \in u + \mathbb{Z}.$$

By (d), we can assume that $S \neq \varnothing$. Let $i_0 \in S$ such that $n_{i_0} + \alpha_{i_0}$ is maximal among all $n_i + \alpha_i$, $i \in S$.

Lemma 4.6 *Let $i \in S$. Let r be given by (2.2). Then we have*

(i) $n_{i_0} + \alpha_{i_0} > n_i + \alpha_i$, $i \neq i_0$.
(ii) *If $L(0, \rho, r) = \infty$, then $n_i + \alpha_i \in 1/2 + \mathbb{Z}$. If $L(0, \rho, r) \neq \infty$, then $n_i + \alpha_i \in \mathbb{Z}$.*
(iii) $\alpha_i - n_i \geq 1$, *for all $i \in S$.*

Proof The segment $\Delta = [|\det|^{-n_{i_0}-\alpha_{i_0}}\rho, |\det|^{n_{i_0}+\alpha_{i_0}}\rho]$ is well-defined. Let δ be the corresponding discrete series [Ze]. Note that $\delta^* \cong \delta$. Now, by Lemma 4.1 $L(1-\alpha_i, \delta \times \tilde{\delta}_i^u)^{-1} \neq 0$, for all i, and $L(-\alpha_{i_0}, \delta \times \tilde{\delta}_{i_0}^u)^{-1} = 0$. Hence $\psi(s, \delta)$ has a pole at $s = 0$. Lemma 4.4 implies that this pole is simple. Then, the defining formula for $\psi(s, \delta)$ implies $L(-\alpha_i, \delta \times \tilde{\delta}_i^u)^{-1} \neq 0$, for $i \neq i_0$. Now, by Lemma 4.1, we obtain (i). Also, since $\psi(s, \delta)$ has a pole at $s = 0$, we must have $L(0, \delta, r)^{-1} \neq 0$. Now, we can apply Lemma 4.2 to obtain (ii).

Finally, suppose that for some $i \in S$ we have $\alpha_i - n_i < 1$. Then, Lemma 4.2 and (ii) imply $L(1, \tilde{\delta}_i, r)^{-1} = 0$. This contradicts (e). ■

The previous lemma enables us to define a segment in the following way. Put $l = \alpha_{i_0} - n_{i_0} - 1$ and $\Delta = [|\det|^{-l}\rho, |\det|^{l}\rho]$. Let δ be the square integrable representation attached to Δ [Ze]. Clearly, we have $\delta^* \cong \delta$.

Lemma 4.7 $L(0, \delta \times \mathcal{T}_1) = \infty$.

Proof Let S' be the set of all $i \in S$ such that $n_i + \alpha_i \geq -n_{i_0} + \alpha_{i_0}$. Clearly, $i_0 \in S'$. Since $\mathcal{T}$ is generic, segments Δ_{i_0} and Δ_i are not linked, for all $i \in S$. Now, Lemma 4.6 (i) implies

$$\begin{cases} -n_i + \alpha_i \geq -n_{i_0} + \alpha_{i_0}, & i \in S', \text{ and} \\ \alpha_{i_0} - n_{i_0} - 1 > n_i + \alpha_i, & i \in S \setminus S'. \end{cases}$$

Now, applying Lemma 4.1, $L(-\alpha_i, \delta \times \tilde{\delta}_i^u)^{-1} \neq 0$, for all i, $1 \leq i \leq m$. Next, again by Lemma 4.1, $L(1-\alpha_{i_0}, \delta \times \tilde{\delta}_{i_0}^u)^{-1} = 0$. It follows that $\psi(0, \delta) = 0$. Further, Lemma 4.4 implies $L(0, \delta \times \mathcal{T}_1)^{-1} L(0, \delta, r)^{-1} = 0$. Note that $L(0, \delta, r)^{-1} \neq 0$ by Lemmas 4.2 and 4.6 (ii). ■

Lemma 4.7 finishes the proof of the theorem. First, combining by [Sh1, Proposition 7.8], we obtain

$$L(0, \tilde{\delta} \times \tilde{\mathcal{T}}_1) = \overline{L(0, \delta \times \mathcal{T}_1)} = \infty.$$

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Next, put $\delta' = \delta^u_{i_0}$. Then the theorem proved in the next section implies that for any $s_0 \in \mathbb{R}$

$$\operatorname{ord}_{s_0} L(s, \tilde{\delta}' \times \tilde{\mathfrak{T}}_1) \geq \operatorname{ord}_{s_0} L(s, \tilde{\delta}' \times \delta),$$

where ord_{s_0} is the order of the pole at s_0 of the corresponding L-function. Now, since Lemma 4.1 implies $L(1 - \alpha_0, \tilde{\delta}' \times \delta) = \infty$, we obtain $L(1, \tilde{\delta}_{i_0} \times \tilde{\mathfrak{T}}_1) = \infty$. This contradicts (e). The theorem is proved.

