ON THE FINITE SIMILARITY GROUPS

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A result contained in a previous paper [1] of the author is

THEOREM 1. *If* (G, K, H) *is a* T_3*-triple,* G *is finite,* a *is an involution contained in* $G-N(K)$ *and* $H \cap K^a = 1$*, then the factor group of* G *over its centre is isomorphic to a group of similarities over a finite field.*

Here we refine this result as follows

THEOREM 2. *Under the hypotheses of Theorem* 1, G *is the direct product of an abelian group and a group isomorphic to a group of similarities over a finite field.*

The converse of this theorem is also true as will be pointed out in section 4.

The concept of a T_3-triple is defined in the next section.

1. Notations and Definitions

$H \lhd G$ means that H is a normal subgroup of the group G; $N(H)$ is the normalizer of H; $C(h)$ is the centralizer of h; $|H|$ is the order of H; $(G; H)$ is the index of H in G; $a^x = x^{-1}ax$; $H^x = x^{-1}Hx$; if $H \lhd G$, then G/H is the factor group of H in G; $Z(G)$ is the centre of G; $G-H$ is the set of elements contained in G but not in H; $A \times B$ is the direct product of the groups A and B; if F is a field, then $S(2, F)$ is the group of similarity mappings $z \to az+b$, $a \neq 0$ over the field F.

DEFINITION. If K and H are subgroups of a group G, $H \lhd K$ and whenever a and a^b are members of $G-K$ there exists exactly one $h \in H$ with the property $a^h = a^b$, then (G, K, H) is called a T_3-triple and G is called a T_3-group.

2. Previous results

In order to avoid repetition we shall use some of the results of [1] without proof. The most important of these is Theorem 1 stated at the beginning of this paper but in addition we have

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Proposition 2.1. $Z(G) = K \cap K^a$.

Proposition 2.2. *With F as defined in Theorem 1, H is isomorphic to the group of mappings $z \to az$, $a \neq 0$, over F.*

Proposition 2.3. $K = H \times Z(G)$.

Proposition 2.4. *G is doubly transitive on the cosets of K in G.*

Proposition 2.5. $N(K) = K$.

Proposition 2.6. $|G| = |K|(|H|+1)$.

All these propositions hold under the assumption of the hypotheses of Theorem 1. In addition we will use the properties of the similarity groups where necessary.

3. The proof of Theorem 2

Throughout this section we will assume that (G, K, H) is a T_3-triple satisfying the hypotheses of Theorem 1. We will denote the order of H by $n-1$ and the order of $Z(G)$ by m. From propositions 2.3 and 2.6 it then follows that $|G| = n|K| = mn(n-1)$.

Lemma 3.1. *K is abelian.*

Proof. By propositions 2.2 and 2.3 K is the direct product of two abelian groups and is hence abelian.

Lemma 3.2. *If $g \in G$ then either*
(1) *$g \in Z(G)$ or*
(2) *g has exactly one conjugate in K and has n conjugates in G, or*
(3) *g has no conjugate in K and has $n-1$ conjugates in G.*

Proof. We will show first that no element of G has two conjugates in K.

Suppose $g^x \in K$, $g^y \in K$, $g^x \neq g^y$. Then $g \in K^{x^{-1}} \cap K^{y^{-1}}$. If $K^{x^{-1}} \neq K^{y^{-1}}$, then, by Proposition 2.4, g has a conjugate in $K \cap K^a$ which is the centre of G (Proposition 2.1). Hence g is contained in the centre of G which is a contradiction.

Alternatively $K^{x^{-1}} = K^{y^{-1}}$ so that $x^{-1}y \in N(K) = K$ by Proposition *2.5. Now $g^x \in K$ so that by Lemma 3.1 $(g^x)^{x^{-1}y} = g^x$. Hence $g^x = g^y$ which* is a contradiction.

Hence no element of G has two conjugates in K.

Now, by the property T_3, if an element of G has a conjugate outside K it has exactly $|H| = n-1$ conjugates outside K. This proves Lemma 3.2.

By Theorem 1, $G/Z(G)$ is isomorphic to a group $S(2, F)$. The group of translations $z \to z+b$ is a normal subgroup of this group so that G has a subgroup D which contains $Z(G)$ and has the property that $D/Z(G)$ is isomorphic to this group of translations. Moreover $D/Z(G) \lhd G/Z(G)$ so that $D \lhd G$.

LEMMA 3.3. *D consists of the centre of G together with all the elements of G which leave no coset of K fixed.*

PROOF. We note first that $D/Z(G)$ consists of the elements of $G/Z(G)$ which leave no coset of $K/Z(G)$ fixed, together with the identity $Z(G)$ of $G/Z(G)$.

If $g \in G$ and g fixes a coset of K in G then the coset $gZ(G)$ fixes a coset of $K/Z(G)$ in $G/Z(G)$. Conversely if g fixes no coset of K in G, then $gZ(G)$ fixes no coset of $K/Z(G)$ in $G/Z(G)$.

LEMMA 3.4. *D is abelian.*

PROOF. Suppose $g \in D-Z(G)$. Then by Lemma 3.2 g has $n-1$ conjugates in G. But G has order $mn(n-1)$ so that $C(g)$ has order mn. Now $Z(G) \subseteqq C(g)$ so that $C(g)/Z(G)$ is defined and is a subgroup of $G/Z(G)$ of order n. But the only subgroup of $G/Z(G)$ having order n is $D/Z(G)$ so that $C(g)/Z(G) = D/Z(G)$ and so $C(g) = D$.

Hence $D-Z(G) \subseteqq Z(D)$ and clearly $Z(G) \subseteqq Z(D)$ which proves the lemma.

