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An Equivalent Form of Picard's Theorem and Beyond

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Abstract. This paper gives an equivalent form of Picard's theorem via entire solutions of the functional equation $f^2 + g^2 = 1$ and then its improvements and applications to certain nonlinear (ordinary and partial) differential equations.

1 Equivalent Form of Picard's Theorem

Picard's theorem states that an entire function, *i.e.,* a complex-valued function differentiable in the complex plane **C**, omitting 0 and 1 must be constant. It also easily implies, by a linear transform, the meromorphic version of the theorem that a meromorphic function in **C** omitting three distinct values must be constant. Picard's theorem is among the most striking results in complex analysis and plays a decisive role in the development of the theory of entire and meromorphic functions and other applications. Different proofs of Picard's theorem are known (see [1, 2, 4–6, 8, 11, 13], etc.). In this article, we connect Picard's theorem to entire solutions of the simple-looking functional equation $f^2 + g^2 = 1$ and give an equivalent form of Picard's theorem through the common zeros f', g' in an elementary way, which is partially motivated by [9], where we characterized meromorphic solutions $f(z_1, z_2)$ and $g(z_1, z_2)$ of the equation $f^2 + g^2 = 1$ in $\mathbf{C}^2$ with an application to certain nonlinear partial differential equations. The theorem will lead to a further result on the equation in Section 2, which improves Picard's theorem and thus also contains another proof of Picard's theorem (with the implications: Theorem 2.1 ⇒ Theorem 1.1 ⇒ Picard's Theorem, and also Theorem 2.1 ⇒ Corollary 2.3 ⇒ Picard's Theorem; see below). Applications to nonlinear differential equations will be given in Section 3.

Theorem 1.1 *Entire solutions f and g of the functional equation $f^2 + g^2 = 1$ in* **C** *are constant if and only if $Z(f') = Z(g')$ (counting multiplicities).*

Here and in the sequel, $Z(h)$ for a function h denotes the set of zeros of h (counting or not counting multiplicities, specified in the question). Thus, the condition in Theorem 1.1 means that f' and g' have the same zeros with counting multiplicities.

Theorem 1.1 is equivalent to Picard's Theorem in the sense that one implies the other.

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Theorem 1.1 implies Picard's Theorem Assume that an entire function F omits 0 and 1. Then $F = e^p$ and $1 - F = e^q$ for two entire functions p, q. Thus, we can write $F = f^2$ and $1 - F = g^2$, where $f = e^{\frac{1}{2}p}$ and $g = e^{\frac{1}{2}q}$. Clearly, $f^2 + g^2 = 1$, and then $ff' = -gg'$, which clearly implies that f' and g' have the same zeros (counting multiplicities), since f and g omit 0. By Theorem 1.1, f and g are constant, and thus F is constant. ■

Picard's Theorem implies Theorem 1.1 For sufficiency of Theorem 1.1, write the given equation to $(f + ig)(f - ig) = 1$. Denote $f + ig = h$. Clearly, $h \not\equiv 0$. We then have that $f - ig = h^{-1}$. Thus,

$$f = \frac{h + h^{-1}}{2}, \quad g = \frac{h - h^{-1}}{2i},$$

which implies that

$$f' = \frac{h'}{2}\frac{h^2 - 1}{h^2}, \quad g' = \frac{h'}{2i}\frac{h^2 + 1}{h^2}.$$

We claim that h omits ± 1 (and also $\pm i$). This can be shown by the following argument, which works also for meromorphic f and g (see Remark 1.2). In fact, if $h^2(z_0) = 1$ for some z_0, then $f'(z_0) = 0$, and thus $g'(z_0) = 0$ by the given condition. But then the order of z_0 as a zero of f' (equal to the sum of the order as a zero of $h^2 - 1$ and the order as a zero of h') is higher than that of z_0 as a zero of g' (equal to the order of z_0 as a zero of h'), a contradiction to the assumption that f' and g' have the same zeros with counting multiplicities. This shows that h^2 omits 1; that is, h omits $-1, 1$. (In the same way, by considering the zeros of $h^2 + 1$, we see that h omits $-i$ and i.) This proves the claim. By Picard's theorem, h must be constant, which implies that f and g are constant, *i.e.*, the sufficiency of Theorem 1.1. The necessity of Theorem 1.1 is obvious. ■

Remark 1.2 As Picard's theorem extends to meromorphic functions, it is natural to ask if Theorem 1.1 holds also for meromorphic functions. The above proof of Theorem 1.1 can go through without any changes if f, g are meromorphic functions, since the function h in the proof actually omits four values and thus must be constant, if h is meromorphic (see the above proof). Thus, Theorem 1.1 does hold for meromorphic functions. We include entire solutions in the statement of Theorem 1.1 for the equivalent form parallel to Picard's theorem.