5 A Result on L-Functions

Theorem 5.1 *Assume that $\mathfrak{T}$ is a generic discrete series of $G = G_n$ $(n \geq 1)$ and $G \neq \mathrm{SO}(1,1)$. Let $\delta \in \mathrm{Irr}$ be in discrete series, and assume that $L(s, \delta \times \mathfrak{T})$ has a pole at $s = 0$. Then, for each discrete series $\delta' \in \mathrm{Irr}$ and for any $s_0 \in \mathbb{R}$,*

$$\operatorname{ord}_{s_0} L(s, \delta' \times \mathfrak{T}) \geq \operatorname{ord}_{s_0} L(s, \delta' \times \tilde{\delta}),$$

where ord_{s_0} is the order of the pole at s_0 of the corresponding L-function.

Remark 5.1 Theorem 5.1 can be explained in terms of the conjectural lifting (see [B]) $\mathrm{Irr}(G_n) \to \mathrm{Irr}\big(\mathrm{GL}(m, F')\big)$, associated to the natural embedding of the Langlands dual groups $\hat{G}_n(\mathbb{C}) \hookrightarrow \hat{H}(\mathbb{C})$, $H = R_{F'/F}\,\mathrm{GL}\big(m(n)\big)$, as follows. (See Definition 2.1 for the notation.) First, the lift of $\mathfrak{T}$ must be a tempered representation of the form $\delta_1 \times \cdots \times \delta_k$, where $\delta_1, \ldots, \delta_k$ are in discrete series, and $\delta_i \ncong \delta_j$ $(i \neq j)$. Next, since L-functions should be preserved under the lifting [B], [Sh1], we have $L(s, \delta \times \mathfrak{T}) = \prod_{i=1}^{k} L(s, \delta \times \delta_i)$. Now, since $L(s, \delta \times \mathfrak{T})$ has a pole at $s = 0$, Lemma 4.1 implies that there exists i_0 such that $\tilde{\delta} \cong \delta_{i_0}$. Finally, since local L-functions never vanish and $L(s, \delta' \times \mathfrak{T}) = \prod_{i=1}^{k} L(s, \delta' \times \delta_i)$, we obtain the theorem.

Proof of Theorem 5.1 Write $\Delta = [|\det|^{-n}\rho, |\det|^{n}\rho]$ and $\Delta' = [|\det|^{-n'}\rho', |\det|^{n'}\rho']$ for the segments attached to δ and δ', respectively. (Here ρ and ρ' are unitary and supercuspidal.) Lemma 4.1 implies that $\operatorname{ord}_{s_0} L(s, \delta' \times \tilde{\delta}) = 0$, for any $s_0 \in \mathbb{R}$, if $\rho' \ncong \rho$. Then Theorem 5.1 holds. Thus, we will assume $\rho' \cong \rho$. We start by the following lemma.

Lemma 5.1 *If $\mathfrak{T}$ and $\delta = \rho$ are supercuspidal, then Theorem 5.1 holds.*

Proof This lemma has exactly the same proof as [M, Proposition 2.1]. There we need a representation [M, p. 711] constructed by Tadić. We can repeat the same proof with the (generic) square integrable subrepresentation of the induced representation of G_{m+n}, induced from $|\det|\rho \otimes \mathfrak{T}$. This representation exists by [Sh1, Theorem 8.1] since $L(s, \rho \times \mathfrak{T})$ has a pole at $s = 0$. ■

In general, choose a standard parabolic subgroup $P = MN$, $M \cong M_\alpha$ (see (2.1)), and a (generic) supercuspidal representation $\omega_1 \otimes \cdots \otimes \omega_k \otimes \sigma$ of M such that $\mathfrak{T}$ is a subquotient of $\mathrm{Ind}_P^G(\omega_1 \otimes \cdots \otimes \omega_k \otimes \sigma)$. The property $(*)$ holds. (See Section 3.)