LEMMA 3.5. *Let σ be a fixed element of H, $\sigma \neq 1$, and let E be the set of elements of G of the form $x^{-1}x^{\sigma}$, $x \in D$. Then E is a complement of $Z(G)$ in D, i.e. $D = Z(G) \times E$.*

PROOF. $D \lhd G$ so that if $x \in D$, also $x^{\sigma} \in D$. Hence $E \subseteqq D$.

If $x^{-1}x^{\sigma} \in D$ and $y^{-1}y^{\sigma} \in D$, then $(x^{-1}x^{\sigma})(y^{-1}y^{\sigma})^{-1} = (xy^{-1})^{-1}(xy^{-1})^{\sigma}$ since D is abelian. Hence $(x^{-1}x^{\sigma})(y^{-1}y^{\sigma})^{-1} \in E$ so that E is a subgroup of D.

If $x^{-1}x^{\sigma} \in Z(G)$ then $x^{\sigma} \in xZ(G)$. Hence $\sigma Z(G)$ commutes with $xZ(G)$ in $G/Z(G)$. Now $\sigma Z(G)$ is not a translation and hence commutes with no translation of $G/Z(G)$. Hence $xZ(G) = Z(G)$ i.e. $x \in Z(G)$. Thus $x^{\sigma} = x$ so that $x^{-1}x^{\sigma} = 1$. Hence $Z(G) \cap E = 1$.

Suppose $g \in D$. Then $gZ(G) \in D/Z(G)$ which is the group of translations of $G/Z(G)$. Hence there exists $x \in D$ such that

$$(xZ(G))^{-1}(xZ(G))^{\sigma Z(G)} = gZ(G).$$

Then $x^{-1}x^{\sigma} \in gZ(G)$ so that $D = Z(G)E$.

Clearly $E \lhd D$ (D abelian) so that $D = Z(G) \times E$.

LEMMA 3.6. *$E \lhd G$.*

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Proof. Suppose $x^{-1}x^{\sigma} \in E$, $x \in D$, $x^{-1}x^{\sigma} \neq 1$ and suppose $g \in G$. By Lemma 3.5, $x^{-1}x^{\sigma} \in D - Z(G)$ so that by Lemma 3.3, $x^{-1}x^{\sigma}$ has no conjugate in K. Hence, by the property T_3 there exists $h \in H$ such that $(x^{-1}x^{\sigma})^h = (x^{-1}x^{\sigma})^g$. But H is abelian so that $(x^{-1}x^{\sigma})^h = (x^h)^{-1}(x^h)^{\sigma} \in E$. Thus $(x^{-1}x^{\sigma})^g \in E$ so that $E \lhd G$.

Lemma 3.7. *HE is a normal subgroup of G and is a complement of $Z(G)$ in G i.e. $G = Z(G) \times HE$.*

Proof. $E \lhd G$ and H is a subgroup of G so that HE is a subgroup of G.

To prove that HE is a normal subgroup of G it will be sufficient to show that HE contains every conjugate of every element of H.

Suppose $h \in H$, $h \neq 1$. Then by Lemma 3.2, h has n conjugates in G. If $x, y \in E$, $x \neq y$ and $h^x = h^y$, then $h^{xy^{-1}} = h$, so that $(xy^{-1})^h = xy^{-1}$. Now, from Lemmas 3.3 and 3.5, xy^{-1} has no conjugate in K so that by the property T_3 there is a unique $h_1 \in H$ with the property $(xy^{-1})^{h_1} = xy^{-1}$, namely $h_1 = 1$. Hence $h = 1$ which is a contradiction. Hence conjugates of h by different elements of E are different.

Now $|D| = mn$ and $|Z(G)| = m$ so that by Lemma 3.5, $|E| = n$. Hence if $h \in H$, $h \neq 1$, h has n conjugates in HE by elements of E and by the above this is the complete set of conjugates of h in G. Hence $HE \lhd G$.

Clearly $HE \cap Z(G) = 1$ and $|HE| = n(n-1)$, $|Z(G)| = m$ so that $Z(G)HE = G$.

Hence we have proved that $G = Z(G) \times HE$.

This is sufficient to prove Theorem 2.

4. The converse of Theorem 2

It remains now to prove

Theorem 3. *Let F be a field and H the subgroup of $S(2, F)$ consisting of similarities of the form $z \to az$, $a \neq 0$. If A is an abelian group then $(A \times S(2, F), A \times H, H)$ is a T_3-triple and moreover satisfies the other hypotheses of Theorem 1.*

Proof. We prove only that $(A \times S(2, F), A \times H, H)$ is a T_3-triple.

$S(2, F)$ may be represented as the group of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, $a, b \in F$, $a \neq 0$ in which case H may be taken as the subgroup of matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $a \in F$, $a \neq 0$. $A \times S(2, F)$ may be then taken as the set of pairs of the form $\left(\alpha, \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right)$ with $\alpha \in A$, $a, b \in F$, $a \neq 0$ and multiplication defined elementwise.

A necessary condition that $\left(\alpha, \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right)$ and $\left(\beta, \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}\right)$ be conjugate is easily seen to be $\alpha = \beta$ and $a = c$.

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Suppose $\left(\alpha, \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right)$, $\left(\alpha, \begin{pmatrix} a & d \\ 0 & 1 \end{pmatrix}\right)$ are conjugate and neither are contained in $A \times H$. Then $b \neq 0$, $d \neq 0$. We must show that the equation

$$\left(1, \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right)\left(\alpha, \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) = \left(\alpha, \begin{pmatrix} a & d \\ 0 & 1 \end{pmatrix}\right)\left(1, \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right)$$

has a unique solution for x. This is easily seen to be $x = b^{-1}d$.

Reference

[1] Peter Lorimer 'On the Finite Two Dimensional Linear Groups'. Journal of Algebra (to appear).

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