It is also natural to ask if the condition "counting multiplicities" may be improved as "ignoring multiplicities", which will be answered in Theorem 2.1 of the next section.

2 Improvements

It turns out that when the multiplicities are ignored, Theorem 1.1 still holds. However, the proof there does not go through any more. We will use a different argument, which is independent of Picard's theorem and actually yields a better result than Theorem 1.1 in two aspects: ignoring multiplicities and relaxing the set equality to set inclusion (see Theorem 2.1). The set inclusion condition turns out to be useful and makes it possible to apply to other questions, as seen in Corollary 2.3 and in applications to (ordinary and partial) differential equations in Section 3.

Theorem 2.1 *Entire solutions f and g of the functional equation $f^2 + g^2 = 1$ in $\mathbf{C}$ are constant if and only if $Z(f') \subseteq Z(g')$ (ignoring multiplicities).*

We give a proof of Theorem 2.1, invoking only the following properties of

$$m(r, f) := \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where $\log^+ x = \max\{0, \log x\}$, for a nonzero entire function f (see *e.g.*, [6, pp. 4, 55]):

(i) $m(r, \frac{1}{f}) \leq m(r, f) + O(1)$;

(ii) $m(r, \frac{f'}{f}) =: S(r, f) = o(m(r, f))$ outside a union of intervals of finite total length.

We note that Properties (i) and (ii) have more general forms, which are not needed in this paper.

Proof of Theorem 2.1 The necessity is obvious. For the sufficiency, write the given equation to $(f + ig)(f - ig) = 1$. Denote $f + ig = h$. Then h is an entire function omitting 0. We then have that $f - ig = h^{-1}$. Thus, we have that

$$f = \frac{h + h^{-1}}{2}, \quad g = \frac{h - h^{-1}}{2i},$$

which implies that

$$f' = \frac{h'}{2} \frac{h^2 - 1}{h^2}, \quad g' = \frac{h'}{2i} \frac{h^2 + 1}{h^2}.$$

Introduce the auxiliary function

$$H(z) = \frac{(h')^2}{(h-1)(h+1)}. \tag{2.1}$$

If $h^2(z_0) = 1$ for some z_0, then $f'(z_0) = 0$, and thus $g'(z_0) = 0$ by the given condition, which implies that $h'(z_0) = 0$. Thus, z_0 is not a pole of H by considering the order of a zero of $(h-1)(h+1) = h^2 - 1$. Hence, H is entire. If $H \equiv 0$, then h is constant and thus f, g are already constant. Assume that $H \not\equiv 0$. Applying property (ii), we have that $m(r, H) = S(r, h)$, and thus

$$m\Big(r, \frac{1}{H}\Big) \leq m(r, H) + O(1) = S(r, h)$$

by property (i). From (2.1) it follows that $h^2 - 1 = \frac{1}{H}(h')^2$ or

$$1 - \frac{1}{h^2} = \frac{1}{H} \frac{(h')^2}{h^2}.$$

Thus, we deduce, by property (ii) again, that

$$\begin{aligned} m(r, h) &= \frac{1}{2} m(r, h^2) \leq \frac{1}{2} m\Big(r, \frac{1}{h^2}\Big) + O(1) \\ &\leq \frac{1}{2} m(r, \frac{1}{H}) + \frac{1}{2} m\Big(r, \frac{(h')^2}{h^2}\Big) + O(1) = S(r, h), \end{aligned}$$

which implies that h is constant, and thus f, g are constant. ■

Remark 2.2 It is natural to ask if Theorem 2.1 could be pushed over to meromorphic solutions. The answer is negative. That is, there are nonconstant meromorphic functions f, g that satisfy the equation $f^2 + g^2 = 1$, and the zeros of f' are also zeros of g'. Here is a counter-example: Take any nonconstant entire function p and set $q = (e^p + 1)/(e^p - 1)$. Then we have that

$$q - 1 = \frac{2}{e^p - 1}, \quad q + 1 = \frac{2e^p}{e^p - 1}.$$

Thus, q does not assume 1 and −1. Now set

$$f = \frac{q + q^{-1}}{2}, \quad g = \frac{q - q^{-1}}{2i}.$$

Then f and g are (non-entire) meromorphic functions. It is easy to verify that $f^2 + g^2 = 1$ and that

$$f' = \frac{q'}{2}\frac{q^2 - 1}{q^2}, \quad g' = \frac{q'}{2i}\frac{q^2 + 1}{q^2}.$$

Clearly, the zeros of f' are also zeros of g' (even counting multiplicities), since q^2 omits 1. But, f and g are not constant.