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Next, the multiplicative properties of γ-factors [Sh2, Corollary 5.6] and Remark 5.2 imply

$$(5.1)\quad \gamma(s,\delta'\times\mathcal{T},\psi_F)=\gamma(s,\delta'\times\sigma,\psi_F)\prod_{i=1}^{k}\gamma(s,\delta'\times\tilde{\omega}_i,\psi_{F'})\gamma(s,\delta'\times\tilde{\omega}_i^*,\psi_{F'}).$$

Remark 5.2 Assume that δ_1 and δ_2 are essentially square integrable representations in Irr. Then $\gamma(s,\delta_1^{\tau}\times\delta_2^{\tau},\psi_{F'})=\gamma(s,\delta_1\times\delta_2,\psi_{F'})$.

Now, we will look at the set S of all i such that $\omega_i^u\cong\rho$. Without the loss of generallity we may assume $S=\{1,\dots,k'\}$ if S is not empty. Then we may form a representation of a general linear group as follows. Take the unique irreducible generic subquotient π of

$$(5.2)\qquad \begin{cases}\tilde{\omega}_1\times\cdots\times\tilde{\omega}_{k'}\times\tilde{\omega}_{k'}^*\times\cdots\times\tilde{\omega}_1^*, & \text{if } L(0,\rho\times\sigma)\neq\infty\\ \tilde{\omega}_1\times\cdots\times\tilde{\omega}_{k'}\times\tilde{\rho}\times\tilde{\omega}_{k'}^*\times\cdots\times\tilde{\omega}_1^*, & \text{if } L(0,\rho\times\sigma)=\infty.\end{cases}$$

We have

Lemma 5.2 *Let $\psi(s,\delta')=\prod_{i=k'+1}^{k}\gamma(s,\delta'\times\tilde{\omega}_i,\psi_{F'})\gamma(s,\delta'\times\tilde{\omega}_i^*,\psi_{F'})$. (If S is empty, $k'=0$, and if $S=\{1,\dots,k\}$, then $\psi(s,\delta')=1$.) The function $\psi(s,\delta')$ has neither real poles nor real zeros.*

Proof This follows from Lemma 4.1. Note that a necessary condition for the existence of a real pole or a real zero is $\omega_i^u\cong\rho$, for some $i\in S$. ■

Lemma 5.3

(i) $L(s,\delta'\times\sigma)$ has a real pole if and only if $L(0,\rho\times\sigma)=\infty$. The unique pole is $s=-n'$, and it is a simple pole.
(ii) The only real pole of $L(s,\delta'\times\tilde{\rho})$ is $s=-n'$. It is a simple pole.

Proof (i) follows from [Sh1, Proposition 7.3], combined with the next formula [Sh2, Theorem 5.5]

$$L(s,\delta'\times\sigma)=L(s+n',\rho\times\sigma).$$

(ii) follows from Lemma 4.1. ■

Now, since $L(0,\delta\times\mathcal{T})=\infty$, we have that $s=0$ is a zero of $\gamma(s,\delta\times\mathcal{T},\psi_F)$. Now, if S is empty, we obtain from (5.1), with $\delta'=\delta$, that $\gamma(0,\delta\times\sigma,\psi_F)=0$. This implies $L(0,\delta\times\sigma)=\infty$. By Lemma 5.3, we obtain $\delta=\rho$. Thus, if S is empty we let $\pi=\tilde{\rho}$. Now, (5.1) implies the following lemma.

Lemma 5.4

(i) If $L(0,\rho\times\sigma)\neq\infty$, then

$$(5.3)\qquad \gamma(s,\delta'\times\mathcal{T},\psi_F)=\gamma(s,\delta'\times\pi)\gamma(s,\delta'\times\sigma,\psi_F)\psi(s,\delta').$$

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(ii) If $L(0,\rho\times\sigma)=\infty$, then

$$\gamma(s,\delta'\times\mathfrak{T},\psi_F)\gamma(s,\delta'\times\tilde{\rho},\psi_{F'})=\gamma(s,\delta'\times\pi)\gamma(s,\delta'\times\sigma,\psi_F)\psi(s,\delta'). \tag{5.4}$$

Lemma 5.5 *π is tempered.*

Proof First, since π is generic, [Ze, Theorem 9.7] enables us to take a sequence of segments

$$\Delta_i=[|\det|^{-k_i+\alpha_i}\tilde{\rho},|\det|^{k_i+\alpha_i}\tilde{\rho}],\quad 2\alpha_i\in\mathbb{Z},\quad 2k_i\in\mathbb{Z}_+,\quad i=1,\dots,l,$$

such that $\pi\cong\delta(\Delta_1)\times\cdots\times\delta(\Delta_l)$. (Note that $\tilde{\rho}\cong(\tilde{\rho})^*$ and $2\alpha_i\in\mathbb{Z}$ follow from property $(*)$.) Next, (see for example [M, Lemma 2.1])

$$L(s,\rho\times\pi)=\prod_{i=1}^{l}L(s+k_i-\alpha_i,\rho\times\tilde{\rho}),$$

and $L(s,\rho\times\pi)$ will have a pole at $s_j=\alpha_j-k_j$. Since $L(s,\rho\times\pi)$ and $L(1-s,\tilde{\rho}\times\tilde{\pi})$ have no common poles [H, Proposition 4.5], we obtain that s_j is a zero $\gamma(s,\rho\times\pi,\psi_F)$. We shall show $s_j\le 0$. If not, then since $\psi(s,\rho)$ is holomorphic and non-zero at $s=s_j$, and the left-hand sides in (5.3) and (5.4) are non-zero, we conclude that $\gamma(s,\rho\times\sigma,\psi_F)$ has a pole for $s=s_j$. This means that $L(1-s_j,\rho\times\sigma)=\infty$. Since $s_j\in\mathbb{R}$, $s_j=1$. Thus we must have $L(0,\rho\times\sigma)=\infty$. Now, since $\gamma(s,\rho\times\tilde{\rho},\psi_F)$ has a pole at $s=1$. We see that $\gamma(s,\rho\times\sigma,\psi_F)$ has a double pole for $s=1$. This a contradiction since γ-factors attached to supercuspidal representations have simple poles [Sh1, Proposition 7.3].

Finally, we shall show that π is tempered. We need to prove $\alpha_j=0$, $j=1,\dots,l$. Suppose that $\alpha_j\neq 0$, for some j, $1\le j\le l$. Because of $\pi\cong\pi^*$, there exists i, $i\neq j$, such that $k_i=k_j$, and $\alpha_i=-\alpha_j$. We may assume that $\alpha_j>0$. Next we know that $k_j-\alpha_j\ge 0$. This, together with $2\alpha_j\in\mathbb{Z}_+$, implies that Δ_i and Δ_j are linked. Then, by [Ze, Thm. 9.7], the induced representation π is reducible. This is a contradiction. ∎

Now, by the definition of local L-functions for tempered representations [Sh1, Section 7], Lemma 5.2, and Lemma 5.4 we obtain the following lemma.

Lemma 5.6 *If $L(0,\rho\times\sigma)=\infty$, then*

$$\operatorname{ord}_{s_0}L(s,\delta'\times\mathfrak{T})+\operatorname{ord}_{s_0}L(s,\delta'\times\rho)=\operatorname{ord}_{s_0}L(s,\delta'\times\pi)+\operatorname{ord}_{s_0}L(s,\delta'\times\sigma).$$

If $L(0,\rho\times\sigma)\neq\infty$, then

$$\operatorname{ord}_{s_0}L(s,\delta'\times\mathfrak{T})=\operatorname{ord}_{s_0}L(s,\delta'\times\pi)+\operatorname{ord}_{s_0}L(s,\delta'\times\sigma).$$

As a consequence of Lemma 5.6, we obtain the following formula

$$\operatorname{ord}_{s_0}L(s,\delta'\times\mathfrak{T})\ge\operatorname{ord}_{s_0}L(s,\delta'\times\pi)=\sum_{i=1}^{l}\operatorname{ord}_{s_0}L\big(s,\delta'\times\delta(\Delta_i)\big), \tag{5.5}$$

for any $s_0\in\mathbb{R}$. Also, we have the following corollary.

Corollary 5.1 *There exists i such that $\delta\cong\delta(\tilde{\Delta}_i)$.*

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Proof First, by the assumption $L(0, \delta \times \mathcal{T}, \psi_F) = \infty$. Then, Lemmas 5.1, 5.3 and 5.6 imply $L(0, \delta \times \pi) = \infty$. Since

$$L(s, \delta \times \pi) = \prod_{i=1}^{l} L\big(s, \delta \times \delta(\Delta_i)\big),$$

there exists i such that $L\big(s, \delta \times \delta(\Delta_i)\big)$ has a pole for $s = 0$. Applying Lemma 4.1, we obtain the lemma. ■

By (5.5) and Corollary 5.1

$$\mathrm{ord}_{s_0} L(s, \delta' \times \mathcal{T}) \geq \mathrm{ord}_{s_0} L(s, \delta' \times \tilde{\delta}).$$ ■

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