We include an improvement of Picard's theorem in a more familiar form as a consequence of Theorem 2.1.

Corollary 2.3 *If f is an entire function that omits* 0*, and any zero of $f - 1$ is multiple, then f is constant.*

We note that Corollary 2.3 can be obtained from the Second Main Theorem of Nevanlinna (see *e.g.*, [6]), which is the core theorem of Nevanlinna theory and will not be used in this paper. We present a proof that follows from Theorem 2.1 in an elementary way.

Proof Make the linear transform $f = 2g - 1$. Then $g \neq \frac{1}{2}$ and the zeros of $g - 1$, if there are any, are all multiple. From the identity $(g-1)^2 = g^2 - 2g + 1$, we obtain an entire function h such that $(g-1)^2 - g^2 = 1 - 2g = e^{2h}$, since $1 - 2g$ omits 0. Thus,

$$\left(\frac{g-1}{e^h}\right)^2 + \left(i\frac{g}{e^h}\right)^2 = 1;$$

that is, $F^2 + G^2 = 1$, where $F = \frac{ig}{e^h}$ and $G = \frac{g-1}{e^h}$. Note that

$$g' = -e^{2h}h' = (2g-1)h'. \tag{2.2}$$

We deduce that

$$F' = i\frac{g' - gh'}{e^h} = \frac{(g-1)h'}{e^h}, \tag{2.3}$$

$$G' = \frac{g' - (g-1)h'}{e^h} = \frac{gh'}{e^h}. \tag{2.4}$$

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If $g(w) = 1$ at a w, then $f(w) = 1$, and thus $g'(w) = 0$ by the given condition. Then $h'(w) = 0$ by (2.2). Thus, any zero of F' must be a zero of h' and thus a zero of G'. That is, the condition of Theorem 2.1 is satisfied for F and G. Hence, F and G are constant, and then, by (2.3) and (2.4), h or g is constant, which implies that f is constant. ∎

3 Applications

Characterizing complex analytic solutions of differential equations is a topic of a long history (see *e.g.*, the monograph [7]). Given a nonlinear differential equation, there is, in general, no systematic way to find its solutions. A slight variation of an equation may require a different method. In fact, the consideration of the above new form of Picard's theorem and its improvement was led by the desire to solve certain nonlinear differential equations in complex variables. We include here two applications, one for ordinary differential equations and the other for partial differential equations, where the results of Section 2 can be applied to characterize their entire solutions. Here we do not intend to give the most general results, but rather to illustrate such applications to certain nonlinear differential equations, which can otherwise be hard to handle using available methods from differential equations.

Corollary 3.1 *Entire solutions of* $f^2 + h^2(f')^{2m} = 1$ *are exactly* $f = \pm 1$, *where* h *is an arbitrary entire function and* $m \geq 2$ *is an integer.*

It is an immediate consequence of Theorem 2.1, since $f^2 + g^2 = 1$ and $Z(f') \subset Z(g')$, where $g(z) = h(z)(f'(z))^m$, which implies that f must be constant and then $f = \pm 1$ from the given equation.

Remark 3.2 Corollary 3.1 fails to hold when $m = 1$. Here is a counterexample. Consider $f = \sin e^z$ and $h = e^{-z}$. Then $f^2 + h^2(f')^2 = 1$. But f is a transcendental entire solution of the equation. We refer the reader to [7] for related Briot–Bouquet type equations.

Next consider the following partial differential equations:

$$u^m(u_x^2 + u_y^2 + e^h) = e^h, \tag{3.1}$$

$(x, y) \in \mathbf{C}^2$, where h is an arbitrary entire function $\mathbf{C}^2$ and $m > 0$ is any integer. We refer the reader to [9, 10, 12] and references therein for related types of PDEs. When $h(z) \equiv 0$ and $m = 2$, equation (3.1) becomes the well-known PDE of tubular surfaces (see *e.g.*, [3, p. 95]); the two parameter family of the surfaces $(x-a)^2 + (y-b)^2 + u^2 = 1$ is a complete integral. The two parameter family has an envelope consisting of the planes $u = 1$ and $u = -1$, which are solutions of the equation. A different family of PDEs including the PDE of tubular surfaces was considered in [11] using relatively more advanced tools from several complex variables, which, however, does not apply to (3.1). We will characterize entire functions of (3.1) with a proof in an elementary manner invoking only the improved Picard's Theorem in Section 2.

Theorem 3.3 *Entire solutions of* (3.1) *in* $\mathbf{C}^2$ *are exactly the* m *unitary complex roots of* $u^m = 1$.

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Remark 3.4 It is natural to ask if the function h on both sides of (3.1) can be relaxed to different entire functions, *i.e.,* if the equation (3.1) can be generalized to

$$u^m\Big(u_x^2+u_y^2+e^{h(x,y)}\Big)=e^{g(x,y)}$$

with two arbitrary entire functions g, h. The answer is negative. To see this, consider the equation

$$u^2\Big((u_x^2+u_y^2)+e^{2(x+y)}\Big)=e^{4(x+y)+\log 3}$$

with $h=2(x+y)$ and $g=4(x+y)+\log 3$. Then the equation admits a transcendental entire solution $u=e^{x+y}$.

Proof of Theorem 3.3 It is clear that each of the m unitary complex roots of $z^m=1$ is a solution of (3.1). Thus, it suffices to show that each entire solution u of (3.1) in $\mathbf{C}^2$ satisfies that $u^m=1$.

Let u be an entire solution of (3.1) in $\mathbf{C}^2$. Make the transformation $x=z+w$, $y=\frac{z-w}{i}$. Denote $v=v(z,w)=u(x,y)$. It is clear from (3.1) that $u(x,y)\neq 0$ for any $(x,y)\in\mathbf{C}^2$ and so that $v(z,w)\neq 0$ for any $(z,w)\in\mathbf{C}^2$. For notational convenience, we continue to use g and h to denote the functions after the transformation. Then $v_z=u_x+\frac{1}{i}u_y$, $v_w=u_x-\frac{1}{i}u_y$, and equation (3.1) reduces to $v^m(v_zv_w+e^h)=e^h$, or

$$v_zv_w=e^hv^{-m}(1-v^m). \tag{3.2}$$

We show that (3.2) implies that $v^m\equiv 1$ in $\mathbf{C}^2$. To this end, we discuss two cases: $m>1$ and $m=1$.

Case (i): $m>1$. First, we assume that at any zero (a,b) of $v(z,w)-1$, $v_w(a,b)=0$. Then for any $c\in\mathbf{C}$, the one-variable function $f_c(w):=v(c,w)$ does not assume 0 and $f_c(w)-1$ has only multiple zeros. By Corollary 2.3, f_c must be constant. We then obtain that $v_w(c,w)=f_c'(w)\equiv 0$ for any c and any w. That is, $v_w\equiv 0$ in $\mathbf{C}^2$, which implies from (3.2) that $v^m\equiv 1$ and thus $u^m\equiv 1$ in $\mathbf{C}^2$. The theorem thus holds in this case.

Next, we only need to consider the case where there exists a zero (a,b) of $v(z,w)-1$ such that $v_w(a,b)\neq 0$. Let η be a unitary root of $z^m=1$. If $v(z,w)-\eta\equiv 0$ in $\mathbf{C}^2$, we are done. Assume that $v(z,w)-\eta\not\equiv 0$ in $\mathbf{C}^2$ for each unitary root η of $z^m=1$. Consider the one variable function $f_b(z):=v(z,b)-1$. We then must have that

$$f_b(z)=v(z,b)-1=0 \tag{3.3}$$

for all $z\in\mathbf{C}$; otherwise, the right-hand side of (3.2), restricted to the complex line $w=b$, is a nonzero entire function of the variable z, which vanishes at a by the choice of (a,b). But then the order of the zero a on the right-hand side of (3.2) is higher than that of a on the left-hand side of (3.2) noting that $v_w(a,b)\neq 0$, which is absurd.

Let $\eta_0\neq 1$ be a unitary root of $z^m=1$. (Such an η_0 exists since $m\geq 2$.) We claim that for each zero (α,β) of $v(z,w)-\eta_0$, we must have that $v_z(\alpha,\beta)=0$. Suppose that there is a zero (α,β) of $v(z,w)-\eta_0$ with $v_z(\alpha,\beta)\neq 0$. Then, in the same argument for (3.3) above, we can deduce that $v(\alpha,w)-\eta_0=0$ for all $w\in\mathbf{C}$. In particular, $v(\alpha,b)-\eta_0=0$, which contradicts (3.3). Using this proved claim, we see that for any $\beta\in\mathbf{C}$, the one-variable function $v_\beta(z):=v(z,\beta)-\eta_0$ has only multiple zeros. But, $v_\beta(z)$ does not assume $-\eta_0$, since v does not assume 0. Thus, we can apply Corollary 2.3 to $v_\beta(z)+\eta_0$

to obtain that $v_\beta(z)$ is constant in $\mathbf{C}$ and thus $v_z(z,\beta) = v'_\beta(z) \equiv 0$ for any β and any z in $\mathbf{C}$. That is, $v_z \equiv 0$ in $\mathbf{C}^2$, which implies from (3.2) that $v^m \equiv 1$ in $\mathbf{C}^2$.

Case (ii): $m = 1$. In this case, the number η_0 in the above proof does not exist any more. Nevertheless, we can make the transformation $v = \frac{1}{s^2}$ to transform the equation (3.2) to the equation

$$s_z s_w = -\frac{1}{4}s^6 e^h(1-s^2).$$

We observe that this equation is of the same form as equation (3.2) with $m = 2$, and the "coefficient" $-\frac{1}{4}s^6e^h$ of $1-s^2$ does not affect the proof in Case (i). Thus, the above same argument yields that $s^2 \equiv 1$, and thus $v \equiv 1$.

Therefore, in any case, we have that $v^m \equiv 1$ and thus $u^m \equiv 1$. This completes the proof of the theorem. ■

References

[1] L. V. Ahlfors, *Conformal invariants: topics in geometrical function theory*. McGraw-Hill Series in Higher Mathematics, McGraw-Hill, New York, 1973.

[2] E. Borel, *Sur les zéros des functions entières.* Acta Math. **20**(1897), no. 1, 357–396. http://dx.doi.org/10.1007/BF02418037

[3] R. Courant and D. Hilbert, *Methods of mathematical physics, II. Partial differential equations*. Wiley Classics Library, John Wiley & Sons, 1991.

[4] B. Davis, *Picard's theorem and Brownian motion.* Trans. Amer. Math. Soc. **23**(1975), 353–362. http://dx.doi.org/10.2307/1998050

[5] W. H. J. Fuchs, *Topics in the theory of functions of one complex variable.* D. Van Nostrand, Princeton, NJ, 1967.

[6] W. K. Hayman, *Meromorphic functions*. Clarendon Press, Oxford, 1964.

[7] E. Hille, *Ordinary differential equations in the complex domain*. Dover, Mineola, NY, 1997.

[8] J. L. Lewis, *Picard's theorem and Richman's theorem by way of Harnack's inequality.* Proc. Amer. Math. Soc. **122**(1994), 199–206. http://dx.doi.org/10.2307/2160861

[9] B. Q. Li, *On meromorphic solutions of $f^2 + g^2 = 1$.* Math. Z. **258**(2008), 763–771. http://dx.doi.org/10.1007/s00209-007-0196-2

[10] ———, *On Fermat-type functional and partial differential equations.* Springer Proceedings in Mathematics, 16, Springer, Milan, 2012, pp. 209–222. http://dx.doi.org/10.1007/978-88-470-1947-8_13

[11] ———, *Estimates on derivatives and logarithmic derivatives of holomorphic functions and Picard's theorem.* J. Math. Anal. Appl. **442**(2016), no. 2, 446–450. http://dx.doi.org/10.1016/j.jmaa.2016.04.060

[12] E. G. Saleeby, *Meromorphic solutions of generalized inviscid Burgers' equations and a family of quadratic PDEs.* J. Math. Anal. Appl. **425**(2015), 508–519. http://dx.doi.org/10.1016/j.jmaa.2014.12.046

[13] G. Y. Zhang, *Curves, domains and Picard's theorem.* Bull. London Math. Soc. **34**(2002), 205–211. http://dx.doi.org/10.1112/S0024609301008712

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